Rigidity of homomorphisms of algebraic groups

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Outline

- Two questions and main result (loose version).
- Some background on families of homomorphisms.
- Main result (precise version) and two applications.
- Zariski tangent spaces and proof sketch of the rigidity result.
- Structure of linearly reductive groups.
- Proof sketch of the existence result.

Introduction

The objects of the talk are the homomorphisms $f : G \to H$, where G and H are algebraic groups over an algebraically closed field k. The group H acts on these homomorphisms by conjugation: $(h \cdot f)(g) = h\,f(g)\,h^{-1}$.

We will discuss the following (loosely stated) questions:

1) Is there a natural geometric structure on the set of homomorphisms $\mathrm{Hom}_{\mathrm{gp}}(G, H)$?
2) How to describe the H-orbits?

Here is a partial answer:

Theorem

Assume that G is linearly reductive. Then $\mathrm{Hom}_{\mathrm{gp}}(G, H)$ has a natural scheme structure. Moreover, every H-orbit is open.

In general, $\mathrm{Hom}_{\mathrm{gp}}(G, H)$ is not an algebraic variety: it may have infinitely many connected components. For example, the multiplicative group $\mathbb{G}_m$ satisfies $\mathrm{Hom}_{\mathrm{gp}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$ via the power maps $t \mapsto t^n$.

Morphisms of algebraic varieties

More generally, we may consider morphisms $f : X \to Y$, where X and Y are algebraic varieties over k, and ask for a natural geometric structure on the set $\mathrm{Hom}(X, Y)$ of such morphisms.

In this direction, we have the following result of Furter and Kraft (2018):

Theorem

Assume that k has characteristic 0 and X, Y are affine. Then $\mathrm{Hom}(X, Y)$ has a natural structure of an affine ind-variety.
If G and H are linear algebraic groups, then $\mathrm{Hom}_{\mathrm{gp}}(G, H)$ has a natural structure of closed subset of $\mathrm{Hom}(G, H)$, and hence of affine ind-variety.
Moreover, $\mathrm{Hom}_{\mathrm{gp}}(G, H)$ is finite-dimensional.
If in addition G is reductive and $H = \mathrm{GL}_n$, then $\mathrm{Hom}_{\mathrm{gp}}(G, H)$ is a countable union of closed H-orbits.

In loose words, an *ind-variety* M is an increasing union of algebraic varieties M_n indexed by the non-negative integers. It is equipped with the *Zariski topology*, for which a subset N is closed if and only if $N \cap M_n$ is closed in M_n for all n. The *dimension* of M is the supremum of the dimensions of the varieties M_n.

Families of morphisms

We now formulate precise definitions and questions.

The ground field k is algebraically closed (for simplicity), of arbitrary characteristic $p \geq 0$.

A *variety* X is a separated reduced scheme of finite type over k. (Equivalently, X is obtained by gluing finitely many affine varieties along open affine subvarieties, such that the diagonal in $X \times X$ is closed).

An *algebraic group* G is a variety equipped with morphisms $m : G \times G \to G$, $i : G \to G$ and with a point e satisfying the group axioms. Then G is smooth, not necessarily connected.

Consider two varieties X, Y. A *family of morphisms* $X \to Y$ *over* S is a morphism $f : X \times S \to Y$. Here S may be a variety, or more generally a scheme.
(Then f yields morphisms $f_s : X \to Y$, $x \mapsto f(x, s)$, where $s \in S(k)$. Also, the data of f is equivalent to that of a morphism $X \times S \to Y \times S$ over S).

Given a family $f : X \times S \to Y$ and a morphism $u : S' \to S$, we may form the *pull-back* $u^*(f) : X \times S' \to Y$, $(x, s') \mapsto f(x, u(s'))$. This is a family of morphisms over S'.

Families of morphisms (continued)

We may now ask whether there is a *universal family* $F : X \times M \longrightarrow Y$ such that every family $f : X \times S \to Y$ is obtained via pull-back by a unique morphism $u : S \to M$.

Then $M(k)$ is identified with $\mathrm{Hom}(X, Y)$ by taking $S = \mathrm{Spec}(k)$. More generally, for any field extension K/k, we have $M(K) \simeq \mathrm{Hom}_K(X_K, Y_K)$.

These notions adapt readily to the setting of homomorphisms of algebraic groups. This yields more precise versions of Questions 1 and 2:

Let G and H be algebraic groups.

1) Is there a universal family of homomorphisms $F : G \times M \to H$?
2) In the affirmative, $M(k) \simeq \mathrm{Hom}_{\mathrm{gp}}(G, H)$ and the action of H by conjugation on itself yields an action on M. How to describe the H-orbits?

The above-mentioned results of Furter and Kraft answer these questions for linear algebraic groups in characteristic 0, and for families over affine ind-varieties. But families over schemes behave differently.

Example: characters of the additive group

Take for G the additive group $\mathbb{G}_a$ (so that $\mathbb{G}_a(k) = k$ equipped with the addition), and for H the multiplicative group $\mathbb{G}_m$ (so that $H(k) = k^*$ equipped with the multiplication).

Consider a family of morphisms (of varieties) $f : \mathbb{G}_a \times S \longrightarrow \mathbb{G}_m$, where S is a variety. Since $\mathbb{G}_m$ is affine, the data of f is equivalent to that of the homomorphism of algebras

$$f^* : \mathcal{O}(\mathbb{G}_m) \longrightarrow \mathcal{O}(\mathbb{G}_a \times S),$$

i.e., of $f^* : k[t, t^{-1}] \longrightarrow \mathcal{O}(S)[x]$, or equivalently, of an invertible element of $\mathcal{O}(S)[x]$. Since $\mathcal{O}(S)$ is reduced, every such element is the constant polynomial h, where $h \in \mathcal{O}(S)$ is invertible. So there is a universal family of morphisms over varieties, namely the constant family

$$F : \mathbb{G}_a \times \mathbb{G}_m \longrightarrow \mathbb{G}_m, \quad (x, y) \longmapsto y.$$

If f is a family of homomorphisms, then $f(0, s) = 1$ for any $s \in S$, and hence $h = 1$. So *every family of homomorphisms over a variety is trivial*.

Characters of the additive group (continued)

But *there exists no family of homomorphisms* $\mathbb{G}_a \to \mathbb{G}_m$ *which is universal for families over schemes.*

Indeed, if such a family $F : \mathbb{G}_a \times N \to \mathbb{G}_m$ exists where N is a scheme, then $N(K)$ is a point for any field extension K/k (since every homomorphism $\mathbb{G}_{a,K} \to \mathbb{G}_{m,K}$ is constant). Thus, N is affine: $N = \mathrm{Spec}(A)$ for some k-algebra A.

Arguing as for varieties, F corresponds to a polynomial $P \in A[x]$ such that $P(x+y) = P(x)P(y)$ identically. By the universal property, for any k-algebra B and for any polynomial $Q \in B[x]$ such that $Q(x+y) = Q(x)Q(y)$ identically, there exists a unique homomorphism of algebras $u : A \to B$ such that $Q = u(P)$. In particular, the degree of Q is bounded independently of the k-algebra B.

But this fails if $p = 0$: take $B = k[t]/(t^{n+1})$ and $Q(x) = \exp(t\,x) = 1 + t\,x + \cdots + \frac{t^n x^n}{n!}$, which has arbitrarily large degree.
This also fails if $p > 0$: take $B = k[t]/(t^2)$ and $Q(x) = 1 + t\,x^{p^n}$, which has arbitrarily large degree as well.

Additive one-parameter subgroups

Assume $p = 0$ and consider homomorphisms $\mathbb{G}_a \to H$, where H is a non-trivial connected linear algebraic group. Let $\mathfrak{h} = \mathrm{Lie}(H)$.

Proposition

(i) *If H is unipotent, then the family*

$$\mathbb{G}_a \times \mathfrak{h} \longrightarrow H, \quad (t, x) \longmapsto \exp(t\,x)$$

is universal for families over schemes.

(ii) *If H is not unipotent, then there exists no family of homomorphisms $\mathbb{G}_a \to H$ which is universal for families over schemes.*

Proof sketch: (i) is proved by a standard argument. For (ii), use the fact that H contains a copy of $\mathbb{G}_m$.

By contrast, for any H there is a universal family over an affine variety, namely $\mathbb{G}_a \times \mathcal{N} \longrightarrow H$, $(t, x) \longmapsto \exp(t\,x)$, where $\mathcal{N} \subset \mathfrak{h}$ denotes the nilpotent variety (Furter and Kraft).

Main result

We say that an algebraic group G (possibly non-linear) is *linearly reductive* if every finite-dimensional representation of G is completely reducible.

Examples include tori, finite groups of order prime to p, and reductive groups if $p = 0$.

Further examples are *abelian varieties*, i.e., connected algebraic groups which are projective varieties (these have only trivial representations).

Theorem

Let G be a linearly reductive group, and H an algebraic group.

(i) *There exists a universal family of homomorphisms $F : G \times M \to H$, where M is a scheme.*

(ii) *M is the union of countably many open H-orbits.*

This result of existence (i) and rigidity (ii) is close to optimal:

Proposition

Let G be an algebraic group. If the assertions of the above theorem hold for any linear algebraic group H, then G is linearly reductive.

An application

Proposition

Let G be a linearly reductive group, and H an algebraic group. Then the natural map

$$\mathrm{Hom}_{\mathrm{gp}}(G, H)/H(k) \longrightarrow \mathrm{Hom}_{K-gp}(G_K, H_K)/H(K)$$

is a bijection for any algebraically closed field extension K/k.

This is due to Vinberg (1996) and Margaux (2009) for G linear.

Proof sketch: recall that the "universal scheme" M satisfies $M(k) = \mathrm{Hom}_{\mathrm{gp}}(G, H)$, and is a disjoint union of open orbits of k-rational points. Thus, the connected components of M are the orbits of the neutral component H^0. As a consequence,

$$M(k)/H(k) = (M(k)/H^0(k))/(H(k)/H^0(k)) = \pi_0(M)/\pi_0(H),$$

where $\pi_0(M)$ denotes the set of connected components of M. But the right-hand side is unchanged when k is replaced with an algebraically closed field extension.

A further application

Proposition

Let G be a finite group of order prime to p, and H an algebraic group. Then there are only finitely many conjugacy classes of homomorphisms $G \to H$.

Proof sketch: There is a universal scheme for morphisms (of varieties) $G \to H$, namely, H^n where $n = |G|$. Thus, there exists a universal scheme M for homomorphisms, which is closed in H^n and hence of finite type. As G is linearly reductive (Maschke's theorem), every H-orbit in M is open.

This applies to the classification of G-actions on a projective variety X. Indeed, such actions correspond bijectively to the homomorphisms $G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ has a natural structure of "locally algebraic group". *If* $\mathrm{Aut}(X)$ *is an algebraic group, then there are only finitely many conjugacy classes of G-actions on X.* The assumption holds if X is of complexity at most 1 under the action of a reductive group.

But this finiteness assertion may fail: there is a smooth projective rational surface X such that $\mathrm{Aut}(X)$ has infinitely many conjugacy classes of involutions (Dinh, Oguiso and Yu, arXiv:2106.05687).

Zariski tangent spaces

Let S be a scheme, and $s \in S(k)$. The Zariski tangent space $T_s S$ is defined as $(\mathfrak{m}/\mathfrak{m}^2)^*$, where $\mathfrak{m}$ denotes the maximal ideal of the local ring $\mathcal{O}_{S,s}$ (then $\mathfrak{m}/\mathfrak{m}^2$ is a k-vector space).

This can be interpreted in terms of the algebra of dual numbers $D = k[t]/(t^2)$. Indeed, $T_s S$ is the preimage of s under the natural map $S(D) \to S(D/tD) = S(k)$.

Next, consider two varieties X, Y and assume that there exists a universal family of morphisms $X \times M \to Y$, where M is a scheme. Then $M(A) = \mathrm{Hom}_A(X_A, Y_A)$ for any k-algebra A. With a little work, this yields:

Lemma

If Y is smooth, then for any $f \in \mathrm{Hom}(X, Y) = M(k)$, we have $T_f M \simeq \Gamma(X, f^* T_Y)$ where T_Y denotes the tangent bundle.

Take for Y an algebraic group H. Since T_H is the trivial bundle with fiber the Lie algebra $\mathfrak{h}$, we obtain under the above assumption

$$T_f M \simeq \mathcal{O}(X) \otimes \mathfrak{h} \simeq \mathrm{Hom}(X, \mathfrak{h}).$$

Zariski tangent spaces (continued)

Let G, H be algebraic groups and assume that there is a universal family of homomorphisms $F : G \times M \to H$, where M is a scheme.

For any $f \in \mathrm{Hom}_{\mathrm{gp}}(G, H) = M(k)$, the Zariski tangent space $T_f M$ is identified with the space of 1-cocycles

$$Z^1(G, \mathfrak{h}) = \{\varphi \in \mathrm{Hom}(G, \mathfrak{h}) \mid \varphi(g_1 g_2) = \varphi(g_1) + g_1 \cdot \varphi(g_2)\}.$$

Here G acts on $\mathfrak{h}$ via $\mathrm{Ad} \circ f$, where $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{h})$ denotes the adjoint representation.

Consider the orbit map

$$H \longrightarrow M, \quad h \longmapsto (g \mapsto h\, f(g)\, h^{-1}).$$

The image of its differential at the neutral element e_H is the subspace of 1-coboundaries

$$B^1(G, \mathfrak{h}) = \{\varphi_z : G \to \mathfrak{h}, \quad g \mapsto g \cdot z - z \quad (z \in \mathfrak{h})\}.$$

So the cohomology group $H^1(G, \mathfrak{h}) = Z^1(G, \mathfrak{h}) / B^1(G, \mathfrak{h})$ is the normal space to the orbit $H \cdot f$ at f in M.

Main result: proof sketch of rigidity

Let G be a linearly algebraic group, and H an algebraic group. Assume that there is a universal family of homomorphisms $F : G \times M \to H$, where M is a scheme, *locally of finite type* (i.e., M is the union of open affine subschemes of finite type).

Since G is linearly reductive, we have $H^1(G, V) = 0$ for any G-module V. Thus, the normal space to $H \cdot f$ in M is zero for any $f \in M(k)$. As a consequence, every H-orbit in M is open.

To show that there are countably many such orbits, note that G, H are defined over a subfield of k which is finitely generated over its prime field, and hence countable. Thus, there exists a countable, algebraically closed subfield $k' \subset k$ and algebraic groups G', H' over k' such that $G = G'_k$ and $H = H'_k$. Then G' is linearly reductive.

By the existence result applied to G', H', there exists a universal scheme M' for homomorphisms $G' \to H'$. Thus, $M \simeq M'_k$. So we may assume that k is countable. Then it suffices to show that the set $\mathrm{Hom}_{\mathrm{gp}}(G, H)$ is countable, or even that $\mathrm{Hom}(G, H)$ is countable. But one easily checks that $\mathrm{Hom}(X, Y)$ is countable for any varieties X, Y.

Main result: remarks on existence

Consider again two varieties X, Y, and assume that there exists a universal scheme M for morphisms $X \to Y$. By results of Grothendieck in EGA IV.8, it follows that M is locally of finite type.

Likewise, if X is a variety having a universal family of *automorphisms* $N \times X \to X$, where N is a scheme, then N is locally of finite type. In particular, the Zariski tangent space $T_{\mathrm{id}} N$ is finite-dimensional.

By an easy argument using the algebra of dual numbers, we obtain $T_{\mathrm{id}} N \simeq \mathrm{Der}(\mathcal{O}_X)$ (the space of derivations of the structure sheaf, i.e., of global vector fields on X). If X is smooth, then $\mathrm{Der}(\mathcal{O}_X) = \Gamma(X, T_X)$.

For an affine variety X, we have $\mathrm{Der}(\mathcal{O}_X) = \mathrm{Der}\, \mathcal{O}(X)$. This is an infinite-dimensional vector space unless X is finite.

As a consequence, $\mathrm{Aut}(X)$ *has no natural scheme structure if X is an affine variety of positive dimension*. But it has a natural ind-variety structure if $p = 0$, by a result of Furter and Kraft.

Affine linearly reductive groups

Recall that an algebraic group G is linear if and only if G is an affine variety. So we will use "affine" and "linear" interchangeably.

The structure of affine linearly reductive groups is due to Nagata:

Theorem

Let G be an affine algebraic group.

(i) *Assume $p = 0$. Then G is linearly reductive iff G^0 is reductive.*

(ii) *Assume $p > 0$. Then G is linearly reductive iff the two following conditions hold: G^0 is a torus and the finite group G/G^0 has order prime to p.*

In particular, the connected affine linearly reductive groups are exactly the reductive groups if $p = 0$, and the tori if $p > 0$.

The class of affine linearly reductive groups is clearly stable under quotients. It is also stable under normal subgroups and extensions. Moreover, an affine algebraic group G is linearly reductive if and only if so is G_K for some algebraically closed field extension K/k (Margaux).

The affinization theorem

Here is another structure result for algebraic groups, due to Rosenlicht and Demazure–Gabriel:

Theorem

Let G be an algebraic group. Then G has a largest affine quotient group $G^{\mathrm{aff}} = G/N$. Moreover, N is a connected algebraic group contained in the center of G^0. We have $\mathcal{O}(N) = k$.

An algebraic group N such that $\mathcal{O}(N) = k$ is called *anti-affine*. Then N is connected and commutative. The structure of anti-affine groups is well-understood; examples include abelian varieties.

Every representation of G factors through G^{aff}. Thus, G is linearly reductive if and only if so is G^{aff}.

As a consequence, the class of linearly reductive groups is stable under extensions. It is clearly stable under quotients (but not under normal subgroups if $p = 0$). Moreover, an algebraic group G is linearly reductive if and only if so is G_K for some algebraically closed field extension K/k.

Structure of linearly reductive groups

Let G be an algebraic group.

Proposition

Assume $p = 0$. Then G is linearly reductive iff it lies in an exact sequence $1 \longrightarrow F_1 \longrightarrow G_1 \times G_2 \longrightarrow G \longrightarrow F_2 \longrightarrow 1$, where F_1 is a finite group scheme, F_2 is a finite group, G_1 is anti-affine, and G_2 is reductive.

A *semi-abelian variety* is an algebraic group G which lies in an exact sequence $1 \to T \to G \to A \to 1$, where T is a torus and A an abelian variety. Then G is connected, commutative, and linearly reductive (as G^{aff} is a torus).

Proposition

Assume $p > 0$. Then G is linearly reductive iff it lies in an exact sequence $1 \longrightarrow N \longrightarrow G \longrightarrow F \longrightarrow 1$, where N is a semi-abelian variety, and F is a finite group of order prime to p.

In particular, the connected linearly reductive groups are exactly the semi-abelian varieties if $p > 0$.

Proof of the existence result: first steps

The starting point is the following observation. Consider an exact sequence of algebraic groups

$$1 \longrightarrow N \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

and an algebraic group H. Then we may identify $\mathrm{Hom}_{\mathrm{gp}}(G, H)$ with the subset of $\mathrm{Hom}_{\mathrm{gp}}(\tilde{G}, H)$ consisting of homomorphisms which restrict trivially to N.

Lemma

If there exists a universal scheme $M_{\tilde{G},H}$ for homomorphisms $\tilde{G} \to H$, then the universal scheme $M_{G,H}$ exists and is closed in $M_{\tilde{G},H}$.

We may thus replace G with a group having a simpler structure. Here is a further observation:

Lemma

If $G = G_1 \rtimes G_2$ and $M_{G_i,H}$ exists for $i = 1, 2$, then $M_{G,H}$ exists and is closed in $M_{G_1,H} \times M_{G_2,H}$.

Proof of the existence result: further reductions

The following result is due to Borel–Serre, Vinberg and others:

Lemma

Let G be an algebraic group, and N a closed normal subgroup of G such that G/N is finite. Then there exists a finite subgroup F of G such that $G = NF$.

Here NF denotes the closed subgroup of G, image of the homomorphism $N \rtimes F \to G$ given by the multiplication.

This yields an exact sequence

$$1 \longrightarrow N \cap F \longrightarrow N \rtimes F \longrightarrow G \to 1.$$

So if there exist universal schemes $M_{N,H}$ and $M_{F,H}$, then $M_{G,H}$ exists as well.

Since $M_{F,H}$ exists and is closed in H^n where $n = |F|$, we may replace G with N.

Using the structure of linearly reductive groups, we may further reduce to G reductive or anti-affine.

Proof of the existence result (continued)

For a reductive group G with maximal torus T, the existence of $M_{G,H}$ follows from that of $M_{T,H}$ by a result of Demazure in SGA3, Exposé XXIV.

This yields a further reduction to the case where G is a torus, which is handled directly by reducing to $H = \mathrm{GL}_n$ and using representation theory.

For an anti-affine group G, one has a more precise result:

Proposition

Let G be an anti-affine group, and H an algebraic group.

(i) $\mathrm{Hom}_{\mathrm{gp}}(G, H) = \mathrm{Hom}_{\mathrm{gp}}(G, Z(H^0)) \simeq \mathbb{Z}^n$ *for some integer* $n \geq 0$.

(ii) *For any family of homomorphisms* $f : G \times S \to H$ *where* S *is a connected scheme, we have identically* $f(g, s) = \varphi(g)$ *where* $\varphi \in \mathrm{Hom}_{\mathrm{gp}}(G, H)$.

(iii) *The universal scheme* $M_{G,H}$ *exists and is isomorphic to* $\mathbb{Z}^n$.

This is well-known for abelian varieties, and follows from a rigidity lemma. A generalization of this lemma due to C. and F. Sancho de Salas yields the proposition.