Топологические тензорные представления алгебры Ли эндоморфизмов счетномерного векторного пространства

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$\mathbb{C}$, V, $\mathfrak{g} = \mathrm{End}V$ – Lie algebra

Conjecture: The only simple countable-dimensional $\mathfrak{g}$-modules are (up to isomorphism) $\mathbb{S}_\mu(V)$, where $\mathbb{S}_\mu$ denotes the Schur functor for a partition μ.

Let $\mathbf{V} = \mathrm{Hom}(V, \mathbb{C}) = V^*$: thin V, fat $\mathbf{V}$

Category $\mathbb{T}_\mathfrak{g}$: its objects are subquotients of finite sums of modules $V^{\otimes p} \otimes \mathbf{V}^{\otimes q}$, homs are homomorphisms of $\mathfrak{g}$-modules.

The category $\mathbb{T}_\mathfrak{g}$ has been studied by Serganova, P. [PS] and A. Chirvasitu, [C].

Main facts

(i) for a fixed basis $v_1, v_2, ...$ of V, the functor $(\cdot)^{wt} \colon \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{gl(\infty)}$ is an equivalence of categories (Chirvasitu)

$$M^{wt} = \bigoplus_{\lambda \in \mathfrak{h}^*} M^{\lambda},\ M^{\lambda} = \{m \in M \mid h \cdot m = \lambda(h)m\}$$

$$\mathfrak{h} = \bigoplus_i ((\mathbb{C}v_i) \otimes (\mathbb{C}v_i^*))$$

The objects of the category $\mathbb{T}_{gl(\infty)}$ are $gl(\infty)$-modules isomorphic to subquotients of direct sums of modules $V^{\otimes p} \otimes (V_*)^{\otimes q}$, where

$$V_* = \text{span}\,\{v_1^*, v_2^*, ...\} \subset V$$

(ii) The category $\mathbb{T}_{gl(\infty)}$ has been studied (relatively) well [DPS]. In more detail:

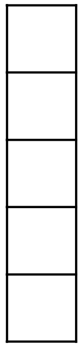
- objects have finite length
- simple objects are $V_{\lambda,\mu}$ where λ, μ are partitions,

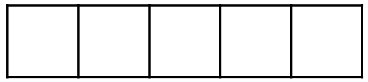
$$V_{\lambda,\mu} := \mathrm{soc}(\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V_*))$$

- equivalent definition: finite-length absolute weight modules
- equivalent definition: finite-length weight modules (for fixed $\mathfrak{h}$) satisfying the LLAC (local large annihilator condition)
- the modules $V^{\otimes p} \otimes (V_*)^{\otimes q}$ are injective (consequently also $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V_*)$ as direct summands of injectives)
- $\mathbb{T}_{gl(\infty)}$ has no projectives
- The category $\mathbb{T}_{gl(\infty)}$ has a universality property as a symmetric tensor category.
- the socle filtration of $V^{\otimes p} \otimes (V_*)^{\otimes q}$ (consequently also of $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V_*)$) is described explicitly in [PStyr] and [DPS].

Examples:

$$\begin{array}{c} V_{\emptyset;\,\emptyset} \\ \hline \vdots \\ \hline V_{(k-1);\,(k-1)} \\ \hline V_{(k);\,(k)} \end{array}$$

$(k) =$ 

$(1^k) =$ 

$$\begin{array}{c} V_{\emptyset;\,\emptyset} \\ \hline \vdots \\ \hline V_{(1^{k-1});\,(1^{k-1})} \\ \hline V_{(1^k);\,(1^k)} \end{array}$$

$$\begin{array}{c} V_{(1);\,(1)} \\ \hline V_{(2);\,(1,1)} \oplus V_{(2);\,(2)} \\ \hline V_{(3);\,(2,1)} \end{array}$$

$$\begin{array}{c} V_{(1);\,\emptyset} \\ \hline V_{(1,1);\,(1)} \oplus 2V_{(2);\,(1)} \\ \hline V_{(2,1);\,(1,1)} \oplus V_{(2,1);\,(2)} \oplus V_{(3);\,(1,1)} \oplus V_{(3);\,(2)} \\ \hline V_{(3,1);\,(2,1)} \end{array}$$

$$V = \varinjlim (\operatorname{span}\{v_1, ..., v_n\} = V_n), \;\; \mathbf{V} = \varprojlim V_n^*$$

V is discrete, $\mathbf{V}$ is linearly compact = inverse limit of finite-dimensional vector spaces, open neighborhoods of zero are $p_n^{-1}(0_n)$, where $0_n \in V_n^*$ and $p_n \colon \mathbf{V} \to V_n^*$ is the projection

We have $\mathbf{V}^* = V$ (the continuous dual of a linearly compact vector space is discrete)

More generally, a vector space W is pro-discrete if $W = \varprojlim W_n$, where W_n are discrete. The topology of W is defined in the same way as above.

Now set $\mathbf{V}^{\widehat{\otimes} p} = (V^{\otimes p})^* = \mathrm{Hom}(V^{\otimes p}, \mathbb{C})$. Then the vector space $\mathbf{V}^{\widehat{\otimes} p}$ is linearly compact since $V^{\otimes p} = \varinjlim (V_n^{\otimes p})$.

Next, consider the space

$$\mathbf{V}^{p,q} := V^{\otimes p} \otimes \mathbf{V}^{\widehat{\otimes} q}$$

This space is ind-linearly compact and complete.

$\mathbf{V}^{p,q}$ can be represented as a countable direct sum of linearly compact spaces W_i (for this it is enough to choose a basis in $V^{\otimes p}$), and subspaces of the form $\bigoplus_i U_i \subset \bigoplus_i W_i$, where U_i are open in W_i, constitute a basis of open neighborhoods of 0 in $\mathbf{V}^{p,q}$

Also, we have a canonical inclusion $V^{\otimes p} \otimes \mathbf{V}^{\otimes q} \hookrightarrow \mathbf{V}^{p,q} = V^{\otimes p} \otimes \mathbf{V}^{\widehat{\otimes} q}$

The category $\mathbf{T}_{\mathfrak{g}}$ is the abelian category of all $\mathfrak{g}$-modules isomorphic to topological subquotients of finite direct sums of the form $\mathbf{V}^{p,q}$. (Topological subquotients are quotients M/N where $N \subset M \subset \mathbf{V}^{p,q}$ are closed submodules)

Modified categories $\mathbb{T}'_{\mathfrak{g}}$ and $\mathbf{T}'_{\mathfrak{g}}$: their objects are isomorphic to submodules of $V^{\otimes p} \otimes \mathbf{V}^{\otimes q}$ and $\mathbf{V}^{p,q}$ respectively

Fact: $\mathbb{T}'_{\mathfrak{g}}$ and $\mathbf{T}'_{\mathfrak{g}}$ are abelian and are equivalent to $\mathbb{T}_{\mathfrak{g}}$ and $\mathbf{T}_{\mathfrak{g}}$ respectively. This follows from the yoga of ordered Grothendieck categories developed in [CP].

There are well-defined functors

$$\mathbb{T}'_{\mathfrak{g}} \underset{\substack{\text{intersection with} \\ \bigoplus_{p,q}(V^{\otimes p} \otimes \mathbf{V}^{\otimes q})}}{\overset{\text{completion}}{\rightleftarrows}} \mathbf{T}'_{\mathfrak{g}}$$

which we prove to be mutually inverse equivalences. The proof reduces to $gl(\infty)$ via the functor $(\cdot)^{wt}$

Corollary. The categories $\mathbb{T}_{\mathfrak{g}}$ and $\mathbf{T}_{\mathfrak{g}}$ are equivalent.

Finally, we define our main object of study, the category $\widehat{\mathbf{T}}_{\mathfrak{g}}$: its objects are continuous duals M^* of $M \in \mathbf{T}_{\mathfrak{g}}$ and morphisms are continuous homomorphisms of $\mathfrak{g}$-modules. The objects of $\widehat{\mathbf{T}}_{\mathfrak{g}}$ are pro-discrete vector spaces.

Example:

$$\begin{array}{ccc} (V \otimes \mathbf{V})^* & \simeq \mathfrak{g} = & \mathrm{End}V \\ \Cup & & \Cup \\ (v \otimes \varphi \mapsto \varphi(\alpha(v))) & \longleftarrow & \alpha \end{array}$$

$$V \otimes \mathbf{V} = \varinjlim_i \varprojlim_j V_i \otimes V_j^* \qquad\qquad (V \otimes \mathbf{V})^* = \varprojlim_i \varinjlim_j V_i^* \otimes V_j$$

$$\varprojlim (V_i^* \otimes V) = \mathrm{End}V$$

All submodules of $\mathrm{End}V$ have been found in [HZ].

Theorem. The functors

$$\mathbf{T}_{\mathfrak{g}} \underset{(\ )^*}{\overset{(\ )^*}{\rightleftarrows}} \widehat{\mathbf{T}}_{\mathfrak{g}}$$

are well-defined anti-equivalences of abelian categories.

Moreover, $\widehat{\mathbf{T}}_{\mathfrak{g}}$ is a tensor category:

$$M \widehat{\otimes} N = (M^* \otimes N^*)^*$$

Corollary: all above mentioned properties of the category $\mathbb{T}_{\mathfrak{g}} \simeq \mathbb{T}_{gl(\infty)}$ hold for the category $\widehat{\mathbf{T}}_{\mathfrak{g}}$ with the arrows reversed.

References

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