Mini-course on K3 surfaces

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Preliminaries on algebraic surfaces

With the word *surface* we will always mean a smooth compact complex connected surface. Most of the time we will be interested in projective surfaces, even if sometimes we will deal with non-projective K3 surfaces, when studying the period domain of such surfaces.

1. Geometry

1.1. Divisors. By a *divisor* on a smooth compact complex variety X we mean a formal finite sum $D := \sum_i a_i C_i$, where the a_i are integers and the C_i are irreducible hypersurfaces of X. The *support* of D is the union $\cup_i C_i$. The divisor D is *effective* if all the $a_i \geq 0$. We say that the divisor is *prime* if it contains just one summand with coefficient 1. In this case we will often identify the divisor with the hypersurface itself. We recall that given a rational function f on X its *principal divisor* is

$$\operatorname{div}(f) := \sum_{C \subset X} \nu_C(f) C,$$

where the sum runs over all the hypersurfaces in X and $\nu_C(f) \in \mathbb{Z}$ is the order of zero/pole of f at C. Two divisors D and D' of X are *linearly equivalent* if $D - D' = \operatorname{div}(f)$ for some rational function f on X. In this case we write $D \sim D'$ to denote that D is linearly equivalent to D'. The set of divisors of X form a free abelian group denoted by $\operatorname{Div}(X)$. It contains the subgroup $\operatorname{PDiv}(X)$ of principal divisors. The quotient

$$\operatorname{Pic}(X) := \operatorname{Div}(X) / \operatorname{PDiv}(X)$$

is the *Picard group* of X. Elements of the Picard group will be called *classes*.

Example 1.1.1. As easy examples one can keep in mind $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}[H]$, where H is a hyperplane, and $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2]$, where each F_i is a fiber of the i-th projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

1.2. Intersection of divisors. Given a divisor D on a surface X there exists an open covering $\{U_i\}$ of X such that the restriction of D to each U_i is a principal divisor $\operatorname{div}(f_i)$. Given an irreducible curve C of X, not contained in the support of D, we define the *restriction* $D|_C$ to be the divisor of C locally defined by $\operatorname{div}(f_i|C)$ on the open subset $U_i \cap C$ of C. Given a curve C and a divisor D on a surface X their *intersection* is the number:

$$D \cdot C := \deg(D|_C),$$

where the right hand side is the degree of a divisor on a curve. Observe that from the previous definition we immediately have $D \cdot C = D' \cdot C$ if D is linearly equivalent to D' and the support of D' does not contain C. We use this property to define the intersection $D \cdot C$ without restrictions on D: if the support of D contains C, then we choose a $D' = D + \mathrm{div}(f)$, where f is a rational function such that $-\nu_C(f)$ is equal to the multiplicity of D at C, and define $D \cdot C := D' \cdot C$. The intersection number of two divisors is defined as $D \cdot \sum_i a_i C_i := \sum_i a_i D \cdot C_i$. It is possible to prove that $A \cdot B = B \cdot A$ for any pair of divisors A and B of X. Since $A' \cdot B' = A \cdot B$ if $A' \sim A$ and $B' \sim B$, then the intersection is well defined on the Picard group of X, that is it induces a bilinear map

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}.$$

The Picard group, modulo torsion, equipped with the quadratic form defined by the intersection pairing is called the *Picard lattice* of X.

Example 1.2.1. The surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ has a Picard group of rank 2. The two generators $[F_1]$ and $[F_2]$ have intersections $F_i \cdot F_j = \delta_{ij}$. Hence the Picard lattice of X is represented by the Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The quadratic form on $\mathbb{Z}^2$ represented by the above matrix is denoted by U.

1.3. The canonical class. Let X be a surface and let ω be a meromorphic 2-form on X. If U is an open affine subset of X with coordinates z_1, z_2, then

$$\omega|_U = f_U \, dz_1 \wedge dz_2$$

were f_U is a meromorphic function on U. If we consider an open affine covering $\{U_i\}$ of X and let $\omega|_{U_i} = f_{U_i} dz_1^i \wedge dz_2^i$, then the collection of principal divisors $\mathrm{div}(f_{U_i})$ defines a divisor of X called a *canonical divisor* of X and denoted by K_X. Now consider two meromorphic forms ω and ω' of X. If we write these forms on two affine open subsets U and V of X, we get $\omega|_U = \alpha_U \, dz_1 \wedge dz_2$ and $\omega'|_U = \alpha'_U \, dz_1 \wedge dz_2$ and similarly on V. Observe that $\alpha_V = J \, \alpha_U$ and $\alpha'_V = J \, \alpha_U$, where J is the Jacobian of the coordinate change from U to V. In particular the quotient α'/α does not change, so that $\mathrm{div}(\alpha'/\alpha)$ is a principal divisor. Hence the class of K_X in $\mathrm{Pic}(X)$ is independent on the choice of the meromorphic form.

Example 1.3.1. We can cover $\mathbb{P}^1$ with two affine coordinate charts U_0 and U_1. The meromorphic form dz_0 on U_0 glues on $U_0 \cap U_1$ with the form $-1/z_1^2 \, dz_1$ of U_1 since $z_1 = 1/z_0$ on $U_0 \cap U_1$. Hence a canonical divisor of $\mathbb{P}^1$ is $K_{\mathbb{P}^1} = -2p$, where p is the zero locus of z_1 in U_1. Similarly one can prove that $K_{\mathbb{P}^n} = -(n+1)H$, where H is a hyperplane of $\mathbb{P}^n$.

1.4. Adjunction formula. If C is a smooth curve on a surface X a canonical divisor for C can be obtained by means of the *adjunction formula*:

$$K_C = (K_X + C)|_C.$$

Since C is a curve, the degree of a canonical divisor is $2g(C) - 2$, where $g(C)$ is the topological genus of C. Thus we have

$$2g(C) - 2 = \deg(K_C) = (K_X + C) \cdot C.$$

It follows that the right hand side intersection number is always even.

Example 1.4.1. If F_i is the fiber of the i-th projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, then F_i is a smooth rational curve with $F_i^2 = 0$. Hence $K_X \cdot F_i = -2$. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. If $[K_X] = a[F_1] + b[F_2]$, then by using the previous observation and our knowledge of the intersections between the F_i's, we get $[K_X] = -2[F_1] - 2[F_2]$.

EXAMPLE 1.4.2. If $K_X \sim 0$ and C is a smooth curve, then $C^2 = 2g(C) - 2$. In particular $C^2 = -2$ if C is rational and $C^2 = 0$ if C is elliptic.

1.5. Riemann-Roch formula. Given a divisor D on a surface X we can form the sheaf $\mathcal{O}_X(D)$, locally defined, on an open subset U of X, as the complex vector space of rational functions f of U such that $\mathrm{div}(f) + D$ is an effective divisor of U. The dimension of the cohomology groups $H^i(X, \mathcal{O}_X(D))$ are denoted by $h^i(\mathcal{O}_X(D))$ and the Euler characteristic by $\chi(\mathcal{O}_X(D)) := h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D))$. The *Riemann-Roch formula* is

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D - K_X) \cdot D + \frac{1}{12}(K_X^2 + e(X)),$$

where $e(X)$ denotes the topological Euler characteristic of X, that is the alternating sum of the ranks of the singular homology groups of X. Observe that $h^0(\mathcal{O}_X) = 1$ since X is a complete variety. Moreover $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X))$ due to an important theorem of Serre, called *Serre's duality theorem*. The Euler characteristic of the sheaf of regular function $\mathcal{O}_X$ is related to the Euler characteristic of the surface X by the following *Noether formula*:

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X)),$$

which is easily obtained by putting $D = 0$ in the Riemann-Roch formula.

EXAMPLE 1.5.1. The Euler characteristic of the projective plane is $e(\mathbb{P}^2) = 3$, since it has cohomology only in even dimension and all these groups are isomorphic to $\mathbb{Z}$. Since $K_{\mathbb{P}^2} = -3H$, then $K_{\mathbb{P}^2}^2 = 9$, so that we have $\chi(\mathcal{O}_{\mathbb{P}^2}) = 1$. Hence the Riemann-Roch formula for the projective plane gives:

$$\chi(\mathcal{O}_{\mathbb{P}^2}(dH)) = \frac{1}{2}(dH + 3H) \cdot dH + 1 = \frac{1}{2}(d+3)d + 1$$

since $H^2 = 1$. Observe that the last formula gives exactly the dimension of the degree d part of the graded polynomial ring $\mathbb{C}[x_0, x_1, x_2]$, when $d \geq 0$. Hence $\chi(\mathcal{O}_{\mathbb{P}^2}(dH)) = h^0(\mathcal{O}_{\mathbb{P}^2}(dH))$. Indeed it is possible to prove that both the higher cohomology groups of the sheaf $\mathcal{O}_{\mathbb{P}^2}(dH)$ vanish for $d > 0$.

2. Topology

2.1. Poincaré duality. Given a compact complex connected surface X we will denote by $H^i(X, \mathbb{Z})$ its i-th singular cohomology group and by $H_i(X, \mathbb{Z})$ the i-th singular homology group . All of them are finitely generated abelian groups. Recall the content of *Poincaré duality* for surfaces [**GH94**, Pag. 53]: for each $i \geq 0$ there is a natural isomorphism

$$H_i(X, \mathbb{Z}) \to H^{4-i}(X, \mathbb{Z}).$$

By the definition of singular homology and cohomology there is a natural map $H^i(X, \mathbb{Z}) \to H_i(X, \mathbb{Z})^*$ coming from the corresponding map at the level of cochain. The *universal coefficient theorem* asserts that the following sequence of abelian group is exact:

$$0 \longrightarrow \mathrm{Ext}^1(H_{i-1}(X, \mathbb{Z}), \mathbb{Z}) \longrightarrow H^i(X, \mathbb{Z}) \longrightarrow H_i(X, \mathbb{Z}) \longrightarrow 0.$$

In particular $H^i(X,\mathbb{Z})/\operatorname{Tors} \cong H_i(X,\mathbb{Z})^*$ and $\operatorname{Tors} H^i(X,\mathbb{Z}) \cong \operatorname{Tors} H_{i-1}(X,\mathbb{Z})$, where Tors denotes the torsion part of an abelian group. As a consequence there is a perfect bilinear symmetric pairing, called the *intersection pairing*:

$$H^2(X,\mathbb{Z})/\operatorname{Tors} \times H^2(X,\mathbb{Z})/\operatorname{Tors} \to \mathbb{Z}.$$

Here the word perfect means that the matrix defining the pairing has determinant ± 1. Other two easy consequences of Poincaré duality and the universal coefficient theorem are

$$\operatorname{Tors} H_2(X,\mathbb{Z}) \cong \operatorname{Tors} H_1(X,\mathbb{Z}) \qquad \operatorname{Tors} H_3(X,\mathbb{Z}) = (0).$$

It is worth noticing that $H^i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^i(X,\mathbb{Z})_{\mathrm{DR}}$, where the right hand side is the De Rham cohomology of X, that is the real vector space of closed forms modulo exact forms. The intersection pairing on $H^i(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ can be thus expressed at the level of forms as:

$$\omega_1 \cdot \omega_2 := \int_X \omega_1 \wedge \omega_2.$$

EXAMPLE 2.1.1. Let $X := \mathbb{C}^2/\Gamma$ be a complex torus obtained by taking the quotient of $\mathbb{C}^2$ with a maximal subgroup $\Gamma \cong \mathbb{Z}^4$. The group Γ acts by translation, so if dx_i is a 1-form on $\mathbb{C}^2$, then it descends to a 1-form on X since it is invariant with respect to the action of Γ, that is $d(x_i + a) = dx_i$ for any $a \in \Gamma$. Moreover one can prove that it is a closed form and that $\{dx_i \wedge dx_j : 1 \le i < j \le 4\}$ gives a basis of $H^2(X,\mathbb{Z})$. Observe that for example $dx_1 \wedge dx_2 \wedge dx_i \wedge dx_j = 0$ whenever dx_i or dx_j is linearly dependent with dx_1, dx_2. Thus the Gram matrix with respect to the given basis is the block matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The standard notation for this type of lattice is $U \oplus U \oplus U$, where U is the rank two lattice whose Gram matrix is the up-left two by two submatrix of the previous matrix.

2.2. The topological index theorem. The intersection form on $H^2(X,\mathbb{Z})/\operatorname{Tors}$ defines a quadratic form $q : H^2(X,\mathbb{Z})/\operatorname{Tors} \to \mathbb{Z}$ by $q(x) := x \cdot x$. Taking tensor product with the real numbers we obtain a real vector space $H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ equipped with a non-degenerate quadratic form. Its signature is a topological invariant of X. If we denote by K_X the canonical divisor of X, with K_X^2 its self-intersection and by $e(X) := \sum_{i=0}^{4} (-1)^i \operatorname{rk} H_i(X,\mathbb{Z})$ the Euler characteristic of X, then we have the following.

THEOREM 2.2.1. *Let b^+ and b^- be respectively the number of positive and negative eigenvalues of the quadratic form q on the real vector space $H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then*

$$b^+ - b^- = \frac{1}{3}(K_X^2 - 2e(X)).$$

EXAMPLE 2.2.2. Let X be a smooth cubic surface of $\mathbb{P}^3$. Since X is birational to the projective plane, then it is possible to show that $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$, so that $\chi(\mathcal{O}_X) = 1$. By adjunction formula we have $K_X = -H|_X$, where H is a plane. Hence $K_X^2 = 3$. By using the previous facts and Noether's formula we get $e(X) = 9$, which gives $b^+ - b^- = -5$. Still by the Euler characteristic of X we deduce that $H^2(X, \mathbb{Z})$ has rank 7, so that $b^+ = 1$ and $b^- = 6$.

2.3. The exponential sequence. Let X be a smooth compact complex variety and denote by $\mathbb{Z}_X$ the sheaf of locally constant integer functions on X. The exponential sequence of X is

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0,$$

where $\exp(f) := e^{2\pi i f}$. We briefly recall that the singular cohomology groups $H^i(X, \mathbb{Z})$ and the sheaf cohomology groups $H^i(X, \mathbb{Z}_X)$ are isomorphic and we will often identify them in the future. Moreover the Picard group of X is isomorphic to the cohomology group $H^1(X, \mathcal{O}_X^*)$. Hence taking cohomology we obain the long exact sequence

$$\begin{array}{ccccccccc} H^1(X,\mathbb{Z}_X) & \longrightarrow & H^1(X,\mathcal{O}_X) & \xrightarrow{\exp} & H^1(X,\mathcal{O}_X^*) & \xrightarrow{\tau} & H^2(X,\mathbb{Z}_X) & \longrightarrow & H^2(X,\mathcal{O}_X) \\ \| & & & & \| & & \| & & \\ H^1(X,\mathbb{Z}) & & & & \mathrm{Pic}(X) & & H^2(X,\mathbb{Z}). & & \end{array}$$

The image of the above exponential map is isomorphic to the quotient of $H^1(X, \mathcal{O}_X)$ by the image of the subgroup $H^1(X, \mathbb{Z})$. It is possible to see that this quotient is an abelian variety, that is a projective complex torus, also denoted by $\mathrm{Pic}(X)^0$, which when X is a curve is exactly the group of degree zero divisors modulo linear equivalence. The image of τ is the *Nerón-Severi group* of X, denoted $\mathrm{NS}(X)$. Hence we can summarize the previous observations in the following exact sequence:

$$0 \longrightarrow \mathrm{Pic}(X)^0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\tau} \mathrm{NS}(X) \longrightarrow 0.$$

It is important to remark that the homomorphism τ is an isometry with respect to the two quadratic forms defined in $\mathrm{Pic}(X)$ and $H^2(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$, that is

$$\tau([D_1]) \cdot \tau([D_2]) = [D_1] \cdot [D_2].$$

2.4. The Lefschetz theorem on hyperplane sections. Let X be a smooth algebraic complex subvariety of dimension n of $\mathbb{P}^N$. Let H be a hyperplane and let $Y = X \cap H$. Then the inclusion map $Y \to X$ induces isomorphisms

$$H_i(Y, \mathbb{Z}) \to H_i(X, \mathbb{Z})$$

for any $i < n - 1$ and is surjective for $i = n - 1$. A similar statement holds for the induced homomorphism

$$\pi_1(Y) \to \pi_1(X).$$

It is an isomorphism when $n \geq 3$ and is surjective when $n = 2$ (see [**EoMb**]).

Exercises

EXERCISE 2.1. Calculate a basis of the Picard group of $\mathbb{P}^a \times \mathbb{P}^b$.

EXERCISE 2.2. Let X be a smooth cubic surface of $\mathbb{P}^3$ which contains a line L. Calculate L^2.

EXERCISE 2.3. Let $X = \mathbb{C}^2/\Gamma$ be a complex torus. Prove that $K_X \sim 0$.

EXERCISE 2.4. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let C be a smooth curve of X whose class in $\operatorname{Pic}(X)$ is $a[F_1] + b[F_2]$. Find the genus of C and $\chi(\mathcal{O}_X(C))$.

EXERCISE 2.5. Let X be a smooth projective surface with $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$. Prove that $\operatorname{Pic}(X)/\operatorname{Tors}$ is unimodular. Moreover, if $n\,K_X \sim 0$ for some positive integer n, show that the previous lattice has signature $(1, 9)$.

CHAPTER 1

The period domain

1. Topological properties

1.1. K3 surfaces. A *K3 surface* is a smooth complex compact surface X which satisfies the following:

$$H^0(X, \mathcal{O}_X(K_X)) = \mathbb{C}\,\omega_X \qquad H^1(X, \mathcal{O}_X) = (0). \tag{1.1.1}$$

The first condition tells us that, modulo scalar multiplication, X admits a unique holomorphic 2-form ω_X, while the second condition is equivalent to ask for the vanishing of the first Betti number of X. As an example of K3 surface consider a smooth quartic surface $X \subset \mathbb{P}^3$, like the Fermat surface defined by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

By adjunction formula the fact that $X \sim 4H$ and the fact that $-K_{\mathbb{P}^3} = -4H$, where H is a plane, we deduce that the canonical class of X is trivial. Hence the first condition in (1.1.1) is satisfied. It is possible to prove that also the second condition is satisfied, by using the Lefschetz theorem on hyperplane sections, which gives the vanishing of $H^1(X, \mathbb{C})$, and the Hodge decomposition that we will introduce later in this chapter. By an iterated application of the previous argument, one can prove that a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$ is again a K3 surface. The same holds for the complete intersection of three quadrics in $\mathbb{P}^5$.

Aim of this mini-course is to introduce the basic theory of K3 surfaces which from many perspectives represent a 2-dimensional generalization of elliptic curves.

1.2. Singular cohomology I. Let X be a K3 surface. Then $h^1(\mathcal{O}_X) = 0$ by definition and $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X)) = h^0(\mathcal{O}_X) = 1$ by Serre's duality. So the Euler characteristic of the structure sheaf $\mathcal{O}_X$ is 2. Hence by Noether's formula and the fact that K_X is trivial we get

$$e(X) = 12\,(\chi(\mathcal{O}_X) + K_X^2) = 24.$$

Since $h^1(\mathcal{O}_X) = 0$, then by the exponential sequence the rank of $H^1(X, \mathbb{Z})$ is zero. Hence the same is true for $H_1(X, \mathbb{Z})$, so that by Poincaré duality also $H^3(X)$ has zero rank. Since X is connected $H^0(X) \cong \mathbb{Z}$ and $H^4(X) \cong \mathbb{Z}$ being X orientable. Thus by our previous calculation of the Euler characteristic of X we deduce that $H^2(X, \mathbb{Z})$, or equivalently $H^2(X, \mathbb{Z})$, has rank 22.

If C is a smooth curve on X, and K_C is the canonical divisor of C, by adjunction formula

$$2g(C) - 2 = \deg(K_C) = (K_X + C) \cdot C = C^2,$$

where $g(C)$ is the topological genus of C. In particular C^2 is an even number. Recall that the curve C has a representative class $[C]$ in $\mathrm{Pic}(X)$ and a class $\tau([C])$ in $H^2(X, \mathbb{Z})$, defined by means of the exponential sequence. Thus we have just shown that all the elements of the Nerón-Severi group of X have even square. It is possible to extend this observation to the whole cohomology group, that is x^2 is an even number for any $x \in H^2(X, \mathbb{Z})$ (see [**BHPVdV04**]).

1.3. The fundamental group. The proof that any K3 surface is simply connected is not easy since it makes use of the full knowledge of the period domain. To sketch the idea, the proof is in two steps. First of all one proves that all K3 surfaces are diffeomorphic. This result depends on the fact that the period domain of K3 surfaces is connected (see Theorems 3.4.1 and 3.5.1) and the fact that a holomorphic family of complex manifolds is a trivial family from the differential point of view [**BHPVdV04**]. In particular it is enough to show that a smooth quartic surface X of $\mathbb{P}^3$ is simply connected.

PROPOSITION 1.3.1. *Any smooth quartic surface of* $\mathbb{P}^3$ *is simply connected.*

PROOF. Consider the degree four Veronese embedding $\nu : \mathbb{P}^3 \to \mathbb{P}^{34}$ and observe that it maps quartic surfaces of $\mathbb{P}^3$ to hyperplane sections of $\nu(\mathbb{P}^3)$ so that $X \cong \nu(\mathbb{P}^3) \cap H$ for some hyperplane H of $\mathbb{P}^{34}$. Then one applies the Lefschetz theorem on hyperplane sections to get

$$\pi_1(X) \cong \pi_1(\nu(\mathbb{P}^3) \cap H) \cong \pi_1(\nu(\mathbb{P}^3)) \cong \pi_1(\mathbb{P}^3),$$

showing that X is simply connected. □

REMARK 1.3.2. If X is a K3 surface then, by the exponential sequence and $h^1(\mathcal{O}_X) = 0$, we know that $H^1(X, \mathbb{Z})$ has rank zero as already observed before. By the universal coefficient theorem also $H_1(X, \mathbb{Z})$ has rank zero. Observe that this argument is not enough to conclude that X is simply connected. Indeed consider the Godeaux surface Y, defined as the quotient of the Fermat quintic S:

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$$

with respect to the action $x_i \mapsto \varepsilon^i x_i$, where ε is a 5-th root of the unity. Since the action has no fixed points, then Y is a smooth surface. Moreover S is simply connected by Lefschetz theorem on hyperplane sections. Hence it is the universal covering space of Y so that $\pi_1(Y) \cong \mathbb{Z}/5\mathbb{Z}$ and $h^1(\mathcal{O}_Y) = 0$ due to the Hodge decomposition (2.3.1) of $H^1(X, \mathbb{C})$.

2. Hodge theory

2.1. Exterior forms. Let V be a complex vector space with basis $\{v_1, \ldots, v_n\}$. The space of (p, q)-forms on V is the complex vector space $V^{p,q}$ generated by the symbols

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \wedge \bar{v}_{j_1} \wedge \cdots \wedge \bar{v}_{j_q},$$

where $v \wedge w = -w \wedge v$ and the indices i_k and j_s vary over all the possible subsets of $\{1, \ldots, n\}$ of cardinalities p and q respectively. The symbol $\wedge V$ denotes the *exterior algebra* of V, meaning with this the vector space

$$\wedge V := \bigoplus_{p+q=0}^{2n} V^{p,q}$$

together with the antisymmetric product $(w, w') \mapsto w \wedge w'$.

EXAMPLE 2.1.1. The exterior algebra of $\mathbb{C}$ is $\wedge\mathbb{C} = \mathbb{C}^{0,0} \oplus \mathbb{C}^{1,0} \oplus \mathbb{C}^{0,1} \oplus \mathbb{C}^{1,1}$ where for example $\mathbb{C}^{1,0} = \langle v \rangle$ and $\mathbb{C}^{1,1} = \langle v \wedge \bar{v} \rangle$. Observe that $\mathbb{C}^{2,0} = \langle 0 \rangle$, since $v \wedge v = 0$ by antisymmetry.

2.2. Dolbeault cohomology. Given a smooth compact complex surface X with cotangent bundle $\mathcal{E}_X$, we define its *exterior bundle* $\wedge\mathcal{E}_X$ to be the vector bundle whose fibers are the exterior algebras $\wedge{\mathcal{E}_X}_p$, for $p \in X$. Its transition functions on the intersection $U_i \cap U_j$ of two open subsets of a trivializing covering of X, are the matrices $\wedge g_{ij}$, where g_{ij} are the transition matrices of the bundle Ω_X. Now, since X has dimension 2, then

$$\wedge\mathcal{E}_X = \bigoplus_{p+q=0}^{4} \mathcal{E}_X^{p,q},$$

moreover the right hand side summands vanish whenever $p > 2$ or $q > 2$. An interesting property of the exterior bundle is that if we have a holomorphic map $f : X \to Y$ of compact complex varieties, then the pull-back $f^* : \wedge\mathcal{E}_Y \to \wedge\mathcal{E}_X$ maps each $\mathcal{E}_Y^{p,q}$ into the corresponding $\mathcal{E}_X^{p,q}$. This property, together with the existence of a linear differential operator

$$\bar{\partial} : \Gamma(X, \mathcal{E}_X^{p,q}) \to \Gamma(X, \mathcal{E}_X^{p,q+1})$$

such that $\bar{\partial} \circ \bar{\partial} = 0$, gives a cohomology theory for compact complex varieties. This is the *Dolbeault cohomology* whose groups are denoted by $H^{p,q}(X)$ and their dimensions by $h^{p,q}(X)$.

Remark 2.2.1. Denote by Ω_X^p the sheaf of *holomorphic p-forms*, that is the sheaf of forms which locally can be written as $\alpha\, dz_{i_1} \wedge \cdots \wedge dz_{i_k}$, with α holomorphic. The sheaf admits the following *acyclic resolution*:

$$\Omega_X^p \longrightarrow \mathcal{E}_X^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{p,1} \xrightarrow{\bar{\partial}} \cdots$$

meaning with this that the sequence is exact and the higher cohomology ($i > 0$) of all the $\mathcal{E}_X^{p,k}$ vanishes. The exactness of the sequence is due to Poincaré Lemma for the operator $\bar{\partial}$. Hence by considering the spectral sequence of the double complex $\check{\mathcal{C}}^i(\mathcal{E}_X^{p,j})$, given by Čech cocycles of the sheaf $\Omega_X^{p,j}$, one proves that

$$H^q(X, \Omega_X^p) \cong H^{p,q}(X).$$

2.3. Hodge decomposition. Let V be a finitely generated free abelian group. a *Hodge structure* of level n, with $n \in \mathbb{Z}$, on $V \otimes_{\mathbb{Z}} \mathbb{C}$ is a direct sum decomposition

$$V \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. Here the overline means the complex conjugation. Denote by $b_i(X)$ the i-th Betti number of X, that is the rank of the singular homology group $H_i(X, \mathbb{Z})$. In case X is a smooth projective variety, or just smooth Kähler variety [**EoMc**], the n-th singular cohomology group of X admits the following Hodge structure of level n:

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X). \tag{2.3.1}$$

In particular each odd Betti number $b_{2k+1}(X)$ is an even number. Observe that by definition a K3 surface is not necessarily projective, but it is always Kähler, as shown in [**Siu83**]. Hence the cohomology of any K3 surface admits a Hodge structure.

We conclude the section by applying the result of the previous proposition and Serre's duality, to the description of the *Hodge diamond* of a K3 surface X. This is a picture containing all the dimensions of the spaces $h^{p,q}(X)$.

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0}(X) & & h^{0,1}(X) & \\ h^{2,0}(X) & & h^{1,1}(X) & & h^{0,2}(X) \\ & h^{2,1}(X) & & h^{1,2}(X) & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

FIGURE 1. The Hodge diamond of a K3 surface

2.4. Singular cohomology II. As a consequence of the previous proposition we have the following [**BHPVdV04**].

PROPOSITION 2.4.1. *Let X be a K3 surface. Then the groups $H^1(X,\mathbb{Z})$ and $H^3(X,\mathbb{Z})$ are trivial. Moreover $H^2(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22 which, endowed with the quadratic form given by the cup product, is an even lattice of signature $(3,19)$.*

PROOF. By the exponential sequence we already know that the first and third Betti numbers of X are zero. Hence $H^1(X,\mathbb{Z}) = (0)$ and

$$\operatorname{Tors} H^2(X,\mathbb{Z}) \cong \operatorname{Tors} H_2(X,\mathbb{Z}) \cong \operatorname{Tors} H_1(X,\mathbb{Z}) \cong \operatorname{Tors} H^3(X,\mathbb{Z})$$

by Poincaré duality and the universal coefficient theorem. Hence it is enough to show that $H_1(X,\mathbb{Z})$ has no torsion (and thus it is trivial). Assume the countrary, then $\pi_1(X)$ would contain a torsion element. This is equivalent to say that X admits a degree $n > 1$ finite unbranched cover $\pi : Y \to X$, where Y is a compact complex surface. Now $e(Y) = n \cdot e(X) = 24\,n$ and $K_Y = \pi^* K_X \sim 0$ so that $h^2(\mathcal{O}_Y) = 1$. Hence by Noether formula we get

$$2 - h^1(\mathcal{O}_Y) = \chi(\mathcal{O}_Y) = \frac{1}{12}(K_Y^2 + e(Y)) = 2\,n,$$

which gives $n = 1$, a contradiction. To conclude the proof, let b^+ and b^- be, respectively, the number of positive and negative eigenvalues of the quadratic form defined by the intersection form on $H^2(X,\mathbb{Z})$. By the topological index theorem and our calculation of the Euler characteristic of X we get

$$b^+ - b^- = \frac{1}{3}(K_X^2 - 2e(X)) = -16.$$

Since $H^2(X,\mathbb{Z})$ has rank 22, then the signature of its quadratic form is $(3,19)$. We have already seen that it is an even lattice. □

REMARK 2.4.2. Observe that even if the argument adopted in the previous proposition shows that $\pi_1(X)_{\mathrm{ab}} \cong H_1(X,\mathbb{Z}) = (0)$, this is not enough to conclude that X is simply connected. There are examples of topological spaces with trivial homology and non-trivial fundamental group, like the Poincaré Homology 3-sphere (`http://goo.gl/sV1Ds`).

2.5. Lattice structure in cohomology. Given a K3 surface X we can consider the Hodge decomposition of its second cohomology group:

$$\begin{array}{ccccccc} H^2(X,\mathbb{C}) & = & H^{2,0}(X) & \oplus & H^{1,1}(X) & \oplus & H^{0,2}(X) \\ & & \| & & & & \| \\ & & \mathbb{C}\,\omega_X & & & & \mathbb{C}\bar{\omega}_X, \end{array}$$

where the two vertical isomorphisms are given at the end of the subsection 2.2 of Chaper 1. Observe that $H^2(X,\mathbb{C})$ is equipped with a quadratic form coming from the cup product defined on singular cohomology of X. This product can be written in terms of differential forms as (see Subsection 2.1 of the Preliminaries):

$$(\omega_1,\omega_2) \mapsto \omega_1 \cdot \omega_2 := \int_X \omega_1 \wedge \omega_2, \tag{2.5.1}$$

where ω_1 and ω_2 are closed 2-forms on X. In this way, if z_1 and z_2 are local coordinates on X, then a local expression of the holomorphic 2-form ω_X is $\alpha\, dz_1 \wedge dz_2$, with α holomorphic. Thus we immediately deduce the *Riemann relations*:

$$\omega_X \cdot \omega_X = 0 \qquad \omega_X \cdot \bar{\omega}_X > 0.$$

Moreover both ω_X and $\bar{\omega}_X$ are orthogonal to any element of $H^{1,1}(X)$, since such an element is locally written as $\beta\, dz_1 \wedge d\bar{z}_2$ or as $\gamma\, d\bar{z}_1 \wedge dz_2$. Observe that if V is the complex vector space spanned by ω_X and $\bar{\omega}_X$, then the two-dimensional real vector space $V_\mathbb{R} := \{x \in V : x = \bar{x}\}$ has a basis made by $\omega_X + \bar{\omega}_X$ and $i(\omega_X - \bar{\omega}_X)$. With respect to this basis the intersection form is diagonal and positive-definite. Also $V = V_\mathbb{R} \otimes \mathbb{C}$ and the quadratic form on V is that induced by the complexification of $V_\mathbb{R}$. We have already seen that the intersection form on $H^2(X,\mathbb{Z})$ is *even*, meaning with this that x^2 is even for any $x \in H^2(X,\mathbb{Z})$, and *unimodular*, which means that the induced map $H^2(X,\mathbb{Z}) \to H_2(X,\mathbb{Z})^*$ is an isomorphism, with signature $(3,19)$. By Milnor Theorem 1.2.1 there is a unique such lattice, modulo isomorphism. We will denote it by Λ_{K3}.

2.6. The Picard lattice. If X is a K3 surface, the long exact cohomology sequence of the exponential sequence of X gives

$$\begin{array}{ccccccc} H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X^*) & \xrightarrow{\tau} & H^2(X,\mathbb{Z}) & \xrightarrow{\pi} & H^2(X,\mathcal{O}_X) \\ \| & & \| & & & & \| \\ (0) & & \mathrm{Pic}(X) & & & & \mathbb{C}\,\omega_X. \end{array}$$

This description fits well with the fact that $\tau(\mathrm{Pic}(X))$ is orthogonal to $\mathbb{C}\,\omega_X$ in $H^2(X,\mathbb{C}) \cong H^2(X,\mathbb{Z}) \otimes \mathbb{C}$ once we interpret the map π of the previous exact sequence as the projection over the first factor in the Hodge decomposition of the cohomology of X.

Since bot ω_X and $\bar{\omega}_X$ are orthogonal to the elements of $H^{1,1}(X)$ with respect to the intersection product (2.5.1) we have that $\psi(\mathrm{Pic}(X))$ is contained in the intersection of $H^{1,1}(X)$ with $H^2(X,\mathbb{Z})$. In fact by the Lefschetz theorem on cohomology [**EoMb**], after identifying $\psi(\mathrm{Pic}(X))$ with $\mathrm{Pic}(X)$, we have:

$$\mathrm{Pic}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}),$$

where we are considering $H^2(X,\mathbb{Z})$ embedded into $H^2(X,\mathbb{C})$. Thus the Picard lattice of a K3 surface can be thought as the sublattice of $H^2(X,\mathbb{Z})$ which is orthogonal to $\omega_X \in H^2(X,\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C} = H^2(X,\mathbb{C})$. In particular $\mathrm{Pic}(X)$ is an even lattice of rank

$$0 \le \rho_X \le 20$$

and signature $(1, \rho_X - 1)$ if X is projective. The number ρ_X is called the *Picard rank* of X. We conclude by recalling that a class $[D] \in \operatorname{Pic}(X)$ is *nef* if $D \cdot C \ge 0$ for any integral curve C of X. The set of nef classes forms the *nef cone*

$$\operatorname{Nef}(X) \subset \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

3. Torelli theorem

In this section we briefly describe the period domain of marked K3 surfaces.

3.1. Deformation theory. A *deformation* of a complex manifold X is a smooth proper flat morphism $\pi : \mathcal{X} \to S$, where both $\mathcal{X}$ and S are connected complex varieties and moreover X is isomorphic to $\mathcal{X}_0 := \pi^{-1}(0)$, where $0 \in S$ is a distinguished point. An *infinitesimal deformation* is defined in a similar way, but this time $S = \operatorname{Spec}(\mathbb{C}[\varepsilon])$, where $\varepsilon^2 = 0$.

Given a morphism $S' \to S$ which maps a distinguished point $0' \in S'$ to $0 \in S$ one can construct the *pull-back* of the deformation as the fibre product

$$\begin{tikzcd} \mathcal{X}' := \mathcal{X} \times_S S' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

The deformation $\mathcal{X} \to S$ of X is *complete* if any other deformation of X is isomorphic to a pull-back by a morphism $S' \to S$. If moreover the morphism is unique then $\mathcal{X} \to S$ is the *universal deformation* of X. If a deformation is complete and just the tangent of the map $S' \to S$ is unique, then the deformation is called *versal*. Observe that a universal deformation, if it exists, is a versal one. The versal deformation of X is denoted by $\mathcal{X} \to \operatorname{Def}(X)$. Hence with $\operatorname{Def}(X)$ we will denote the complex manifold whose points "represent" the deformations of X. We refer to [**Huy12**, Theorem 2.5, pag. 76] for more details about the following.

Theorem 3.1.1. *Every compact complex manifold X has a versal deformation. Moreover $T_0 \operatorname{Def}(X) \cong H^1(X, T_X)$.*

i) If $H^2(X, T_X) = (0)$, then a smooth versal deformation exists.
ii) If $H^0(X, T_X) = (0)$, then a universal deformation exists.
iii) The versal deformation of X is versal and complete for any of its fibers $\mathcal{X}_t$ if $h^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ is constant.

It is possible to prove that the infinitesimal deformations of X are in bijection with the elements of $H^1(X, T_X)$. Hence they represent the tangent vectors to $\operatorname{Def}(X)$ at the point $0 \in \operatorname{Def}(X)$.

Now if X is a K3 surface the existence of a holomorphic 2-form ω_X, which vanishes nowhere, gives an isomorphism between the tangent and the cotangent sheaf:

$$T_X \to \Omega_X \qquad \tau \mapsto \omega_X(\tau, -).$$

Thus $H^0(X, T_X)$ vanishes being isomorphic to $H^0(X, \Omega_X)$, whose dimension is $h^{0,1}(X) = h^{1,0}(X) = h^1(\mathcal{O}_X) = 0$. Hence X has a universal deformation. By a similar argument one proves that $H^2(X, T_X)$ vanishes, so that the universal deformation of X is smooth. Moreover

$$h^1(X, T_X) = -\chi(T_X) = 10\,\chi(\mathcal{O}_X) = 20,$$

where the middle equality is by the Riemann-Roch theorem for vector bundles on an algebraic surface [**Fri98**, Theorem 2(ii), pag. 31] (see below for another calculation when X is a quartic surface). Observe that since $\mathrm{Def}(X)$ is smooth, then its dimension is the dimension of its tangent space at $0 \in \mathrm{Def}(X)$, so that $\dim \mathrm{Def}(X) = 20$, by our previous calculation and Theorem 3.1.1. It is possible to show that fibers $\mathcal{X}_t$ in a sufficiently small neighborhood of $0 \in \mathrm{Def}(X)$ are still K3 surfaces. Hence by Theorem 3.1.1 the universal deformation of X is also a universal deformation of any such fiber $\mathcal{X}_t$.

EXAMPLE 3.1.2. If we consider smooth quartic surfaces of $\mathbb{P}^3$, they form a $\mathbb{P}^{34}$, the dimension being obtained just by counting the elements of a monomial basis of quartics minus one. Two such quartics X and Y are isomorphic if there exists an element f of the projective linear group $G := \mathrm{PGL}(3, \mathbb{C})$ such that $f(X) = Y$. Since G has dimension 15, then the GIT quotient $\mathbb{P}^{34}//G$ is 19-dimensional. This in particular implies that not all K3 surfaces are quartic surfaces. A similar conclusion can be obtained by considering the map γ coming from the exact sequence of the normal sheaf of X:

$$\begin{array}{ccccccccccc} H^0(T_X) & \longrightarrow & H^0(T_{\mathbb{P}^3}|_X) & \longrightarrow & H^0(\mathcal{N}_X) & \xrightarrow{\gamma} & H^1(T_X) & \longrightarrow & H^1(T_{\mathbb{P}^3}|_X) & \longrightarrow & H^1(\mathcal{N}_X) \\ \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\ (0) & & \mathbb{C}^{15} & & \mathbb{C}^{34} & & \mathbb{C}^{20} & & \mathbb{C} & & (0) \end{array}$$

We have already seen that the first space vanishes. The third and sixth equalities are due to $\mathcal{N}_X \cong \mathcal{O}_X(4)$ and Riemann-Roch. The second and fifth equalities are due to the Euler sequence of $\mathbb{P}^3$ for the tangent sheaf $T_{\mathbb{P}^3}$ tensored with $\mathcal{O}_X$. The space $\gamma(H^0(\mathcal{N}_X))$ represents the infinitesimal deformations of X inside $\mathbb{P}^3$, that is it can be regarded as the space of embedded infinitesimal deformations of X. Hence X has a 19-dimensional family of such deformations, which corresponds to the tangent space at the point $[X]$ of the GIT quotient $\mathbb{P}^{34}//G$.

3.2. The period domain. Recall that the second cohomology of any K3 surface X is isometric to the K3 lattice Λ_{K3}. A *marking* is an isometry:

$$\Phi : H^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}.$$

Taking the complexification of Φ we obtain a $\mathbb{C}$-linear map which we will denote by the same symbol. Thus we can consider the image of the period line $\Phi(\mathbb{C}\,\omega_X)$ in $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes_{\mathbb{Z}} \mathbb{C})$. The *period domain* is the open subset of the 20-dimensional projective quadric hypersurface:

$$\mathcal{Q} := \{\mathbb{C}\,\omega \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) : \omega \cdot \omega = 0 \text{ and } \omega \cdot \bar{\omega} > 0\}.$$

Observe that due to the Riemann conditions $\Phi(\mathbb{C}\,\omega_X) \in \mathcal{Q}$ for any K3 surface X and any marking Φ. Consider now the universal deformation $\mathcal{X} \to \mathrm{Def}(X)$ of X. A marking Φ for X induces a marking for all the fibers of the deformation. This allows us to define the *period map* to be the holomorphic map:

$$\mathcal{P}_X : \mathrm{Def}(X) \to \mathcal{Q} \qquad t \mapsto \mathbb{C}\,\omega_t,$$

where ω_t is the image, via the marking induced by Φ, of a holomorphic 2-form of $\mathcal{X}_t = \pi^{-1}(t)$.

3.3. The local Torelli theorem.

PROPOSITION 3.3.1 (local Torelli theorem). *Let X be a K3 surface and let $\mathcal{X} \to \mathrm{Def}(X)$ be the universal deformation of X. Then the period map $\mathcal{P}_X : \mathrm{Def}(X) \to \mathcal{Q}$ is a local isomorphism.*

For a complete proof of this proposition see [**Huy12**, Proposition 2.9, pag. 77]. Here we just observe that it is possible to show that the differential $d\mathcal{P}_X$ at the point $0 \in \mathrm{Def}(X)$ is given by the $\mathbb{C}$-linear map induced by the contraction homomorphism (contraction by means of the holomorphic symplectic form ω):

$$\begin{tikzcd} H^1(X, T_X) \arrow[r, "\text{contr.}"] & H^1(X, \Omega_X) \arrow[d, "\cong"] \\ T_0\,\mathrm{Def}(X) \arrow[u, "\cong"] & \mathrm{Hom}(H^{2,0}(X), H^{2,0}(X)^{\perp}/H^{2,0}(X)), \end{tikzcd}$$

by showing that the right bottom expression is the tangent space of $\mathcal{Q}$ at $\mathcal{P}(0)$. Hence $d\mathcal{P}_X$ is an isomorphism at $0 \in \mathrm{Def}(X)$ and the local period map is a local isomorphism since in a small analytic neighborhood of $0 \in \mathrm{Def}(X)$ all the fibers are K3 surfaces with the same universal deformation space. Observe that we already knew that both $\mathrm{Def}(X)$ and $\mathcal{Q}$ have dimension 20, which is the dimension of $H^{1,1}(X)$.

3.4. The global Torelli theorem. The following theorem has been proved by Pjateckiĭ-Šapiro, Šhafarevič [**PŠŠ71**].

THEOREM 3.4.1 (global Torelli theorem). *Let X and X' be two K3 surfaces and let $\sigma : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ be an isometry such that*

(i) $\sigma(\mathbb{C}\,\omega_X) = \mathbb{C}\,\omega_{X'}$;
(ii) $\sigma(\mathrm{Nef}(X)) = \mathrm{Nef}(X')$.

Then there exists a unique isomorphism $\varphi : X' \to X$ such that $\varphi^* = \sigma$.

Condition (ii) of the theorem is usually formulated in terms of the Kähler cone of X. Instead of introducing this cone here, we prefer to use the nef cone of X which is its closure. When X is projective, the nef cone is defined as done at the end of the previous section.

Let us denote now with $\mathrm{Def}(X)'$ the moduli space of pairs (X, Φ), where X is a K3 surface and Φ is a marking for X modulo the natural notion of isomorphism between pairs. This can be formally obtained as $\mathrm{Isom}(R^2 f_*\mathbb{Z}_{\mathcal{X}}, \Lambda_{\mathrm{K3}})$, where $f : \mathcal{X} \to \mathrm{Def}(X)$ is the universal deformation of X. It is possible to show (see [**Huy12**]) that the forgetful map $\mathrm{Def}(X)' \to \mathrm{Def}(X)$ is an infinite étale covering and that the period map lifts to the *global period map* $\mathcal{P} : \mathrm{Def}(X)' \to \mathcal{Q}$.

3.5. Surjectivity of the global period map. The following theorem is due to Todorov [**Tod79**].

THEOREM 3.5.1 (Surjectivity of the global period map). *Let $\mathbb{C}\,\omega \in \mathcal{Q}$. Then there exists a K3 surface X and a marking $\Phi : H^2(X,\mathbb{Z}) \to \Lambda_{\mathrm{K3}}$ such that $\mathbb{C}\Phi(\omega_X) = \mathbb{C}\,\omega$.*

Observe that the global period map $\mathcal{P}$ is locally injective due to the local injectivity of the period map $\mathcal{P}_X$ but it is not necessarily injective. Indeed if $\sigma : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an isometry which satisfies condition (i) but not condition

(ii) of Theorem 3.4.1, then the pairs (X, Φ) and $(X, \Phi \circ \sigma)$ are not isomorphic but $\mathcal{P}((X, \Phi)) = \mathcal{P}((X, \Phi \circ \sigma))$. Thus two such pairs are in the same fiber of the global period map. It is possible to show that if the K3 surface is very general and σ is any isometry of its second cohomology group which satisfies condition (i) of Theorem 3.4.1, then either σ or $-\sigma$ satisfies condition (ii) of the Theorem.

PROPOSITION 3.5.2. *For any positive integer* $0 \leq n \leq 20$ *there exists a K3 surface* X *with* $\rho_X = n$.

PROOF. Let $\mathbb{C}\,\omega \in \mathcal{Q}$ be a period such that $S := \omega^{\perp} \cap \Lambda_{\mathrm{K3}}$ is a lattice of rank n. Observe that this depends just on the coefficients of ω with respect to a basis of Λ_{K3}. Now by Theorem 3.5.1 there exists a K3 surfaces X and a marking $\Phi : H^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{K3}}$ such that $\Phi(\mathbb{C}\,\omega_X) = \mathbb{C}\,\omega$. The Picard lattice of X is $\omega_X^{\perp} \cap H^{1,1}(X)$, so that its image in Λ_{K3} is S. Thus X has Picard number n. $\square$

Exercises

EXERCISE 3.1. Shows that any non-isotrivial family of K3 surfaces, that is a family $\pi : \chi \to S$ whose period map is non-constant, admits a dense subset of K3 surfaces of rank 20.

EXERCISE 3.2. Show that for any even, positive integer number n there exists a K3 surface whose Picard lattice is generated by a class x with $x^2 = n$.

EXERCISE 3.3. Let X be the *Fermat quartic* surface of $\mathbb{P}^3$ defined by:

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

(i) Show that X contains 48 lines contained in the 12 planes of equations:

$$x_3 = \pm\zeta x_i \qquad x_3 = \pm\zeta^3 x_i,$$

where $i \in \{0, 1, 2\}$ and ζ is a 8-th primitive root of unity.

(ii) Show that the intersection matrix of the classes of the lines in $\mathrm{Pic}(X)$ has rank 20 and signature $(1, 19)$. In particular $\mathrm{Pic}(X)$ is a maximal sub lattice of the 20-dimensional vector space $H^{1,1}(X)$.

(iii) Deduce that the intersection form on $H^2(X, \mathbb{Z})$ has signature $(3, 19)$.

(iv) Reproduce the calculation of the intersection matrix of the lines of S by means of the following Magma code [**BCP97**].

```
K<a>:=CyclotomicField(8);
P<x,y,z,w>:=ProjectiveSpace(K,3);
X:=Scheme(P,x^4+y^4+z^4+w^4);
lines:=&cat[PrimeComponents(Scheme(X,x+p*q)): p in [a,-a,a
    ^3,-a^3], q in [y,z,w]];
M:=Matrix(#lines,[Degree(p meet q): p,q in lines]);
for i in [1..#lines] do M[i,i]:=-2; end for;
Rank(M);
```

CHAPTER 2

Lattices

1. Even lattices

A lattice is a finitely generated free abelian group Λ together with a quadratic form $q : \Lambda \times \Lambda \to \mathbb{Z}$. Basic invariants of a lattice Λ are its *rank*, defined as the dimension of the real vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, and its *signature*, defined to be the pair of numbers of positive and negative eigenvalues of the extension of the quadratic form q to $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. A lattice is *even* if $q(x) \in 2\mathbb{Z}$ for any $x \in \Lambda$. Recall that a lattice is *unimodular* if the determinant of a Gram matrix of q with respect to a basis is ± 1. An *isometry* of lattices is an homomorphism of abelian groups $\sigma : \Lambda_1 \to \Lambda_2$ such that $q_2(\sigma(x)) = q_1(x)$, for any $x \in \Lambda_1$, where q_i is the quadratic form of Λ_i. A good reference for the whole section is [**Dol83**].

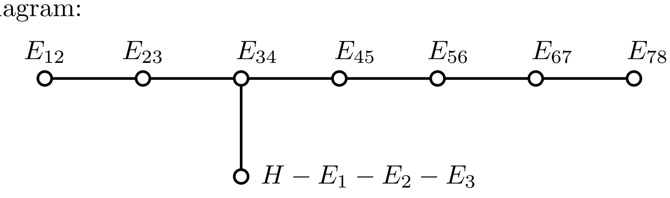
1.1. The U and E_8 lattices. The *U lattice* is the rank two unimodular lattice of signature $(1,1)$, whose Gram matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We define the E_8 lattice by means of the following geometric construction. Let $\pi : Y \to \mathbb{P}^2$ be the blow-up of the projective plane at $r \leq 8$ distinct general points. The surface Y is a *del Pezzo* surface, that is its anticanonical class is ample, and its Picard group is a lattice of signature $(1,r)$. It is not difficult to see that $\mathrm{Pic}(Y)$ is unimodular, as it admits a basis done by the classes of the pull-back of a line plus the exceptional divisors, whose Gram matrix is diagonal with determinant ± 1. Inside $\mathrm{Pic}(Y)$ consider the sublattice

$$K_Y^{\perp} := \{x \in \mathrm{Pic}(Y) : x \cdot K_Y = 0\}.$$

Since $K_Y^2 > 0$, then $K_Y^{\perp}$ is a negative definite lattice. If we concentrate on the case $r = 8$, we see that $K_Y^{\perp}$ is the lattice spanned by the classes of the vertices of the following diagram:



Each vertex E_{ij} is the class of the difference $E_i - E_j$ of the i-th and j-th exceptional divisor of the blow-up. The vertex $H - E_1 - E_2 - E_3$ is the class of the pull-back of a line minus the first three exceptional divisors. Finally each edge represents an intersection between the classes of the corresponding vertices. For example we have an edge from E_{12} to E_{23} since $(E_1 - E_2) \cdot (E_2 - E_3) = -E_2^2 = 1$. The vertices of the above picture form a basis of the lattice. Its Gram matrix with respect to the given basis is

$$\begin{bmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

It is called the E_8-*lattice*. Since the previous matrix has determinant 1, the E_8-lattice is unimodular.

1.2. The K3 lattice. Whenever we have two lattices Λ_1 and Λ_2, we can form their direct sum $\Lambda_1 \oplus \Lambda_2$. This is a lattice with respect to the product $(x_1, x_2) \cdot (y_1, y_2) := x_1 \cdot y_1 + x_2 \cdot y_2$. With Λ^n we wil mean the direct sum of n copies of Λ. Recall the following theorem of J. Milnor.

THEOREM 1.2.1 ([**Mil58**]). *Let Λ be an indefinite unimodular lattice. If Λ is even, then $\Lambda \cong E_8(\pm 1)^m \oplus U^n$ for some m and n integers. If Λ is odd, then $\Lambda \cong (1)^m \oplus (-1)^n$ for some m and n integers.*

A remark about notation is due here. Our notation for the lattice E_8 is not the standard one adopted in the theory of Lie Groups. To relate with this notation we should write $E_8(-1)$ instead, meaning with this the lattice whose entries of the Gram matrix are the opposite of those that we have given for our E_8. As a consequence of the previous theorem we have the following.

PROPOSITION 1.2.2. *The K3 lattice Λ_{K3} is isometric to $E_8^2 \oplus U^3$.*

PROOF. Since both U and E_8 are unimodular and even, then also their sum is. Moreover the lattice $E_8^2 \oplus U^3$ has signature $(3, 19)$, so that it is not definite. Hence we conclude by Theorem 1.2.1, recalling that the K3 lattice Λ_{K3}, which is isomorphic to $H^2(X, \mathbb{Z})$ for any K3 surface X, is even unimodular with signature $(3, 19)$. $\square$

1.3. The discriminant group. Given a lattice Λ we define its *dual lattice* to be the subset of elements of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ which have integer intersection with any element of Λ. In symbols it is:

$$\Lambda^* := \{x \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} : x \cdot z \in \mathbb{Z} \text{ for any } z \text{ in } \Lambda\}.$$

Observe that the dual lattice may be not a lattice with respect to our original definition, since it can contain elements whose intersection is not integer. By abuse of language we will keep calling it lattice. For example consider the rank one lattice whose Gram matrix is (2). Then its dual lattice has Gram matrix $(1/2)$. With abuse of language we will still call it lattice. Given a non-degenerate even lattice Λ, its *discriminant group* is the quotient

$$\mathrm{d}(\Lambda) := \Lambda^*/\Lambda$$

equipped with the quadratic form $q_\Lambda : \mathrm{d}(\Lambda) \to \mathbb{Q}/2\mathbb{Z}$, induced by the quadratic form q on Λ. Observe that if M is a Gram matrix for Λ, then the order of the discriminant group $d(\Lambda)$ is the absolute value of the determinant of M. In particular $\Lambda^* = \Lambda$ if and only if Λ is unimodular.

EXAMPLE 1.3.1. Consider the rank 2 lattice Λ whose Gram matrix with respect to a basis $\{e_1, e_2\}$ is

$$M := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

The vector $v := (e_1 - e_2)/3 \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ has integer intersection with e_1 and e_2, hence with all the elements of Λ, so that $v \in \Lambda^*$. Since M has determinant 3, then the discriminant group $d(\Lambda)$ has order three by the previous observation, so that it is generated by v. Since $q_d(v) = -2/3$, a Gram matrix for the discriminant is $(-2/3)$.

1.4. Primitive embeddings. An inclusion of lattices $\Lambda_1 \subset \Lambda$ is a *primitive embedding* if the quotient Λ/Λ_1 is a torsion-free abelian group. For example if Λ is a lattice with basis $\{e_1, e_2\}$, then the sublattice Λ_1 spanned by $\{e_1 + e_2, e_1 - e_2\}$ is not primitive in Λ, as the quotient $\Lambda/\Lambda_1 \cong \mathbb{Z}/2\mathbb{Z}$. Given a sublattice $\Lambda_1 \subset \Lambda$ we define its *orthogonal lattice* to be $\Lambda_1^\perp := \{x \in \Lambda : x \cdot y = 0 \text{ for any } y \in \Lambda_1\}$. Observe that $\Lambda_1^\perp$ is always primitive in Λ.

PROPOSITION 1.4.1. *Let Λ be a unimodular lattice, $\Lambda_1 \subset \Lambda$ be a primitive embedding and $\Lambda_2 := \Lambda_1^\perp$ be its orthogonal complement. Then, for $i = 1, 2$, there are natural isomorphisms of abelian groups*

$$\gamma_i : \Lambda/(\Lambda_1 \oplus \Lambda_2) \to \mathrm{d}(\Lambda_i).$$

In particular $d(\Lambda_1) \cong d(\Lambda_2)$.

PROOF. First of all observe that an element $x \in \Lambda$ can be written in a unique way as $x = x_1 + x_2$, with $x_i \in \Lambda_i^*$, since $\Lambda_1 \oplus \Lambda_2$ has finite index in Λ. Consider now the homomorphism $\varphi_i : \Lambda \to \Lambda_i^*$ defined by $\varphi_i(x) = x_i$. We want to prove that it is surjective. Since Λ_i is primitive in Λ, then the inclusion map $i : \Lambda_i \to \Lambda$ admits a projection $\pi : \Lambda \to \Lambda_i$, that is $\pi \circ i = \mathrm{id}$. This implies that the map $i^* : \Lambda^* \to \Lambda_i^*$, which coincides with φ_i since Λ is unimodular, is surjective. Moreover if for example $\varphi_2(z) = 0$, then $z \in \Lambda_2^\perp = \Lambda_1$, since Λ_1 is primitive in Λ. Hence we get an isomorphism of abelian groups $\Lambda/\Lambda_1 \to \Lambda_2^*$ and similarly exchanging 1 with 2. Hence the induced maps $\Lambda/(\Lambda_1 \oplus \Lambda_2) \to \Lambda_i^*/\Lambda_i = d(\Lambda_i)$ are isomorphisms. $\square$

1.5. Lifting isometries. Consider now a primitive embedding $L \subset \Lambda$ of non-degenerate even lattices of the same rank. This gives inclusions $L \subset \Lambda \subset \Lambda^* \subset L^*$. The quadratic form q_L on $\mathrm{d}(L)$ restricts to the null form on Λ/L since $q(x)$ is an even integer for any $x \in \Lambda$. On the other hand, if we have an *isotropic subgroup* H of $\mathrm{d}(L)$, that is a subgroup such that $q_L|_H \equiv 0$, then there exists a non degenerate lattice $\Lambda \supset L$ such that $\Lambda/L \cong H$. Hence there is a bijection

$$\{\Lambda :\ \Lambda \supset L \text{ with } \operatorname{rk}(\Lambda) = \operatorname{rk}(L)\} \leftrightarrow \{\text{Subgroups } H \subset \mathrm{d}(L) :\ q_L|_H \equiv 0\}.$$

We are interested in understanding when an isometry σ of such an L extends to an isometry η of a lattice $\Lambda \supset L$ of the same rank. Observe that an isometry σ of L induces an isometry of its dual lattice L^*, which in turns gives an isometry σ^* of the discriminant lattice $\mathrm{d}(L)$. Consider the inclusions

$$L \subset \Lambda \subset \Lambda^* \subset L^*.$$

It is not difficult to show that σ admits an extension η if and only if $\sigma^*(\Lambda/L) = \Lambda/L$.

1.6. Gluing isometries. We have already seen by Proposition 1.4.1 that given a primitive sublattice Λ_1 of a unimodular lattice Λ and its orthogonal Λ_2, there is a natural isomorphism $\gamma : \mathrm{d}(\Lambda_1) \to \mathrm{d}(\Lambda_2)$ which allows us to identify the two discriminant groups. Now we consider when a pair of isometries of Λ_1 and Λ_2 give an isometry of Λ. More precisely we have the following.

PROPOSITION 1.6.1. *Let $\Lambda_1 \subset \Lambda$ be a primitive sublattice of a unimodular lattice and let $\Lambda_2 := \Lambda_1^{\perp}$ be its orthogonal sublattice. Let σ_1 and σ_2 be two isometries of Λ_1 and Λ_2 respectively. Then the following are equivalent.*

(i) *There exists a unique isometry σ of Λ such that $\sigma|_{\Lambda_1} = \sigma_1$ and $\sigma|_{\Lambda_2} = \sigma_2$.*

(ii) *If σ_i^* is the isometry of the discriminant lattice induced by σ_i, then the following diagram is commutative*

$$\begin{tikzcd} \mathrm{d}(\Lambda_1) \arrow[r, "\gamma"] \arrow[d, "\sigma_1^*"'] & \mathrm{d}(\Lambda_2) \arrow[d, "\sigma_2^*"] \\ \mathrm{d}(\Lambda_1) \arrow[r, "\gamma"'] & \mathrm{d}(\Lambda_2). \end{tikzcd}$$

PROOF. Recall that the elements of the quotient group $H := \Lambda/(\Lambda_1 \oplus \Lambda_2)$ are of the form $x + \gamma(x)$, with $x \in \mathrm{d}(\Lambda_1)$. Assume that (i) holds. Then $\sigma^*(x + \gamma(x)) = \sigma_1^*(x) + \sigma_2^*(\gamma(x))$ is an element of H, so that $\sigma_2^*(\gamma(x)) = \gamma(\sigma_1^*(x))$, which proves (ii).

Assume now that (ii) holds. Then given an element $x + \gamma(x)$ of H we have that $(\sigma_1^* \oplus \sigma_2^*)(x + \gamma(x)) = \sigma_1^*(x) + \sigma_2^*(\gamma(x)) = \sigma_1^*(x) + \gamma(\sigma_1^*(x))$ is again in H. Hence we conclude by our previous discussion. □

As a last remark, observe that the two quadratic forms q_{Λ_1} and q_{Λ_2} on the two discriminant groups are related. If $x + \gamma(x)$ is an element of $\Lambda/(\Lambda_1 \oplus \Lambda_2)$, then $0 = q(x + \gamma(x)) = q_{\Lambda_1}(x) + q_{\Lambda_2}(\gamma(x))$. Hence we have

$$q_{\Lambda_1} = -q_{\Lambda_2} \circ \gamma.$$

2. Automorphisms

Now we want to apply our knowledges of even lattices and Torelli theorem to the study of automorphisms of K3 surfaces. To this aim we will denote by $\mathrm{Aut}(X)$ the group of automorphisms of X and by $\mathrm{Aut}(X)_0$ the subgroup of $\mathrm{Aut}(X)$ which induces the identity on the Picard group. Given an automorphism φ of X, denote by φ^* its action on $H^2(X, \mathbb{Z})$. If Φ is a marking for X, we get a commutative diagram:

$$\begin{tikzcd} \Lambda_{\mathrm{K3}} \arrow[r, "\sigma"] & \Lambda_{\mathrm{K3}} \\ H^2(X,\mathbb{Z}) \arrow[u, "\Phi"] \arrow[r, "\varphi^*"'] & H^2(X,\mathbb{Z}) \arrow[u, "\Phi"'] \end{tikzcd}$$

where σ is an isometry of the K3 lattice Λ_{K3} which maps the period line $\mathbb{C}\,\omega = \Phi(\mathbb{C}\,\omega_X)$ into itself and preserves the image of the nef cone. Conversely, given such a σ, by the global Torelli theorem, there exists a unique automorphism φ of X such that $\sigma = \varphi^*$. Hence, after identifying $H^2(X, \mathbb{Z})$ with the K3 lattice Λ_{K3}, we have

$$\mathrm{Aut}(X) = \{\sigma \in O(\Lambda_{\mathrm{K3}}) : \sigma(\mathbb{C}\,\omega_X) = \mathbb{C}\,\omega_X, \sigma \text{ preserves the ample cone of } X\}.$$

2.1. The transcendental lattice. If we denote by $S \subset \Lambda_{\mathrm{K3}}$ the Picard lattice of X and by $T := S^{\perp}$ its *transcendental lattice*, then we can apply the results of the previous section to construct automorphisms of a given X. Observe that $\omega_X \in T \otimes_{\mathbb{Z}} \mathbb{C}$.

EXAMPLE 2.1.1. Assume that S is isomorphic to U. Since S is unimodular, then also T is. Thus $T \cong U^2 \oplus E_8^2$, by Theorem 1.2.1. Let $\sigma_T = -\operatorname{id}$ and $\sigma_S = \operatorname{id}$. Since the discriminant groups of S and T are trivial, then the hypothesis of Proposition 1.6.1 is automatically satisfied, so that there exists an isometry σ of Λ_{K3} inducing both σ_S and σ_T. Moreover $\sigma(\omega_X) = -\omega_X$ and σ is the identity on the whole Picard lattice, so that in particular it preserves the nef cone. Whence there exists an isomorphism φ of X which induces σ in cohomology. It is possible to prove that the quotient surface $Y := X/\langle\varphi\rangle$ is smooth projective. In particular, since ω_X is not preserved by φ^*, then $H^{2,0}(Y) = (0)$. Moreover the Picard lattice of Y has rank 2. Hence by the classification of smooth algebraic surfaces Y is a rational surface. In the next chapter we will see that Y is a Hirzebruch surface $\mathbb{F}_4$.

2.2. Symplectic automorphisms. Given an automorphism φ of a K3 surface X it must preserve the period line. Hence we have

$$\varphi^*(\omega_X) = \zeta\, \omega_X,$$

for some complex number ζ. if $\zeta = 1$, the automorphism φ is *symplectic* and *non-symplectic* otherwise. Assume that φ is symplectic. Given an element $z \in T$ in the transcendental lattice we have $\varphi^*(z) \cdot \omega_X = z \cdot \varphi^*(\omega_X) = z \cdot \omega_X$, so that $\varphi^*(z) - z$ is orthogonal to the period ω_X. hence $\varphi^*(z) - z$ belongs to both the Picard and the transcendental lattices of X so that $\varphi^*(z) - z = 0$. Thus

$$\varphi^*|_T = \operatorname{id}.$$

On the other hand if φ is an automorphism which induces the identity on the transcendental lattice, then it is obviously symplectic as $\omega_X \in T \otimes \mathbb{C}$.

If we denote by $\mathrm{G}(X)$ the subgroup of $\mathrm{Aut}(X)$ whose elements are symplectic automorphisms, then we get an exact sequence

$$0 \longrightarrow \mathrm{G}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(X)|_{\mathbb{C}\,\omega_X} \longrightarrow 0. \tag{2.2.1}$$

Mukai proved in [**Muk88**] that if $G(X)$ is finite then it is isomorphic to a subgroup of the Mathieu group M_{23}.

2.3. Nikulin involutions. An important example of symplectic automorphism is the case of involutions, that is $\varphi^2 = \operatorname{id}$. These are also called *Nikulin involutions*, after the work of Nikulin [**Nik79**]. In this case the only possible eigenvalues of φ^* are ± 1. We have already seen that φ^* restricts to the identity on the transcendental lattice T. This implies that the induced action of the discriminant lattice $\mathrm{d}(T)$ is the identity. Hence $\varphi^*|_S$ must induce the identity on $\mathrm{d}(S)$, where S is the Picard lattice. Observe that if (z_1, z_2) is a fixed point, in local coordinates, of a Nikulin involution φ, then $\varphi(z_1, z_2) = (-z_1, -z_2)$ since $\varphi^*(\omega_X) = \omega_X$, where $\omega_X = \alpha\, dz_1 \wedge dz_2$ in local coordinates. Thus any such fixed point is isolate. By applying the *holomorphic Lefschetz fixed point formula* [**EoMa**]:

$$\sum_{p \in \mathrm{Fix}(\varphi)} \frac{1}{\det(I - d\varphi_p)} = \sum_{q=0}^{2} (-1)^q \operatorname{Tr}(\varphi^*|_{H^{0,q}(X)})$$

we conclude that φ has exactly 8 fixed points, since the right hand side has just two summands equal to 1, while the left hand side has n summands equal to $1/4$, where n is the number of fixed points of φ. In particular the quotient surface $Y = X/\langle\varphi\rangle$ is singular exactly at the images of these points, where it has ordinary double points. A minimal resolution of singularities $Y' \to Y$ gives another K3 surface.

2.4. 2-elementary lattices. A lattice is 2-*elementary* if its discriminant group is isomorphic to a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$. Let X be a K3 surface with 2-*elementary* Picard lattice, and let $S \subset \Lambda_{\mathrm{K3}}$ be the image of the Picard lattice via a marking so that

$$\mathrm{d}(S) \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

If σ_S is an involution of S then the corresponding σ_S^* acts as the identity on the discriminant group $d(S)$, since $\sigma_S^*(x) = \pm x = x$. Hence by Proposition 1.6.1 there is an isometry σ of the K3 lattice Λ_{K3} which induces σ_S on S and $\sigma_T = \mathrm{id}$ on T. Thus as soon as σ_S^* preserves the nef cone, it induces an automorphism of X. Since the eigenvalues of σ_S are ± 1, then we are just asking for the ample cone of X to have non-empty intersection with the eigenspace of σ_S corresponding to eigenvalue 1. In particular we have

$$\{\text{Nikulin involutions of } X\} = \{\sigma_S \in O(S) : \sigma_S(\mathrm{Nef}(X)) = \mathrm{Nef}(X)\}.$$

Example 2.4.1. As an explicit example one can consider the involution of the Fermat quartic surface $V(x_0^4 + x_1^4 + x_2^4 + x_3^4)$ given by $(x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : -x_1 : x_2 : x_3)$, a direct calculation shows that it is symplectic and its fixed points are $(0:0:-\zeta:1)$, $(0:0:\zeta:1)$, $(0:0:-\zeta^3:1)$, $(0:0:\zeta^3:1)$, $(-\zeta:1:0:0)$, $(\zeta:1:0:0)$, $(-\zeta^3:1:0:0)$, $(\zeta^3:1:0:0)$, where ζ is a primitive 8-th root of unity.

2.5. Non-symplectic automorphisms. Given an automorphism φ of a K3 surface X, it induces an isometry φ^* of the transcendental lattice T. This gives a homomorphism $\gamma : \mathrm{Aut}(X) \to O(T)$ whose kernel is the subgroup of symplectic automorphisms $\mathrm{G}(X)$, as we have already observed. Now we are interested in the image of the previous homomorphism.

Theorem 2.5.1. *Let X be a K3 surface with transcendental lattice T. The image of the homomorphism $\mathrm{Aut}(X) \to O(T)$ is a finite group. In particular $\mathrm{Aut}(X)$ is finite if and only if $G(X)$ is finite.*

Proof. Let σ be the image of an automorphism φ of X. Then σ preserves the two-dimensional complex vector space $\langle\omega_X, \bar{\omega}_X\rangle$ and its orthogonal. The restriction of the quadratic form to both spaces is definite (positive on the first and negative on the second). Hence the eigenvalues of σ have module 1. On the other hand σ is an isometry of an integer lattice T, so its eigenvalues are algebraic integers. Thus they are roots of unity. Since the degree of the characteristic polynomial of σ equals the rank of T, in particular it is bounded, then only a finite number of roots of unity can appear as eigenvalues of σ. Thus the representation of $\mathrm{Aut}(X)$ on $\mathbb{C}\,\omega_X$, given by (2.2.1), assumes only a finite number of roots of unity. Hence we get the statement. □

Given a non-symplectic automorphism φ of a K3 surface X we know that $\varphi^*(\omega_X) = \zeta\,\omega_X$ for some $\zeta \neq 1$. If φ has finite order p, then ζ must be a p-th root of unity, non necessarily primitive since some proper power of φ can be symplectic.

It is possible to prove [**MO98**] that the transcendental lattice T has the structure of free $\mathbb{Z}[\zeta]$-module induced by the multiplication $\zeta \cdot x := \varphi^*(x)$.

Example 2.5.2. Let X be a K3 surface whose Picard lattice S has rank 20, so that the transcendental lattice T has rank 2. If φ is a non-symplectic automorphism of X of finite order, then its action on the transcendental lattice T is represented by a 2×2 matrix with integer entries. Thus the eigenvalue ζ of $\sigma := \varphi^*$ relative to ω_X is a root of unity which lives in a degree 2 extension of $\mathbb{Q}$. Thus $\zeta \in \{-1, \varepsilon, \varepsilon^2, \pm i\}$, where ε is a primitive third root of unity. Assume $\zeta = \varepsilon$, and let $\{e, \sigma(e)\}$ be a basis of the transcendental lattice T. If $e^2 = 2n$, then, by using the fact that $\sigma^2 + \sigma + \mathrm{id} = 0$, we get $e \cdot \sigma(e) = \sigma(e) \cdot \sigma^2(e) = \sigma(e) \cdot (-e - \sigma(e)) = -e \cdot \sigma(e) - 2n$, so that $e \cdot \sigma(e) = -n$. One can reason in a similar way when $\zeta = i$ obtaining the Gram matrices

$$\begin{bmatrix} 2n & -n \\ -n & 2n \end{bmatrix} \qquad \begin{bmatrix} 2n & 0 \\ 0 & 2n \end{bmatrix}$$

of transcendental lattices which admit respectively a non-symplectic involution of order three and a non-symplectic involution of order four (acting on the period as the multiplication by i, that is φ^2 is still non-symplectic). As an example, consider the Fermat surface X. In Exercise 3.3 you showed that the lines of X span a lattice of rank 20. It is not hard to show that the discriminant group is $(\mathbb{Z}/8\mathbb{Z})^2$. Observe that X admits non-symplectic automorphisms of order four, like for example $(x_0 : x_1 : x_2 : x_3) \mapsto (i\, x_0 : x_1 : x_2 : x_3)$. Hence by our previous argument the transcendental lattice of X is diagonal with eigenvalues $2n$. In particular its determinant $4n^2 \mid 64$, so that $n \in \{1, 2, 4\}$. It is possible to show that $n = 4$, so that the transcendental lattice of X has Gram matrix:

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

We conclude by observing that since T has determinant 64, which is not divisible by 3, then by our previous observations X does not admit a non-symplectic automorphism of order three.

Non-symplectic automorphisms of order two have been extensively studied. In particular if $\varphi \in \mathrm{Aut}(X)$ is such an automorphism, then the quotient surface $Y := X/\langle\varphi\rangle$ is either an Enriques surface (if φ does not have fixed points) or a rational surface. In the next section we will analyze the case of Enriques surfaces in more detail (see also Example 2.1.1).

2.6. The Weyl group. An element e of a lattice S is a *root* if $e^2 = -2$. Given a root $e \in S$ define the *Picard-Lefschetz reflection* associated to e as the isometry $s_e : S \to S$ given by $x \mapsto x + (x \cdot e)\, e$. The *Weyl group* of the lattice S is:

$$\mathrm{W}(S) := \langle s_e : e \text{ is a root of } \Lambda_{\mathrm{K3}} \rangle.$$

If X is a K3 surface, then no element of the Weyl group of the Picard lattice S can be induced by an automorphism. Indeed $s_e(e) = -e$ and it is not difficult to show, as a consequence of the Riemann-Roch theorem, that either e or $-e$ has to be an effective class, but an automorphism can not map an effective class into its (non-effective) opposite. Also, if h is an ample class of X and e is effective, then $h \cdot e > 0$, so that $s_e(h) \cdot e < 0$, which implies $s_e(h)$ non-ample. The effect of applying a Picard-Lefschetz reflection with respect to a root e is to make a reflection with respect to the hyperplane $e^\perp$. This reflection moves the whole ample cone, by our previous observation. It can be proved that the closure of the ample cone, that is the nef cone $\mathrm{Nef}(X)$, is a *fundamental chamber* for the action of $\mathrm{W} := \mathrm{W}(\mathrm{Pic}(X))$ on the Picard lattice, meaning with this that $\mathrm{W}\cdot\mathrm{Nef}(X)$ defines a decomposition of the positive light cone $\{x \in \mathrm{Pic}(X)\otimes\mathbb{C} : x^2 > 0 \text{ and } x\cdot h > 0 \text{ with } h \text{ ample}\}$ into chambers which are congruent to $\mathrm{Nef}(X)$ and W acts freely and transitively on this set of chambers. On the other hand an isometry of the Picard lattice coming from an automorphism clearly preserves the Nef cone. Hence if we denote by $O(\mathrm{Nef}(X))$ the isometries of $\mathrm{Pic}(X)$ which preserve the Nef cone, we have a map

$$\mathrm{Aut}(X) \to O(\mathrm{Nef}(X)).$$

We want to show that this map has finite kernel and cokernel. The first is an immediate consequence of Theorem 2.5.1. To prove the second, observe that the set of $\sigma \in O(\mathrm{Nef}(X))$ which induce the identity on the discriminant lattice $\mathrm{d}(\mathrm{Pic}(X))$ has finite index in $O(\mathrm{Nef}(X))$. Each such σ admits a lifting to an isometry σ' of $H^2(X,\mathbb{Z})$, just choosing the identity on the transcendental lattice. By the Global Torelli Theorem, σ' is induced by an automorphism of X, since σ' preserves both the period and the Nef cone of X. This proves what claimed.

2.7. Finite automorphisms groups. Summarizing our previous observations we have the following.

THEOREM 2.7.1 ([**Nik79**]). *The automorphism group of a K3 surface is finite if and only if the Weyl group $W(\mathrm{Pic}(X))$ has finite index in the isometry group of* $\mathrm{Pic}(X)$.

By using the previous theorem it is possible to classify all the Picard lattices of $K3$ surfaces which admit a finite automorphism group (see [**Nik79**,**Nik75**,**Nik79**]). The number n of these lattices for any Picard rank ρ_X is given in the following table (see [**Dol83**, Theorem 2.2.2]).

ρ_X	3	4	$5-6$	7	8	9	10	$11-12$	$13-14$	$15-19$	20
n	27	17	10	9	12	10	9	4	3	1	0

The great part of these lattices are 2-elementary, that is the discriminant group is a sum of copies of $\mathbb{Z}/2\mathbb{Z}$. This is related with the fact that a K3 surface X with that Picard lattice admits a non-symplectic involution.

EXAMPLE 2.7.2. The lattice $S = U \oplus (-2) \oplus E_8^2$ is known to belong to the previous list. Since a Gram matrix of S has determinant -2, then $d(S) \cong \mathbb{Z}/2\mathbb{Z}$, so that S is 2-elementary. The orthogonal complement of S in the K3 lattice is the 2-elementary lattice $T = U \oplus (2)$. After imposing the Riemann conditions on the period $\omega \in T \otimes \mathbb{C}$, we see that the period line $\mathbb{C}\,\omega$ lies on a 1-dimensional variety. Hence there is a 1-dimensional family of K3 surfaces, each member of which admits a Picard lattice which contains a copy of S and whose very general element has a Picard lattice isomorphic to S. These last very general surfaces have finite automorphism group.

3. Enriques surfaces

An *Enriques surface* is a smooth projective surface Y with $H^1(Y,\mathcal{O}_Y)$ trivial, $2K_Y \sim \mathcal{O}_Y$ and K_Y not linearly equivalent to zero.

3.1. Topological invariants. If Y is an Enriques surface, then from the exponential sequence and $h^1(\mathcal{O}_Y)=0$ we deduce that both $H^1(X,\mathbb{Z})$ and $H^3(X,\mathbb{Z})$ have zero rank. Moreover the first group is trivial since it is always torsion-free. Now recall that $\operatorname{Tors} H_1(X,\mathbb{Z}) \cong \operatorname{Tors} H_2(X,\mathbb{Z}) \cong \operatorname{Tors} H^2(X,\mathbb{Z})$, so that it is enough to determine the first group. Since the class of K_X is non trivial but $2K_X \sim 0$, then $[K_X]$ is a 2-torsion element of $\operatorname{Pic}(X)$ which gives 2-torsion element of $H^2(X,\mathbb{Z})$ by the injectivity of the map τ in the exponential sequence of Y. By the previous isomorphisms between torsion groups we deduce that $H_1(X,\mathbb{Z})$ contains a 2-torsion element which in turn implies that $\pi_1(Y)$ contains such an element. Hence Y admits an unbranched double covering

$$\pi : X \to Y,$$

where X is a compact complex surface with $K_X \cong \pi^* K_Y \sim 0$ and $h^1(\mathcal{O}_X)=0$. Thus X is a K3 surface. In particular $\pi_1(Y)\cong \mathbb{Z}/2\mathbb{Z}$ since X is simply connected. Hence

$$\operatorname{Tors} H^2(Y,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

is generated by the class of K_Y. By Noether formula $e(Y) = 12(\chi(\mathcal{O}_Y)+K_Y^2) = 12$, since K_Y is numerically trivial, $h^1(\mathcal{O}_Y)=0$ and $h^2(\mathcal{O}_Y)=h^0(K_Y)=0$, where the last equality is due to the fact that K_Y is not linearly equivalent to zero. Hence $H^2(Y,\mathbb{Z})/\operatorname{Tors}$ is a unimodular lattice of rank 10 which, by Poincaré duality and the universal coefficient theorem. Moreover $H^2(Y,\mathbb{Z})\cong \operatorname{Pic}(Y)$, by the exponential sequence. Hence it is an even lattice by the adjunction formula and the fact that K_X is numerically trivial. Hence the signature of $H^2(Y,\mathbb{Z})$ is $(1,9)$ by the Hodge index theorem. Thus by Milnor theorem 1.2.1 this lattice has to be $U\oplus E_8$. We summarize the previous observations in the following proposition.

PROPOSITION 3.1.1. *Let Y be an Enriques surface. Then* $\operatorname{Pic}(Y)\cong \Lambda\oplus\mathbb{Z}/2\mathbb{Z}$, *where Λ is isomorphic to the rank 10 even unimodular lattice $U\oplus E_8$.*

3.2. The Enriques lattice. Let Y be an Enriques surface. The Picard lattice of the K3 surface X, in the double cover $\pi : X \to Y$, contains the pull-back $\pi^*\operatorname{Pic}(Y)$. This is the following 2-elementary lattice called the *Enriques lattice*:

$$\Lambda_{\mathrm{En}} := U(2)\oplus E_8(2).$$

By the global Torelli theorem the moduli space of Enriques surfaces is birational to the moduli space of pairs (X,σ), where X is a K3 surface such that $\operatorname{Pic}(X)$ contains a lattice isomorphic to Λ_{En} and σ is a non-symplectic involution whose induced homomorphism σ^* on $\operatorname{Pic}(X)$ is the identity on Λ_{En}.

3.3. Projective constructions. Let $Q=\mathbb{P}^1\times\mathbb{P}^1$ be a smooth quadric surface of $\mathbb{P}^3$. Consider the involution τ of Q given by

$$((x_0:x_1),(y_0:y_1)) \mapsto ((x_0:-x_1),(y_0:-y_1)).$$

It has 4 fixed points p_1,p_2,p_3,p_4. Choose now an irreducible curve B of Q cut out by a quartic surface, that is B has class $(4,4)$ in $\operatorname{Pic}(Q)$, which passes through the p_i's and is invariant with respect to τ. It is possible to show, by using the Riemann-Hurwitz formula, that the double cover of Y branched along B is a K3 surface X. Due to our choice of B the involution τ lifts to an involution τ' of X. If ν is the double cover automorphism of $X\to Q$, then $g=\nu\circ\tau'$ is an involution of X without fixed points. The quotient surface $Y = X/\langle g\rangle$ is an Enriques surface. We summarize the construction in the following diagram

$$\begin{tikzcd} X \arrow[r, "\tau'"] \arrow[d, "/g"'] & X \arrow[d, "/\nu"] \\ Y & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

REMARK 3.3.1. The very general Enriques surface Y is double covered by a K3 surface X whose Picard lattice is isomorphic to $U(2) \oplus E_8(2)$. Since an element x of this lattice has square $x^2 \in 4\mathbb{Z}$, then X does not contain (-2)-curves, so that the same is true for Y.

Now, if Y is not very general then the Picard lattice of the K3 surface X which double covers Y can have rank > 10 so that X may contain a (-2)-curve C. The image Γ of C in Y is a (-2)-curve of Y. Observe that even if the Picard rank of X is bigger than 10, that of Y remains constant, since any Enriques surface has Picard lattice of rank 10. What happened is that the class of Γ, which in the very general case was not effective, now becomes effective. Hence deforming an Enriques surface one expects to change the shape of the cone of effective divisors without changing the lattice structure on the Picard lattice.

3.4. Automorphisms. We conclude the section by discussing finite automorphism groups of Enriques surfaces. If ψ is an automorphism of an Enriques surface Y and $\pi : X \to Y$ is the K3 double covering, then $\psi \circ \pi$ lifts to a covering automorphism $\varphi \in \mathrm{Aut}(X)$, since X is simply connected. This means that if σ is the involution of X which exchanges the two sheets of the covering π, then $\sigma \circ \varphi = \varphi \circ \sigma$. On the other hand, any automorphism φ of X which commutes with σ induces an automorphism of Y. Hence we have an isomorphism

$$\mathrm{Aut}(Y) \to \{\varphi \in \mathrm{Aut}(X) : \varphi \circ \sigma = \sigma \circ \varphi\}.$$

This representation of $\mathrm{Aut}(Y)$ into a subgroup of automorphisms of a K3 surface, allows one to use the global Torelli theorem to classify which Y admit a finite automorphism group. The complete result, found by Kondo, is contained in the following theorem.

THEOREM 3.4.1 ([**Kon86**]). *Let Y be an Enriques surface whose automorphism group is finite. Then the transcendental lattice T_X of the general K3 surface X which double covers Y belongs to the following list.*

type	T_X	$\mathrm{Aut}(Y)$
I	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	D_4
II	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$	S_4
III	$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$	$D_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^4$
IV	$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$	$N \ltimes (\mathbb{Z}/2\mathbb{Z})^4$
V	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$	$S_4 \ltimes \mathbb{Z}/2\mathbb{Z}$
VI	$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$	S_5
VII	$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$	S_5

Exercises

EXERCISE 3.1. Let $L \subset \Lambda$ be an inclusion of non-degenerate even lattices. Show that $L^\perp$ is primitive in Λ and that $(L^\perp)^\perp = L$ if and only if L is primitive in Λ.

EXERCISE 3.2. Let Λ_1 and Λ_2 be non-degenerate even lattices. Prove that $\mathrm{d}(\Lambda_1 \oplus \Lambda_2) = \mathrm{d}(\Lambda_1) \oplus \mathrm{d}(\Lambda_2)$.

EXERCISE 3.3. Let L be a non-degenerate even lattice and let H be a subgroup of its discriminant $\mathrm{d}(L)$ such that $q_L|_H \cong 0$. Show that $\Lambda := \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} : x \bmod L \in H\}$ is a lattice which contains L and such that $\Lambda/L \cong H$.

EXERCISE 3.4. Let L be the lattice $(-2)^{16}$ with basis $\{e_1, \dots, e_{16}\}$. Consider the set $\mathcal{K}$ of affine functions $(\mathbb{Z}/2\mathbb{Z})^{16} \to \mathbb{Z}/2\mathbb{Z}$. Find the discriminant group of the *Kummer lattice*:

$$\Lambda_{\mathrm{Km}} := \left\{ \frac{1}{2} \sum_i a(i) e_i : a \in \mathcal{K} \right\}.$$

CHAPTER 3

Projective properties

1. Preliminaries

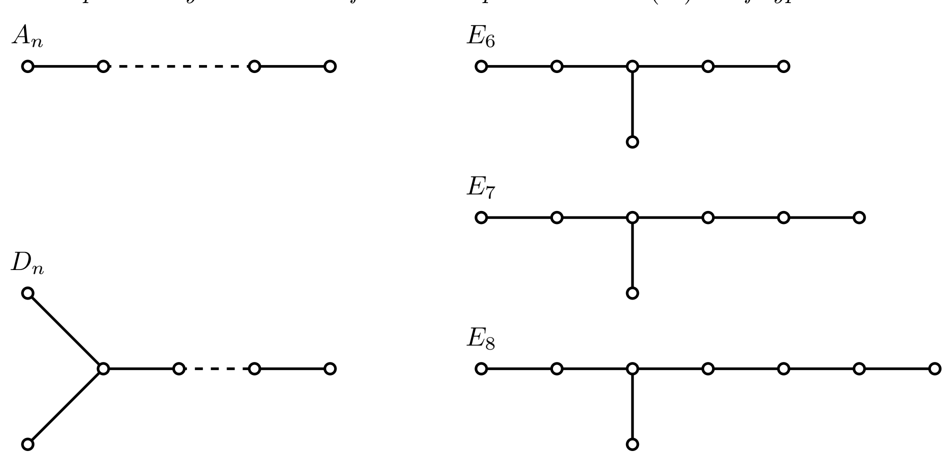
1.1. Zariski decomposition. Let X be a smooth projective surface and let D be an effective divisor of X. The *Zariski decomposition* of D is

$$D = P + N,$$

where both P and N are divisors with rational coefficients, P is nef, $P \cdot N = 0$ and N is a sum, with positive coefficients of prime divisors N_i, such that the intersection matrix $(N_i \cdot N_j)$ on the components of its support is negative-definite. As proved in [**Laz04**, Theorem 2.3.19] any effective divisor admits a unique Zariski decomposition.

1.2. Negative curves. Now, let us assume that D is an effective divisor on a K3 surface X and that $D = P + N$ is its Zariski decomposition. Since the components N_i of the support of N are prime divisors of negative self-intersection, by adjunction formula $2g(N_i) - 2 = N_i^2 < 0$, so that each N_i is a smooth rational curve with $N_i^2 = -2$. Such curves are called (-2)-*curves*. In particular each connected component Γ of the support of N is a union of (-2)-curves and the intersection matrix of such curves is negative definite. Due to this condition Γ must be a tree, since otherwise Γ contains a cycle and its components $N_1, \ldots, N_r$ satisfy $(\sum_i N_i)^2 = 0$, a contradiction. In fact a lot more can be said about the structure of such a Γ.

THEOREM 1.2.1. *Let Γ be a connected curve on a K3 surface X. Assume that the intersection form on the prime components of Γ is negative-definite. Then the lattice spanned by the classes of these components in* $\mathrm{Pic}(X)$ *is of type*



1.3. Morphisms. Recall that given a divisor D on a projective variety X, the *complete linear series* $|D|$ is the projective space whose points are the effective divisors D' of X linearly equivalent to D. By means of $|D|$ one can define a rational map, denoted by $\varphi_{|D|} : X \to \mathbb{P}^n = |D|^*$, defined by $p \mapsto |D - p|$, where the last symbol means the linear subspace of elements $D' \in |D|$ which contain p. Equivalently, given a basis $\{s_0, \dots, s_n\}$ of $H^0(X, \mathcal{O}_X(D))$, we have

$$\varphi_{|D|}(p) := (s_0(p) : \cdots : s_n(p)).$$

Consider now a smooth irreducible curve C on a K3 surface X whose class $[C] \in \mathrm{Pic}(X)$ is ample. If $C^2 > 0$, then by the Kawamata-Viehweg vanishing theorem [**Laz04**, Theorem 4.3.1] the higher cohomology groups of $\mathcal{O}_X(C)$ vanish. Thus

$$h^0(\mathcal{O}_X(C)) = \frac{C^2}{2} + 2$$

by the Riemann-Roch theorem. It is not hard to prove that the same holds if $C^2 \leq 0$. Hence particular the complete linear series $|C|$ has dimension $C^2/2 + 1 = g(C)$, where $g(C)$ is the topological genus of C. Moreover, by adjunction formula and the triviality of K_X, the restriction of $\mathcal{O}_X(C)$ to C is the canonical divisor K_C of the curve. Hence there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0. \tag{1.3.1}$$

By passing to the long exact sequence in cohomology and recalling that $h^1(\mathcal{O}_X) = 0$, we observe that the restriction map $H^0(\mathcal{O}_X(C)) \to H^0(\mathcal{O}_C(K_C))$ is surjective. Equivalently this means that the rational map

$$\varphi_{|C|} : X \to \mathbb{P}^g,$$

where $g = g(C)$, defined by the complete linear series $|C|$, induces the canonical embedding on all the smooth members of $|C|$. In particular if C is non-hyperelliptic then φ is an embedding, so that C is a very ample divisor on X. More generally we have the following.

THEOREM 1.3.1. *Let C be a smooth curve on a K3 surface X with $C^2 > 0$. Then the complete linear series $|C|$ is base point free. The morphism $\varphi_{|C|} : X \to \mathbb{P}^g$ has degree 1 or 2. Moreover it has degree 2 if and only if any smooth member of $|C|$ is a hyperelliptic curve.*

EXAMPLE 1.3.2. Let X be a K3 surface which contains a smooth curve C with $C^2 = 2$, whose classe $[C] \in \mathrm{Pic}(X)$ is ample. By the Riemann-Roch theorem $h^0(\mathcal{O}_X(C)) = 3$, that is the dimension of the complete linear series $|C|$ is 2. By adjunction formula C has genus 2, so that it is hyperelliptic. Hence the morphism $\varphi_{|C|} : X \to \mathbb{P}^2$ is a double cover. If $B \in |C|$ is a general smooth member, the restriction of $\varphi_{|C|}$ to B is the canonical map of B, hence it is a double cover branched at six points. Since $\varphi_{|C|}(B)$ is a line this implies that the degree of the branch divisor of $\varphi_{|C|}$ is a plane curve of degree six.

1.4. Semiample divisors. We recall that a divisor D is *semiample* if the complete linear series $|nD|$ is base point free for some positive integer n, that is for any $p \in X$ there exists an element $D' \in |D|$ such that $x \notin D'$. The following theorem shows that any nef divisor on a K3 surface is semiample, the converse being obvious.

THEOREM 1.4.1. *Let D be a nef divisor on a smooth K3 surface, then D is semiample.*

PROOF. Since D is a nef divisor its class lies in the closure of the ample cone of X by Kleiman theorem [**Laz04**, Theorem 1.4.23]. Hence $D^2 \geq 0$ and $h^0(\mathcal{O}_X(D)) > 1$ by Riemann-Roch. Assume $D^2 > 0$. First of all we prove that the linear series $|D|$ does not contain fixed components. Indeed, if E is the fixed divisor of the linear series, then $h^2(\mathcal{O}_X(E)) = h^0(\mathcal{O}_X(-E)) = 0$, where the first equality is by Serre's duality and the second is because E is effective. Thus

$$1 = h^0(\mathcal{O}_X(E)) \geq \frac{E^2}{2} + 2,$$

where the inequality is by Riemann-Roch, implies $E^2 < 0$. Observe that for any nef and big divisor P, that is $P^2 > 0$, we have $h^0(\mathcal{O}_X(P)) = P^2/2 + 2$, by Riemann-Roch and the Kawamata-Viewheg vanishing theorem. Since both D and $D-E$ are nef and big divisors with linear series of the same dimension, then by the previous observation $D^2 = (D-E)^2$, so that $0 \leq 2D\cdot E = E^2 < 0$, a contradiction. Hence $|D|$ does not have fixed components and we conclude that D is semiample by Zariski's theorem [**Laz04**, Remark 2.1.32].

Assume now that $D^2 = 0$. It is enough to show, as above, that $|D|$ does not contain fixed components. If E is the fixed part of the linear series, then $E^2 < 0$ and $D-E$ is a nef divisor. Hence $0 \leq (D-E)^2 = -2D\cdot E + E^2 < 0$, a contradiction. □

It is possible to prove more in general that if D is a nef divisor on a K3 surface, then $|3D|$ is base point free [**SD74**].

2. The Mori cone

Let X be a K3 surface; recall that $\mathrm{Pic}(X)$ injects into $H^2(X, \mathbb{Z})$. Denote by $N_1(X)$ the image of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ into $H^2(X, \mathbb{R})$, that is the real vector space of 1-cycles modulo homological equivalence. The *Mori cone* of X is the closure, in the Euclidean topology, of the cone of $N_1(X)$:

$$\mathrm{NE}(X) := \{\sum_i a_i[C_i] : C_i \text{ is a curve of } X \text{ and } a_i \geq 0\}.$$

We will denote the Mori cone by $\overline{\mathrm{NE}(X)}$. Since we are dealing with a surface, then curves are also divisors, so that the Mori cone of X coincides with the closure of the cone of effective divisors of X. A vector v of a cone V is said to span an *extremal ray* of V if v can not be written as a sum $v = v_1 + v_2$ of vectors $v_i \in V$ which are not multiples of v. Now observe that if E is an effective divisor such that, for any integer $n > 0$, the components of any reducible element of the linear series $|nE|$ are linearly equivalent to a multiple of E, then the class $[E]$ of E spans an extremal ray of $\mathrm{NE}(X)$. Indeed, if $[E] = x_1 + x_2$, with x_1 and x_2 in $\mathrm{NE}(X)$, then $nE \sim D_1 + D_2$, for some integer $n > 0$, where D_1 and D_2 are not linearly equivalent to a multiple of E. Hence $|nE|$ contains a reducible element whose prime components are not all linearly equivalent to a multiple of E, a contradiction. As a consequence, the class of a (-2)-curve spans an extremal ray of $\mathrm{NE}(X)$. If $\rho_X \geq 2$, denote by V the *light cone* of X, that is:

$$V := \{x \in \mathrm{Pic}(X)_{\mathbb{Z}} \otimes \mathbb{R} : x^2 \geq 0\}.$$

The reason for the name "light cone" is that the Picard lattice has a quadratic form of signature $(1, \rho_X - 1)$, like the Minkowski space-time. Let V_X^+ be the closure of the connected component of $V - \{0\}$ which contains the nef cone.

The following theorem has been proved in [**Kov94**]

THEOREM 2.0.2. *Let X be a K3 surface with $\rho_X \geq 2$. Then one of the following holds:*

(i) $\rho_X = 1$ *and the Mori cone is generated by an ample class;*
(ii) $\rho_X = 2$ *and the Mori cone is generated by the classes of a* (-2)*-curve and an elliptic curve;*
(iii) $2 \leq \rho_X \leq 4$*, the surface X does not contain elliptic curves and* (-2)*-curves, and the Mori cone is* V_X^+*;*
(iv) $2 \leq \rho_X \leq 11$ *and the Mori cone is* V_X^+*, which is also the closure of the cone spanned by classes of elliptic curves;*
(v) $2 \leq \rho_X \leq 20$ *and the Mori cone is the closure of the cone generated by classes of* (-2)*-curves.*

All the previous cases occur for any indicated value of ρ_X*.*

It is interesting to observe that, in case $\rho_X \geq 3$, if X contains a (-2)-curve then the Mori cone of X is the closure of the cone generated by the classes of (-2)-curves of X.

2.1. K3 Surfaces without (-2)-curves. If X does not contain (-2)-curves, then any effective class x has non-negative self intersection. The main point here is: does there exist a class $x \in \mathrm{Pic}(X)$ with $x^2 = 0$? The theorem gives an affirmative answer if $\rho_X \geq 5$. In this case by Riemann-Roch x or $-x$ is effective. Let us say x. Thus x must span an extremal ray of the effective cone of X, since otherwise $x = \sum_i \alpha_i x_i$, with x_i classes of effective integral curves of X and α_i positive rational coefficients. From

$$x \cdot (\sum_i \alpha_i x_i) = x^2 = 0$$

and the fact that X does not contain curves of negative self-intersection, we conclude $x = x_i$ by the Hodge index theorem. Let $[D]$ be a primitive generator of the ray spanned by x in $\mathrm{Pic}(X)_{\mathbb{R}}$, that is it has integer coefficients with greatest common divisor 1. Since x is an extremal ray, then D is an integral curve. By adjunction formula $2p_a(D) - 2 = D^2 = 0$, so that either D is an elliptic curve or it is a singular rational curve. In both cases the morphism $\varphi_{|D|} : X \to \mathbb{P}^1$ defined by the complete linear series $|D|$ is an *elliptic fibration*, meaning with this that its general fiber is a smooth elliptic curve. The fact that x is an extremal ray of the Mori cone implies that the elliptic fibration $\varphi_{|D|}$ does not have reducible fibers.

2.2. The case $\rho_X \geq 12$. In this case it is possible to prove [**Kon86**, Lemma 4.1] that there exists a class $x \in \mathrm{Pic}(X)$ with $x^2 = -2$. By Riemann-Roch either x or $-x$ must be effective. Let us assume x to be effective. Then $x = \sum_i \alpha_i x_i$, with x_i classes of effective integral curves of X and α_i positive rational coefficients. From

$$x \cdot (\sum_i \alpha_i x_i) = x^2 = -2$$

we deduce that $x_i^2 < 0$ for some i, so that X contains the (-2)-curve whose class is x_i. According to Theorem 2.0.2 the Mori cone of X is generated by classes of (-2)-curves.

2.3. The case $\rho_X \leq 2$. If $\rho_X = 1$, the Mori cone is spanned by the primitive ample class of X and there is not much to say. If $\rho_X = 2$, by Theorem 2.0.2, the Mori cone has two extremal rays. which can be generated by the classes of two (-2)-curves, one (-2)-curve and an elliptic curve, two elliptic curves, two non-effective classes x_1, x_2 with $x_i^2 = 0$. The following are four examples of Gram matrices of Picard lattices for each of the four possibility.

$$\begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}$$

It is not difficult to show that each such lattice embeds into the K3 lattice Λ_{K3} so that by Theorem 3.5.1 there exists a K3 surface X in each case with that Picard lattice. Moreover it is possible to give a projective model in each case. In the first case X is a quartic surface of $\mathbb{P}^3$ which admit a hyperplane section which is the union of two conics C_1 and C_2. Since the C_i are smooth rational curves on X, then by adjunction formula they are (-2)-curves. Moreover two plane conics intersect at 4 points by Bezout's theorem.

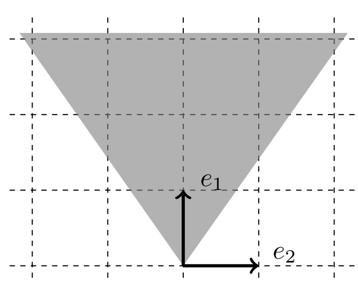
In the second case X contains two classes x_1 and x_2 which intersect at one point. One might be tempted to sat that both the x_i are classes of elliptic curves C_1 and C_2, but this can not be the case. Indeed if so, there would be two elliptic fibrations on X, given by $|C_1|$ and $|C_2|$. Since $C_1 \cdot C_2 = 1$, then $\varphi_{|C_1|}|_{C_2} : C_2 \to \mathbb{P}^1$ would be one to one, that is an isomorphism, a contradiction. Hence one of the two C_i must be reducible, for example $C_2 = C_1 + E$. Now E is a (-2)-curve of X and the Mori cone of X is spanned by the classes of E and C_1. A projective model is given by the double cover of a Hirzebruch surface $Y = \mathbb{F}_4$ branched along an element of $B + \Gamma \in |-2K_Y|$, where Γ is the (-4)-curve of Y. In this case the class of C_1 is the pul-back of the class of a element of the ruling of Y and the class of E is the pull-back of that of Γ.

The third case is the double cover of $Y = \mathbb{P}^1 \times \mathbb{P}^1$ branched along $|-2K_Y|$. The two elliptinc fibrations come from the two rulings of Y.

Finally in the fourth case $x^2 \in 4\mathbb{Z}$ for any $x \in \mathrm{Pic}(X)$ so that X does not contain (-2)-curves. Moreover, Moreover if e_1 and e_2 generate $\mathrm{Pic}(X)$ and $x = ae_1 + be_2$, with a, b integers, then

$$x^2 = 4a^2 - 8b^2$$

can not vanish, so that X does not contain elliptic curves. Hence we are in case (iv) of Theorem 2.0.2, so that the Mori cone of X is generated by the classes of non-effective curves.



2.4. Mori dream K3 surfaces. We now wish to deepen our knowledge of K3 surfaces which admit a finitely generated Mori cone. A smooth algebraic surface X is *Mori dream* if the following conditions hold:

(i) $h^1(\mathcal{O}_X) = 0$;
(ii) $\mathrm{Nef}(X)$ is generated by a finite number of semiample classes.

Observe that since the nef cone is dual to the Mori cone, then condition (ii) is equivalent to ask that the Mori cone is generated by finitely many classes of effective curves and that each nef divisor is semiample.

THEOREM 2.4.1. *Let X be a K3 surface. The X is Mori dream if and only if its automorphism group is finite.*

PROOF. We have already proved in Theorem 1.4.1 that on any K3 surface every nef divisor is semiample. Recall that in the previous chapter we showed that the homomorphism $\mathrm{Aut}(X) \to O(\mathrm{Nef}(X))$ has finite kernel and cokernel. As a consequence $\mathrm{Aut}(X)$ is finite if and only if $O(\mathrm{Nef}(X))$ is finite. This happens if and only if $\mathrm{Nef}(X)$ is polyhedral. $\square$

Observe that if $\mathrm{Nef}(X)$ is polyhedral, to each maximal face F of this cone there corresponds an extremal ray of the effective cone. This ray has to be spanned by the class e of a curve E which is orthogonal to all the nef classes in F. If $\rho_X \geq 3$, then F contains at least two elements x_1 and x_2. Observe that $(x_1 + x_2)^2 > 0$, since Since the signature of $\mathrm{Pic}(X)$ is $(1, \rho_X - 1)$. By the same reason $e^2 < 0$, being orthogonal to a class of positive self intersection. hence E is a (-2)-curve. This shows that the Mori cone of X is spanned by a finite number of (-2)-curves.

3. Cox rings

In this last section we consider Cox rings of K3 surfaces. Briefly recall the definition of the *Cox ring* of X:

$$\mathcal{R}(X) := \bigoplus_{[D]\in \mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

It is possible to show that $\mathcal{R}(X)$ is finitely generated if and only if X is Mori dream, hence if and only if $\mathrm{Aut}(X)$ is a finite group. The action of $\mathrm{Aut}(X)$ can be used in some cases to determine a presentation for $\mathcal{R}(X)$. In particular if $\varphi \in \mathrm{Aut}(X)$ is a non-symplectic involution one wish to relate the Cox rings of the two surfaces of the double covering:

$$\pi : X \to Y := X/\langle \varphi \rangle.$$

We know that Y is either a rational surface or an Enriques surface. If Y is an Enriques surface, then we have an injective homomorphism $\mathrm{Aut}(Y) \to \mathrm{Aut}(X)$ since X is the universal covering of Y. It is possible to show, by means of Theorem 3.4.1, that if $\mathrm{Aut}(Y)$ is finite then $\mathrm{Aut}(X)$ is not finite. Hence there are no Mori dream K3 surfaces which double cover an Enriques surface. Observe that this does not mean that there are no Mori dream Enriques surfaces. Indeed it is possible to prove that Y is Mori dream if and only if $\mathrm{Aut}(Y)$ is finite [**AHL10**]. If Y is a rational surface it is possible to relate its Cox ring with that of X in the following case.

THEOREM 3.0.2. *Let X be a K3 surface which admits a double cover $\pi : X \to Y$ on a Mori dream rational surface Y. If $\pi^*(\mathrm{Pic}(Y))$ has finite index in* $\mathrm{Pic}(X)$*, then*

X is Mori dream. Moreover If $\pi^*(\mathrm{Pic}(Y)) = \mathrm{Pic}(X)$, then there is an isomorphism of $\mathrm{Pic}(X)$-graded rings:

$$\mathcal{R}(X) \cong \mathcal{R}(Y)[t]/(t^2 - x_B),$$

where x_R is a defining section for the branch divisor B of π.

The proof makes use of the fact that there is an isomorphism of sheaves $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-1/2\,B)$, where B is the branch divisor of π. By the hypothesis, if D is a divisor of X, then $D = \pi^*L$, for some divisor L of Y. It s possible to show that there is an isomorphism of sheaves:

$$\mathcal{O}_X(D) \cong \pi^*\mathcal{O}_Y(L) \oplus \sqrt{x_B} \cdot \pi^*\mathcal{O}_Y(L - 1/2\,B).$$

By taking global sections and observing that $H^0(\pi^*\mathcal{O}_X(L)) \cong H^0(\mathcal{O}_X(L))$, one proves the statement.

3.1. Examples of Cox rings. Consider the K3 surface X whose Picard lattice has Gram matrix

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

We have already seen that X is double cover of $Y := \mathbb{P}^1 \times \mathbb{P}^1$ branched along a smooth curve $B \in |-2K_Y|$. Since the Picard lattice of Y is generated by two classes f_1, f_2 of zero self intersection with $f_1 \cdot f_2$, if we set $e_i : 0\pi^*(f_i)$, then $\{e_1, e_2\}$ is a basis of $\mathrm{Pic}(X)$. Hence the condition $\mathrm{Pic}(X) = \pi^*(\mathrm{Pic}(Y))$ holds, so that

$$\mathcal{R}(X) \cong \mathbb{C}[x_1, \ldots, x_4, t]/(t^2 - x_B),$$

since the Cox ring of Y is a polynomial ring, being Y a toric variety (see [**ADHL**, Chapter II]). As a second example consider the K3 surface whose Picard lattice has Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As we have already explained X is double cover of a Hirzebruch surface $Y = \mathbb{F}_4$ branched along a smooth element $B \in |-2K_Y|$. Observe that B is a union of two disjoint curves $C \cup \Gamma$, where Γ is the unique rational curve of self intersection -4 of Y. Since Γ is in the branch locus of π, we have that $E := \pi^{-1}(\Gamma)$ is still a smooth rational curve, so that it is a (2)-curve. Observe that $\pi^*\Gamma = 2E$, so that the condition of Theorem 3.0.2 is not satisfied since the class of $[E]$ does not belong to $\pi^*(\mathrm{Pic}(Y))$. It is still possible [**AHL10**] to find a presentation for the Cox ring of X:

$$\mathcal{R}(X) \cong \mathbb{C}[x_1, \ldots, x_4, t]/(t^2 - x_C),$$

where one of the x_i, let us say the fourth is the square root of the generator of the Cox ring of Y which corresponds to a defining section of Γ.

Exercises

EXERCISE 3.1. Let D be a nef and big divisor on a K3 surface X. Show that a multiple of D defines a morphism $X \to X'$, where X' is a normal surface with *Du Val singularities*, that is singularities whose minimal resolution is a tree of rational curves of type A_n, D_n, E_6, E_7 or E_8.

EXERCISE 3.2. Let D be an elliptic curve on a K3 surface. Show that the complete linear series $|D|$ has dimension 1.

EXERCISE 3.3. Let D be a divisor on a K3 surface with $D^2 \geq -2$. Show that either $-D$ or D is linearly equivalent to an effective divisor.

EXERCISE 3.4. Show that if D is a nef divisor on a K3 surface, with $D^2 = 0$, then D is linearly equivalent to nE, where E is an elliptic curve and n is a positive integer.

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