Finite multiplicities beyond spherical pairs

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- $\mathbf{G}$: reductive group over $\mathbb{R}$, $\mathbf{X}$:= algebraic $\mathbf{G}$-manifold, $\mathfrak{g} := Lie(\mathbf{G})$, $\mathcal{N}(\mathfrak{g}^*)$:=nilpotent cone, $G := \mathbf{G}(\mathbb{R})$, $X := \mathbf{X}(\mathbb{R})$,
- $\mathcal{S}(X)$:= infinitely smooth functions on X, flat at infinity (Schwartz).
- $\mathbf{X}$ is called spherical if it has an open orbit of a Borel subgroup $\mathbf{B} \subset \mathbf{G}$.
- X is called real spherical if it has an open orbit of a minimal parabolic subgroup $P_0 \subset G$.

Major Goal: study $L^2(X)$, $C^\infty(X)$, $\mathcal{S}(X)$ as rep-s of G.
Studied by Bernstein, Delorme, van den Ban, Schlichtkrull, Kroetz, Kobayashi, Oshima, Knop, Beuzart-Plessis, Kuit, Wan,...

Theorem (Kobayashi-Oshima, 2013)

Let $\mathbf{X} = \mathbf{G}/\mathbf{H}$. Then

1. *$\mathbf{X}$ is spherical $\iff$ $\mathcal{S}(X)$ has bounded multiplicities.*
2. *X is real-spherical $\iff$ $\mathcal{S}(X)$ has finite multiplicities.*

$$m_\sigma(\mathcal{S}(X)) := \dim \operatorname{Hom}(\mathcal{S}(X), \sigma), \quad m_\sigma(\mathcal{S}(G/H)) = \dim(\sigma^{-\infty})^H$$

Today: finite multiplicities for "small enough" representations in wider generality.

Ξ-spherical spaces

$\forall x \in \mathbf{X}$, have action map $\mathbf{G} \to \mathbf{X}$, thus $\mathfrak{g} \to T_x\mathbf{X}$, and $T_x^*\mathbf{X} \to \mathfrak{g}^*$.
This gives the moment map $\mu : T^*\mathbf{X} \to \mathfrak{g}^*$.
For $\mathbf{X} = \mathbf{G}/\mathbf{H}$: $T^*\mathbf{X} \cong \mathbf{G} \times_{\mathbf{H}} \mathfrak{h}^\perp$ and $\mu(g, \alpha) = g \cdot \alpha$

Definition

- *For a nilpotent orbit* $\mathbf{O} \subset \mathcal{N}(\mathfrak{g}^*)$, *say* $\mathbf{X}$ *is* $\mathbf{O}$*-spherical if*

$$\dim \mu^{-1}(\mathbf{O}) \leq \dim \mathbf{X} + \dim \mathbf{O}/2$$

- *For a* $\mathbf{G}$*-invariant subset* $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$, *say* $\mathbf{X}$ *is* Ξ*-spherical if* $\mathbf{X}$ *is* $\mathbf{O}$*-spherical* $\forall \mathbf{O} \subset \Xi$.

For $\mathbf{X} = \mathbf{G}/\mathbf{H}$, $\mathbf{X}$ is $\mathbf{O}$-spherical $\iff \dim \mathbf{O} \cap \mathfrak{h}^\perp \leq \dim \mathbf{O}/2$.
For parabolic $\mathbf{P} \subset \mathbf{G}$, $\mathbf{O_P}$:=the unique orbit s.t. $\mathfrak{p}^\perp \cap \mathbf{O_P}$ is dense in $\mathfrak{p}^\perp$.

Theorem 1 (Aizenbud - G.)

$\mathbf{X}$ *is* $\overline{\mathbf{O_P}}$*-spherical* $\iff$ $\mathbf{P}$ *has finitely many orbits on* $\mathbf{X}$.

$\forall x \in \mathbf{X}$, have action map $G \to X$, thus $\mathfrak{g} \to T_x\mathbf{X}$, and $T_x^*\mathbf{X} \to \mathfrak{g}^*$.
This gives the moment map $\mu : T^*X \to \mathfrak{g}^*$.
For $\mathbf{X} = \mathbf{G}/\mathbf{H} : T^*\mathbf{X} \cong \mathbf{G} \times_{\mathbf{H}} \mathfrak{h}^\perp$ and $\mu(g\mathbf{H}, \alpha) = g \cdot \alpha$

Definition

- *For a nilpotent orbit $\mathbf{O} \subset \mathcal{N}(\mathfrak{g}^*)$, say $\mathbf{X}$ is $\mathbf{O}$-spherical if*

$$\dim \mu^{-1}(\mathbf{O}) = \dim X + \dim \mathbf{O}/2$$

- *For a $\mathbf{G}$-invariant subset $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$, say $\mathbf{X}$ is Ξ-spherical if $\mathbf{X}$ is $\mathbf{O}$-spherical $\forall \mathbf{O} \subset \Xi$.*

For $\mathbf{X} = \mathbf{G}/\mathbf{H}$, $\mathbf{X}$ is $\mathbf{O}$-spherical $\iff \dim \mathbf{O} \cap \mathfrak{h}^\perp \leq \dim \mathbf{O}/2$.
For parabolic $\mathbf{P} \subset \mathbf{G}$, $\mathbf{O_P}$:=the unique orbit s.t. $\mathfrak{p}^\perp \cap \mathbf{O_P}$ is dense in $\mathfrak{p}^\perp$.

Theorem 1 (Aizenbud - G. 2021)

$\mathbf{X}$ is $\overline{\mathbf{O_P}}$-spherical $\iff$ $\mathbf{P}$ has finitely many orbits on $\mathbf{X}$.

Corollary (following Wen-Wei Li)

- *$\mathbf{X}$ is $\mathcal{N}(\mathfrak{g}^*)$-spherical $\iff$ $\mathbf{X}$ is spherical*
- *$\mathbf{X}$ is $\{0\}$-spherical $\iff$ $\mathbf{G}$ has finitely many orbits on $\mathbf{X}$.*

Casselman - Wallach representations - the category $\mathcal{M}(G)$

- $\mathcal{M}(G)$:=Serre subcategory of the category of continuous representations in Fréchet spaces generated by representations of the form $C^\infty(G/P_0, \mathcal{E})$, where $G \supset P_0$-minimal parabolic subgroup, $\mathcal{E}$-any smooth vector bundle over G/P_0.
  $\mathrm{Irr}(G)$:=irreducible representations in $\mathcal{M}(G)$.
- $\sigma \in \mathrm{Irr}(G) \iff \sigma$ =space of smooth vectors in a continuous irreducible representation in a Hilbert space.
- $\mathcal{M}(G) =$ continuous representations π in Fréchet spaces s.t.:
  1. π is smooth and has moderate growth
  2. π has finite length
  3. $\pi|_K$ has finite multiplicities, where $G \supset K$-maximal compact subgroup.
- $\mathcal{M}(G)$ is abelian category, equivalent to admissible $(\mathfrak{g}, K)$-modules.
- $\mathcal{S}(X) \notin \mathcal{M}(G)$ for most X - it is "too big" to be admissible or to have finite length.

Associated variety of the annihilator & the main theorem

- $\mathcal{U}_n(\mathfrak{g})$ - PBW filtration on the universal enveloping algebra.
- $\operatorname{gr}\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \operatorname{Pol}(\mathfrak{g}^*)$.
- For an ideal $I \subset \mathcal{U}(\mathfrak{g})$, $\mathcal{V}(I) :=$ zero set of symbols of I in $\mathfrak{g}^*$.
- For a $\mathfrak{g}$-module M, $\mathrm{Ann}(M) \subset \mathcal{U}(\mathfrak{g})$ - annihilator, $\mathcal{V}(\mathrm{Ann}(M)) \subset \mathfrak{g}^*$
- For $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$, $\mathcal{M}_\Xi(G) = \{\pi \in \mathcal{M}(G) \,|\, \mathcal{V}(\mathrm{Ann}(\pi)) \subset \Xi\}$

Theorem 2 (Aizenbud - G. 2021)

Let $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$ closed **G***-invariant. Let* **X** *be Ξ-spherical* **G***-manifold, and let $\sigma \in \mathcal{M}_\Xi(G)$. Then* $\dim \operatorname{Hom}(\mathcal{S}(X), \sigma) < \infty$

Applications to branching problems

Corollary

Let $\mathbf{H} \subset \mathbf{G}$ *be reductive subgroup. Let* $\mathbf{P} \subset \mathbf{G}$ *and* $\mathbf{Q} \subset \mathbf{H}$ *be parabolic subgroups with* $|\mathbf{P} \backslash \mathbf{G} / \mathbf{Q}| < \infty$ *Then* $\forall \pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(G)$ *and* $\tau \in \mathcal{M}_{\overline{\mathbf{O_Q}}}(H)$,

$$\dim \mathrm{Hom}_H(\pi|_H, \tau) < \infty$$

Corollary

1. *Let* $\mathbf{P} \subset \mathbf{G}$ *be a parabolic subgroup s.t.* $\mathbf{G}/\mathbf{P}$ *is a spherical* $\mathbf{H}$*-variety. Then* $\forall \pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(G)$, $\pi|_H$ *has finite multiplicities.*
2. *Let* $\mathbf{Q} \subset \mathbf{H}$ *be a parabolic subgroup that is spherical as a subgroup of* $\mathbf{G}$. *Then for any* $\tau \in \mathcal{M}_{\overline{\mathbf{O_Q}}}(H)$, $\mathrm{ind}_H^G \tau$ *has finite multiplicities.*

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For simple $\mathbf{G}$ and symmetric $\mathbf{H}\subset\mathbf{G}$, all $\mathbf{P}\subset\mathbf{G}$ satisfying (i), and all $\mathbf{Q}\subset\mathbf{H}$ satisfying (ii) are classified by He, Nishiyama, Ochiai, Oshima. A strategy for other spherical $\mathbf{H}$ + implementation for classical $\mathbf{G}$: Avdeev-Petukhov.

Corollary

Let $\mathbf{H}$ *be a reductive group, and* $\mathbf{P}, \mathbf{Q} \subset \mathbf{H}$ *be parabolic subgroups s.t.* $\mathbf{H}/\mathbf{P} \times \mathbf{H}/\mathbf{Q}$ *is a spherical* $\mathbf{H}$*-variety, under the diagonal action.*
Then $\forall \pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(H)$*, and* $\tau \in \mathcal{M}_{\overline{\mathbf{O_Q}}}(H)$*,* $\pi \otimes \tau$ *has finite multiplicities.*

All such triples $(\mathbf{H}, \mathbf{P}, \mathbf{Q})$ were classified by Stembridge.
Example: $\mathbf{H} = \mathrm{GL}_n$, $\tau \in \mathcal{M}_{\overline{\mathbf{O}_{\min}}}(H)$, or classical $\mathbf{H}$ and $\pi, \tau \in \mathcal{M}_{\overline{\mathbf{O}_{2^n}}}(H)$.

- Our results also extend to certain representations of non-reductive H.

Example (Generalized Shalika model)

Let $\mathbf{G} = \mathrm{GL}_{2n}$*,* $\mathbf{R} = \mathbf{LU} \subset \mathbf{G}$ *with* $\mathbf{L} = \mathrm{GL}_n \times \mathrm{GL}_n$ *and* $\mathbf{U} = \mathrm{Mat}_{\mathbf{n}\times\mathbf{n}}$*,*
$\mathbf{M} = \Delta\,\mathrm{GL}_n \subset \mathbf{L}$*,* $\mathbf{H} := \mathbf{MU}$*.*
Let $\mathfrak{m}^* \supset \mathbf{O}_{\min} :=$ *minimal nilpotent orbit, and* $\pi \in \mathcal{M}_{\overline{\mathbf{O}_{\min}}}(M)$*.*
Let ψ *be a unitary character of* H*.*
Then $\mathrm{ind}_H^G(\pi \otimes \psi)$ *has finite multiplicities.*

Similar case: $\mathbf{G} = O_{4n}$, $\mathbf{L} = \mathrm{GL}_{2n}$, $\mathbf{M} = \mathrm{Sp}_{2n}$, $\mathbf{O}_{\mathrm{ntm}} \subset \mathfrak{m}^*$.

Some necessary conditions for finite multiplicities

Theorem (Tauchi)

Let $P \subset G$ be a parabolic subgroup. If all degenerate principal series representations of the form $\operatorname{Ind}_P^G \rho$, with $\dim \rho < \infty$, have finite H-multiplicities, then H has finitely many orientable orbits on G/P.

Corollary

Let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup defined over $\mathbb{R}$. Suppose that for all but finitely many orbits of $\mathbf{H}$ on $\mathbf{G}/\mathbf{P}$, the set of real points is non-empty and orientable. Then the following are equivalent.

1. $\mathbf{H}$ *is* $\overline{\mathbf{O_P}}$*-spherical.*
2. *Every* $\pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(G)$ *has finite multiplicities in* $\mathcal{S}(G/H)$*.*
3. H *has finitely many orbits on* G/P*.*
4. $\mathbf{H}$ *has finitely many orbits on* $\mathbf{G}/\mathbf{P}$*.*

The assumption of the corollary holds if H and G are complex reductive groups.

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1. $\mathbf{H}$ *is* $\overline{\mathbf{O_P}}$*-spherical.*
2. *Every* $\pi \in \mathcal{M}_{\overline{\mathbf{O_P}}}(G)$ *has finite multiplicities in* $\mathcal{S}(G/H)$.
3. H *has finitely many orbits on* G/P.
4. $\mathbf{H}$ *has finitely many orbits on* $\mathbf{G}/\mathbf{P}$.

The assumption of the corollary holds if H and G are complex reductive groups. In general however, the finiteness of $\mathbf{H}\backslash\mathbf{G}/\mathbf{P}$ is not necessary, but the finiteness of $H\backslash G/P$ is not sufficient for finite multiplicities.

Reduction to distributions

- $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ - space of tempered distributions.
- Examples for $X = \mathbb{R}$: $\delta_0, \delta_x^{(n)}, f \mapsto \int_{-\infty}^{\infty} f(x)g(x)dx$, where g-smooth and tempered f-n.
- Non-example: $f \mapsto \int_{-\infty}^{\infty} f(x)\exp(x)dx$

Theorem 3 (Aizenbud - G. 2021)

Let $I \subset \mathcal{U}(\mathfrak{g})$ be a two-sided ideal such that $\mathcal{V}(I) \subset \mathcal{N}(\mathfrak{g}^*)$. Let $\mathbf{X}, \mathbf{Y}$ be $\mathcal{V}(I)$-spherical $\mathbf{G}$-manifolds. Let $\mathcal{S}^*(X \times Y)^{\Delta G, I}$ denote the space of ΔG-invariant tempered distributions on $X \times Y$ annihilated by I. Then

$$\dim \mathcal{S}^*(X \times Y)^{\Delta G, I} < \infty$$

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$$\dim \mathcal{S}^*(X \times Y, \mathcal{E})^{\Delta G, I} < \infty$$

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$$\dim \mathcal{S}^*(X \times Y, \mathcal{E})^{\Delta G, I} < \infty$$

Proof of Theorem 2.

$\Xi \subset \mathcal{N}(\mathfrak{g}^*)$, $\mathbf{X}$ is Ξ-spherical, $\sigma \in \mathcal{M}_\Xi$. Need: $\dim \operatorname{Hom}_G(\mathcal{S}(X), \sigma) < \infty$. Let $\mathcal{E}$ be a bundle on $Y := G/K$ s.t. $\mathcal{S}(Y, \mathcal{E}) \twoheadrightarrow \sigma$. Let $I := \operatorname{Ann}(\sigma)$. Then $\mathcal{V}(I) \subset \Xi$, and

$$\operatorname{Hom}_G(\mathcal{S}(X), \sigma) \hookrightarrow \operatorname{Hom}_G(\mathcal{S}(X), \mathcal{S}(Y, \mathcal{E}))^I \hookrightarrow \mathcal{S}^*(X \times Y, \mathbb{C} \boxtimes \mathcal{E})^{\Delta G, I}$$

$\square$

Main technique: D-modules

Theorem 3

Let $I \subset \mathcal{U}(\mathfrak{g})$ be a two-sided ideal such that $\mathcal{V}(I) \subset \mathcal{N}(\mathfrak{g}^*)$. Let $\mathbf{X}, \mathbf{Y}$ be $\mathcal{V}(I)$-spherical $\mathbf{G}$-manifolds. Let $\mathcal{E}$ be an algebraic vector bundle on $X \times Y$. Let $\mathcal{S}^*(X \times Y, \mathcal{E})^{\Delta G, I}$ denote the space of ΔG-invariant tempered $\mathcal{E}$-valued distributions on $X \times Y$ annihilated by I. Then

$$\dim \mathcal{S}^*(X \times Y, \mathcal{E})^{\Delta G, I} < \infty$$

- $D_{\mathbf{X}}$:=sheaf of algebraic differential operators. $\operatorname{Gr} D_{\mathbf{X}} \cong \mathcal{O}(T^*\mathbf{X})$.
- For a fin.gen. sheaf M of $D_{\mathbf{X}}$-modules,
  $\operatorname{SingS}(M) := \operatorname{Supp} \operatorname{Gr}(M) \subset T^*\mathbf{X}$.
- Bernstein: if $M \neq 0$ then $\dim \operatorname{SingS}(M) \geq \dim \mathbf{X}$.
- M is called holonomic if $\dim \operatorname{SingS}(M) = \dim \mathbf{X}$.

Theorem (Bernstein-Kashiwara)

For any holonomic M, $\dim \operatorname{Hom}_{D_{\mathbf{X}}}(M, \mathcal{S}^*(X)) < \infty$.

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- For a f.gen. sheaf M of $D_{\mathbf{X}}$-modules, $\mathrm{SingS}(M) := \mathrm{Supp}\,\mathrm{Gr}(M) \subset T^*\mathbf{X}$.
- M is called holonomic if $\dim \mathrm{SingS}(M) = \dim \mathbf{X}$.
- Bernstein-Kashiwara: $\forall$ holonomic M, $\dim \mathrm{Hom}_{D_{\mathbf{X}}}(M, \mathcal{S}^*(X)) < \infty$.

Lemma

Let $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$ and let $\mathbf{X}, \mathbf{Y}$ be Ξ-spherical $\mathbf{G}$-manifolds. Then

$$\dim \mu^{-1}_{\mathbf{X} \times \mathbf{Y}}(\Xi \cap (\Delta\mathfrak{g})^{\perp}) \leq \dim \mathbf{X} + \dim \mathbf{Y}$$

Proof of Theorem 3.

$M := D_{\mathbf{X} \times \mathbf{Y}}$-module with $\mathcal{S}^*(X \times Y)^{\Delta G, I} \hookrightarrow \mathrm{Hom}(M, \mathcal{S}^*(X, Y))$.
By the lemma, M is holonomic. □

Proof of the geometric lemma

Lemma

Let $\Xi \subset \mathcal{N}(\mathfrak{g}^*)$ *and let* $\mathbf{X}, \mathbf{Y}$ *be* Ξ*-spherical* $\mathbf{G}$*-manifolds. Then*

$$\dim \mu^{-1}_{\mathbf{X}\times\mathbf{Y}}(\Xi \cap (\Delta\mathfrak{g})^{\perp}) \leq \dim \mathbf{X} + \dim \mathbf{Y}$$

Proof.

$\forall$ orbit $\mathbf{O} \subset \Xi$ we have

$$\dim \mu^{-1}_{\mathbf{X}\times\mathbf{Y}}((\mathbf{O}\times\mathbf{O}) \cap (\Delta\mathfrak{g})^{\perp}) = \dim \mu^{-1}_{\mathbf{X}}(\mathbf{O}) + \dim \mu^{-1}_{\mathbf{Y}}(\mathbf{O}) - \dim \mathbf{O} \leq$$
$$\dim \mathbf{X} + \dim \mathbf{O}/2 + \dim \mathbf{Y} + \dim \mathbf{O}/2 - \dim \mathbf{O} = \dim \mathbf{X} + \dim \mathbf{Y}$$

$\square$

Ingredients of the proof of Theorem 1

Classical: $\forall$ parabolic $\mathfrak{p}\subset\mathfrak{g}$, $\forall$ orbit $\mathbf{O}\subset\overline{\mathbf{O}_{\mathbf{P}}}$, $\dim \mathfrak{p}^{\perp}\cap\mathbf{O}\leq \dim \mathbf{O}/2$.

Lemma

Let $\mathbf{P}$ *act on* $\mathbf{X}$ *and* $\mathbf{S}:=\mu_{\mathbf{P},\mathbf{X}}^{-1}(\{0\})$. *Then the following are equivalent:*

1. $\mathbf{P}$ *has finitely many orbits on* $\mathbf{X}$.
2. $\mathbf{S}\subset T^*\mathbf{X}$ *is a Lagrangian subvariety.*
3. $\dim \mathbf{S}\leq \dim \mathbf{X}$

Proof.

$\mathbf{S}$ = union of conormal bundles to orbits. □

If $\mathbf{G}\supset\mathbf{P}$ acts on $\mathbf{X}$, $\mu_{\mathbf{P},\mathbf{X}}: T^*\mathbf{X}\to\mathfrak{g}^*\to\mathfrak{p}^*$, and
$\mathbf{S}=\mu_{\mathbf{P},\mathbf{X}}^{-1}(\{0\})=\mu_{\mathbf{G},\mathbf{X}}^{-1}(\mathfrak{p}^{\perp})$.

Theorem 4 (Strengthening of Theorem 1)

Let $\mathbf{P}\subset\mathbf{G}$ *be a parabolic subgroup, and* $\mathbf{O_P}\subset\mathcal{N}(\mathfrak{g}^*)$ *be its Richardson orbit. Then the following are equivalent:*

1. $\mathbf{P}$ *has finitely many orbits on* $\mathbf{X}$.
2. $\forall\mathbf{O}\subset\overline{\mathbf{O_P}}, \Gamma_{\mathbf{O}} := \{(x, -\mu(x)) \in T^*(X) \times \mathfrak{g}^* \,|\, \mu(x) \in \mathbf{O}\}$ *is an isotropic subvariety of* $T^*(\mathbf{X}) \times \mathbf{O}$.
3. $\mathbf{X}$ *is* $\overline{\mathbf{O_P}}$*-spherical.*

Proof of (i)⇒(ii).

Let $\nu : T^*(\mathbf{G}/\mathbf{P}) \to \mathfrak{g}^*$. Let $\mathbf{T_O} \subset T^*\mathbf{X} \times T^*(\mathbf{G}/\mathbf{P})$ denote the preimage of $\Gamma_{\mathbf{O}}$ under $\mathrm{Id} \times \nu$. Enough to prove that $\mathbf{T_O}$ is isotropic.

$$T_{\mathbf{O}}\subset T := \{(x, y) \in T^*\mathbf{X} \times T^*(\mathbf{G}/\mathbf{P}) \,|\, \mu(x) = -\nu(y)\} = (\mu \times \nu)^{-1}(\Delta\mathfrak{g})^{\perp},$$

where $(\Delta\mathfrak{g})^{\perp}\subset(\mathfrak{g} \times \mathfrak{g})^*$. Thus $\mathbf{T} = \mu_{\Delta\mathbf{G}}^{-1}(\{0\})$, where

$$\mu_{\Delta\mathbf{G}} : T^*(\mathbf{X} \times \mathbf{G}/\mathbf{P}) \to (\mathfrak{g} \times \mathfrak{g})^*.$$

$|\mathbf{P}\backslash\mathbf{G}| < \infty \Rightarrow |(\mathbf{X} \times \mathbf{G}/\mathbf{P})/\mathbf{\Delta G}| < \infty \Rightarrow \mathbf{T}$ is Lagrangian. $\square$

- $\forall$ parabolic $\mathfrak{p}\subset\mathfrak{g}$,orbit $\mathbf{O} \subset \overline{\mathbf{O}_\mathbf{P}}$ we have $\dim \mathfrak{p}^\perp \cap \mathbf{O} \leq \dim \mathbf{O}/2$.
- For $\mathbf{S} := \mu^{-1}_{\mathbf{P},\mathbf{X}}(\{0\})$:
  $|\mathbf{P}\backslash\mathbf{X}| < \infty \iff \dim \mathbf{S} \leq \dim \mathbf{X} \iff \mathbf{S}$ is Lagrangian.

Proof of (iii)$\Rightarrow$(i), *i.e.* $\mathbf{X}$ is $\overline{\mathbf{O}_\mathbf{P}}$-spherical $\Rightarrow |\mathbf{P}\backslash\mathbf{G}| < \infty$.

Let $\mathbf{S} := \mu^{-1}(\mathfrak{p}^\perp) = \mu^{-1}_{\mathbf{P},\mathbf{X}}(\{0\})$. Then $|\mathbf{P}\backslash\mathbf{X}| < \infty \iff \dim \mathbf{S} \leq \dim \mathbf{X}$. Further, $\dim \mathbf{S} = \max_{\mathbf{O}\subset\overline{\mathbf{O}_\mathbf{P}}} \dim \mu^{-1}(\mathfrak{p}^\perp \cap \mathbf{O})$. $\forall \mathbf{O}\subset\overline{\mathbf{O}_\mathbf{P}}$ we have

$$\dim \mu^{-1}(\mathfrak{p}^\perp \cap \mathbf{O}) = \dim \mu^{-1}(\mathbf{O}) + \dim \mathfrak{p}^\perp \cap \mathbf{O} - \dim \mathbf{O} \leq$$
$$\leq \dim \mu^{-1}(\mathbf{O}) - \dim \mathbf{O}/2.$$

Thus

$$\dim \mu^{-1}(\mathbf{O}) \leq \dim \mathbf{X} + \dim \mathbf{O}/2 \ \forall \mathbf{O}\subset\overline{\mathbf{O}_\mathbf{P}} \Rightarrow \dim \mathbf{S} \leq \dim \mathbf{X}.$$

□

(ii)$\Rightarrow$(iii): $\Gamma_\mathbf{O} = \{(x, -\mu(x)) \in T^*(X) \times \mathfrak{g}^* \mid \mu(x) \in \mathbf{O}\}$ is isotropic, thus

$$\dim \mu^{-1}(\mathbf{O}) = \dim \Gamma_\mathbf{O} \leq (\dim T^*\mathbf{X} + \dim \mathbf{O})/2 = \dim \mathbf{X} + \dim \mathbf{O}/2$$

Main geometric questions:

- Give a criterion for $\overline{O}$-sphericity of X for non-Richardson O.
- What should be the definition of O-real spherical X?

Further geometric questions:

- Does O-spherical $\Rightarrow$ $\overline{O}$-spherical?
- Does $\dim O \cap \mathfrak{h}^{\perp} \leq \dim O/2$ imply $O \cap \mathfrak{h}^{\perp}$ being isotropic in O? Wen-Wei Li: for spherical $\mathfrak{h}$, and any O: $O \cap \mathfrak{h}^{\perp}$ is isotropic in O.
- For $X = U \coprod Z$, when is X $\overline{O}$-spherical in terms of U and Z?

More questions:

- Can we bound $m_{\sigma}(\mathcal{S}(X))$? Have to use some invariant of σ.
- By Theorem 3, relative characters given by $\mathcal{S}(X) \to \sigma$ and $\mathcal{S}(X) \to \tilde{\sigma}$ for $\mathcal{V}(\mathrm{Ann}(\sigma))$-spherical $\mathbf{X}$ are holonomic. Are they regular holonomic? Wen-Wei Li: for spherical $\mathbf{X}$ they are.
- If $\mathbf{G}/\mathbf{H}$ is $\mathcal{V}(\mathrm{Ann}(\sigma))$-spherical, is $\sigma^{HC}|_{\mathfrak{h}}$ finitely generated? Holds for real spherical G/H (Aizenbud-G.-Kroetz-Liu, Kroetz-Schlichtkrull).
- Conjecture: Theorem 2 holds over non-archimedean fields as well.

Thank you for your attention!

Examples by I.Karshon

Example ($\bar{\mathbf{O}}$-spherical $\mathbf{X}$ for non-Richardson $\mathbf{O}$)

$\mathbf{G} := \mathrm{GL}(V) \times \mathrm{GL}(W) \times \mathrm{Sp}(V \otimes W \oplus V^* \otimes W^*)$,
$\iota : \mathrm{GL}(V) \times \mathrm{GL}(W) \to \mathrm{Sp}(V \otimes W \oplus V^* \otimes W^*)$, $\mathbf{H} := Graph(\iota) \subset \mathbf{G}$.
Then $\mathbf{G}/\mathbf{H}$ *is* $\overline{\mathbf{O}_{\mathrm{reg}} \times \mathbf{O}_{\mathrm{reg}} \times \mathbf{O}_{\min}}$*-spherical.*

Example (Strict inequality)

W *:=symplectic vector space,* $\mathbf{G} := \mathrm{Sp}(W) \times \mathrm{Sp}(W \oplus W)$, *and*

$$\mathbf{H} := \{(Y, \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix})\} \subset \mathbf{G}$$

Let $\mathbf{O} := O_{\min} \times O_{\min} \subset \mathfrak{g}^* = \mathfrak{sp}^*(W) \times \mathfrak{sp}^*(W \oplus W)$.
Then $\dim \mathfrak{h}^\perp \cap \mathbf{O} = \dim W + 1$, *while* $\dim \mathbf{O} = 3 \dim W$.
Thus for $\dim W > 2$ *we have* $\dim \mathfrak{h}^\perp \cap \mathbf{O} < \dim \mathbf{O}/2$ *and thus*

$$\dim \mu_{\mathbf{G}/\mathbf{H}}^{-1}(\mathbf{O}) < \dim \mathbf{G}/\mathbf{H} + \dim \mathbf{O}/2.$$