# Brick varieties, postroids, and Legendrian links

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# Braid groups and braid matrices

## Definition

The **braid group** $\mathrm{Br}_n$ and the **positive braid monoid** $\mathrm{Br}_n^+ \subset \mathrm{Br}_n$:

- Generators: $\sigma_i$, $i \in [1, n-1]$;
- Relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2.$$

Let $z \in \mathbb{C}, i \in [1, n-1]$. The **braid matrix** $B_i(z) \in \mathrm{GL}(n, \mathbb{C}[z])$ :

$$B_i(z) := \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & z & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix} \begin{matrix} \\ \\ i \\ i+1 \\ \\ \\ \end{matrix}$$

# Braid matrices

Given a positive braid word $\beta = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathrm{Br}_n^+$ and $z_1, \ldots, z_r \in \mathbb{C}$, we define the **braid matrix** $B_\beta(z_1, \ldots, z_r) \in \mathrm{GL}(n, \mathbb{C}[z_1, \ldots, z_r])$ to be the product

$$B_\beta(z_1, \ldots, z_r) = B_{i_1}(z_1) \cdots B_{i_r}(z_r).$$

Replace each $\sigma_i$ by the transposition $s_i$. This defines a projection $\pi : \mathrm{Br}_n \to S_n$.

**Example**

$B_\beta(0, \ldots, 0)$ is the permutation matrix of $\pi(\beta)$.

**Lemma**

- $B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1 z_3)B_{i+1}(z_1), \quad (\star)$ $\forall i \in [1, n-2]$.
- $B_i(z_1)B_j(z_2) = B_j(z_2)B_i(z_1)$, *for* $|i - j| \geq 2$.

# Half-twist

$\Delta = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)\sigma_1$. It is a lift of the longest element $w_0 = (n \;\; (n-1) \;\; \dots \;\; 1) \in S_n$.

$$B_\Delta\left(z_1,\dots,z_{\binom{n}{2}}\right) = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & \dots & 1 & z_1 \\ \vdots & \dots & \dots & z_{n-2} \\ 1 & z_{\binom{n}{2}} & \cdots & z_{n-1} \end{pmatrix}.$$

Let $\Delta' \in \mathrm{Br}_n^+$ be *any* positive braid lift of $w_0$ (**half-twist**). By $(\star)$,

$$B_{\Delta'}\left(z_1,\dots,z_{\binom{n}{2}}\right) = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & \dots & 1 & z_{2,n} \\ \vdots & \dots & \dots & z_{n-1,n} \\ 1 & z_{n,2} & \cdots & z_{nn} \end{pmatrix},$$

where the $z_{i,j} \in \mathbb{C}[z_1,\dots,z_{\binom{n}{2}}]$ are algebraically independent polynomials.

# Full twist

Let $\Delta^2 \in \mathrm{Br}_n^+$ represent the **full-twist** braid, i.e. the square of the positive braid lift of $w_0 \in S_n$ to the braid group. Then its braid matrix can be decomposed as

$$B_{\Delta^2}\left(z_1,\dots,z_{\binom{n}{2}},w_1,\dots w_{\binom{n}{2}}\right)=LU=$$

$$=\begin{pmatrix}1 & 0 & \dots & 0\\ c_{21} & 1 & \dots & 0\\ \vdots & \dots & \ddots & 0\\ c_{n1} & \dots & \dots & 1\end{pmatrix}\begin{pmatrix}1 & u_{12} & \dots & u_{1n}\\ 0 & 1 & \dots & u_{2n}\\ 0 & \dots & \ddots & u_{n-1,n}\\ 0 & \dots & \dots & 1\end{pmatrix},$$

where $c_{ij} \in \mathbb{C}[z_1,\dots,z_{\binom{n}{2}}]$ and $u_{ij} \in \mathbb{C}[w_1,\dots,w_{\binom{n}{2}}]$ are algebraically independent.

# Braid varieties

## Definition

Let $\beta = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathrm{Br}_n^+$ be a positive braid word. The **braid variety** $X_0(\beta) \subseteq \mathbb{C}^r$ is the affine closed subvariety given by

$$X(\beta) := \{(z_1, \ldots, z_r) : B_\beta(z_1, \ldots, z_r) \text{ is upper-triangular}\} \subseteq \mathbb{C}^r .$$

Let $\pi \in S_n$ be considered as a permutation matrix. The **braid variety** $X_0(\beta; \pi) \subseteq \mathbb{C}^r$ as

$$X(\beta; \pi) := \{(z_1, \ldots, z_r) : B_\beta(z_1, \ldots, z_r)\pi \text{ is upper-triangular}\} \subseteq \mathbb{C}^r .$$

It follows from the braid relation $(\star)$ that different presentations of the same braid $[\beta] \in \mathrm{Br}_n$ yield algebraically isomorphic braid varieties.

- $X(\Delta^2) \cong \mathbb{C}^{\binom{n}{2}}$.
- $X(\Delta; w_0) = \{\mathrm{pt}\}$.

# Appearances of braid matrices and braid varieties

- [Euler]: Continuants;
- [Stokes]: Study of irregular singularities;
- [Broué-Michel]: Deligne-Lusztig varieties;
- [Deligne]: Braid invariants;
- ...
- [Kálmán]: study of Legendrian Contact DGAs (under the name of **path matrices**);
- [Mellit]: proof of the curious Lefschetz property for character varieties.

# Trefoil

Consider $\beta = \sigma_1^3 \in \mathrm{Br}_2^+$. Its closure is the (right-handed) trefoil knot. $X(\sigma_1^5) = X_0(\sigma_1^3 \cdot \Delta^2)$ is defined by the condition:

$$B(z_1)B(z_2)B(z_3)B(z_4)B(z_5) \text{ is upper-triangular.}$$

By rewriting the matrix product, we get

$$X(\sigma_1^3 \cdot \Delta^2) \cong X(\sigma_1^3 \cdot \Delta; w_0) \times \mathbb{C}.$$

$$X(\sigma_1^3 \cdot \Delta; w_0) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : (z_1 + z_3 + z_1 z_2 z_3) \neq 0\} \subset \mathbb{C}^3.$$

This shows that $X(\sigma_1^3 \cdot \Delta; w_0)$ is smooth. We can also write

$$X(\sigma_1^3 \cdot \Delta; w_0) \cong \{(z_1, z_2, z_3, t) : (z_1 + z_3 + z_1 z_2 z_3)t = 1\} \subset \mathbb{C}^3 \times \mathbb{C}^*,$$

so there exists a $\mathbb{C}^*$-action on $X(\sigma_1^3 \cdot \Delta; w_0)$ whose quotient yields an affine surface.

# Torus action

## Definition

Let $\beta \in \mathrm{Br}_n^+$ of length $r = \ell(\beta)$. The torus action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^{\ell(\beta)}$ is given by

$$(t_1, \dots, t_n).(z_1, \dots, z_r) := (c_1 z_1, \dots, c_r z_r), \quad (z_1, \dots, z_r) \in \mathbb{C}^r,$$

where $c_k = t_{w_k(i_k+1)} t_{w_k(i_k)}^{-1}$, $w_k = s_{i_1} \cdots s_{i_{k-1}}$, and $w = w_{r+1}$ is the permutation corresponding to $\beta$.
This torus action preserves $X_0(\beta) \subseteq \mathbb{C}^r$ thanks to $(\star)$.

$T := (\mathbb{C}^*)^n / \mathbb{C}^*_{diag} \cong (\mathbb{C}^*)^{n-1}$.
$\mathbb{C}^*_{diag}$ acts trivially on $X_0(\beta)$. This induces the $T$-torus action $T \times X_0(\beta) \to X_0(\beta)$.

If $[\beta] = [\beta'] \in \mathrm{Br}_n^+$, then there exists an algebraic isomorphism $X_0(\beta) \cong X_0(\beta')$ which is equivariant w.r.t. this torus action.

# HOMFLY-PT homology

- With $\beta$ one can associate a **Rouquier complex** $T_\beta$ in the category of complexes of Soergel bimodules.
- Up to homotopy, it depends only on $[\beta]$.
- HOMFLY-PT (= Khovanov-Rozansky) homology of $\beta$ : $HHH(\beta) := H^*(HH^*(T_\beta))$.

**Theorem (Khovanov-Rozansky)**

*$HHH(\beta)$ is, up to shifts in gradings, a topological invariant of the closure of $\beta$.*

- $a = 0$ part is not a topological invariant. But it is invariant under conjugation, positive (de)stabilization ($\gamma < - > \gamma\sigma_k$, for $\gamma \in Br_k$), and Reidemester II and III moves.
- Webster-Williamson,…, Mellit, Trinh: $\mathbf{gr}^W H^T_{*,BM}(X(\beta, w_0)) = \mathbf{gr}^W H^*_T(X(\beta\Delta) = HHH^{a=n}(\beta\Delta)$.
- E. Gorsky-Hogancamp-Mellit-Nakagane: $HHH^{a=n}(\beta\Delta) = HHH^{a=0}(\beta\Delta^{-1})$.

# Markov theorem for braid varieties

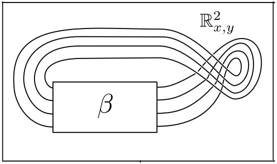
## Corollary

*$H_T^*(X(\beta, w_0))$ with its weight filtration is invariant under conjugation and positive (de)stabilization I (and Reidemeister II and III moves) for $\beta\Delta^{-1}$.*

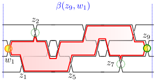
## Theorem (Casals - E. Gorsky - MG - Simental)

*$X(\beta, w_0)$, up to $\mathbb{C}^*$ factors, is invariant under conjugation and positive (de)stabilization (and Reidemeister II and III moves) for $\beta\Delta^{-1}$.*

- [Casals-Ng]: The "pigtail closure" of $\beta\Delta^{-1}$ can be realized as a Legendrian link in $\mathbb{R}^3$ (with the standard contact structure $\xi_{\text{st}} = \ker(dz - ydx)$).



- [Chekanov,...]: A DGA for any Legendrian link. Generators correspond to crossings, differentials count certain discs.
- Here:
  - $n^2$ generators in degree 1 (crossings in the pigtail);
  - A generator $z_i$ of degree 0 for each positive crossing;
  - A generator $w_j$ of degree (-1) for each negative crossing.




- Conjugations, positive (de)stabilizations, and Reidemeister II and III moves induce *stable tame isomorphisms* of DGAs $A(\gamma)$ of braid closures. In particular, they do not change $H^*(A(\gamma))$.
- [Kálmán] If $\beta$ is positive, then $X(\beta, w_0) \cong \mathrm{Spec}(H^0(A(\beta\Delta))$. (**augmentation variety**).
- [CGGS] If $\beta$ is equivalent to a positive braid, then $H^i(A(\beta\Delta)) = 0, i \neq 0$ and $\mathrm{Spec}(H^0(A(\beta\Delta)) \cong X/V$, where $V$ is a collection of commuting vector fields on $X$ parameterized by negative crossings.

# Closed Bott-Samelson varieties and brick manifolds

(i) Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$ be a positive braid word. The (closed) Bott-Samelson variety $\mathbf{BS}(\beta) \subseteq \mathcal{F}\ell^{\ell+1}$ associated to $\beta$ is the moduli space of $(\ell+1)$-tuples of flags $(\mathcal{F}_0, \dots, \mathcal{F}_\ell)$ such that consecutive flags $\mathcal{F}_{k-1}, \mathcal{F}_k$ coincide or differ only in $V_{i_k}$, for each $k \in [1, \ell]$.

(ii) Assume that $\beta$ contains a reduced expression of $w_0$ as a subword. The **brick manifold** is the intersection

$$\mathrm{brick}(\beta) := \mathbf{BS}(\beta) \cap p_0^{-1}(\mathcal{F}^{st}) \cap p_\ell^{-1}(\mathcal{F}^{ast}).$$

Warning: These depend on the word $\beta$, not only on the braid $[\beta]$.

**Theorem (Escobar)**

brick$(\beta)$ *is smooth, irreducible and of dimension* $\ell - \binom{n}{2}$.

# Open Bott-Samelson varieties and brick manifolds

(i) Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$ be a positive braid word. The open Bott-Samelson variety $\mathrm{OBS}(\beta) \subseteq \mathcal{F}\ell^{\ell+1}$ associated to $\beta$ is the moduli space of $(\ell+1)$-tuples of flags $(\mathcal{F}_0, \ldots, \mathcal{F}_\ell)$ such that consecutive flags $\mathcal{F}_{k-1}, \mathcal{F}_k$ are in relative position $s_{i_k}$ (i.e. differ precisely in $V_{i_k}$), for each $k \in [1, \ell]$.

(ii) Assume that $\beta$ contains a reduced expression of $w_0$ as a subword. The **open brick manifold** is the intersection

$$\mathrm{brick}(\beta)^\circ := \mathrm{brick}(\beta) \cap \mathrm{OBS}(\beta).$$

[Broué-Michel, Deligne,...] These depend only on the braid $[\beta]$ !!!

## Theorem (Escobar)

- $\mathrm{brick}(\beta) = \coprod \mathrm{brick}(\beta')^\circ$, *for* $\beta'$ *subwords of* $\beta$ *containing* $w_0$.
- *The adjacency of the strata is described by the* **dual subword complex** *of* $(\beta, w_0)$ *introduced by [Knutson-Miller].* $\mathrm{brick}(\beta)^\circ$ *is the unique top dimensional stratum.*

# Torus actions and moment polytopes

Bott-Samelson varieties are Hamiltonian symplectic manifolds with respect to the natural action of $(\mathbf{C}^*)^{n-1}$.

Escobar: the image of $\mathrm{brick}(\beta)$ under the corresponding moment map is a *brick polytope* of $\beta$ [Pilaud-Stump].

$\mathrm{brick}(\beta)$ is a toric variety of this polytope with respect to this torus action if and only if the word $\beta$ is *root independent*.

[Pilaud-Stump]: The brick polytope of a root independent word $\beta$ realizes its spherical subword complex; this is not true for an arbitrary braid word $\beta$.

## Theorem (CGGS)

*Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_n$ be a positive braid word, $\reflectbox{$\beta$} \in \mathcal{B}_n$ its opposite braid word, $\delta(\reflectbox{$\beta$})$ its Demazure product, and consider the truncations $\beta_j := \sigma_{i_1} \cdots \sigma_{i_j}$, $j \in [1, \ell]$. The following holds:*

(*i*) *The algebraic map*

$$\Theta : \mathbf{C}^\ell \longrightarrow \mathcal{F}\ell_n^{\ell+1}, \quad (z_1, \ldots, z_\ell) \mapsto (\mathcal{F}^{st}, \mathcal{F}^1, \ldots, \mathcal{F}^\ell),$$

*where $\mathcal{F}^j$ is the flag associated to the matrix $B^{-1}_{\reflectbox{$\beta$}_j}(z_{\ell-j+1}, \ldots, z_\ell)$, restricts to an isomorphism*

$$\Theta : X(\reflectbox{$\beta$}; \delta(\beta)) \xrightarrow{\cong} \mathrm{brick}^\circ(\beta),$$

*of affine varieties. It is compatible with the torus actions.*

(*ii*) *Suppose that the Demazure product of $\reflectbox{$\beta$}$ is $\delta(\reflectbox{$\beta$}) = w_0$. Then, the complement to $X(\reflectbox{$\beta$}; w_0)$ in $\mathrm{brick}(\beta)$ is a normal crossing divisor. Its components correspond to all possible ways to remove a letter from $\reflectbox{$\beta$}$ while preserving its Demazure product.*

Consider the equivalent braid words

$$\beta_1 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1, \quad \beta_2 = \sigma_1\sigma_2\sigma_2\sigma_1\sigma_2.$$

In both cases, the braid varieties are algebraic tori

$$X(\beta_1; w_0) \cong X(\beta_2; w_0) \cong (\mathbf{C}^*)^2.$$

The variety $\mathrm{brick}(\beta_1)$ has

- $X(\beta_1; w_0)$ as an open stratum;
- 5 strata of codim 1 (isomorphic to $\mathbf{C}^*$);
- 5 strata of codim 2 (points).

$\mathrm{brick}(\beta_1)$ is a toric degree 5 del Pezzo surface, i.e. the toric variety associated to the pentagon, and these various strata correspond to toric orbits.

For $\mathrm{brick}(\beta_2)$, $X(\sigma_1\sigma_2^3; w_0)$ is empty, so there can only be four codimension 1 strata and four codimension 2 strata:

$\mathrm{brick}(\beta_2) \cong \mathbf{P}^1 \times \mathbf{P}^1$.

At least in the toric case, all such compactifications of $X(\beta; w_0)$ are related by of blow-up sand blow-downs, corresponding to braid moves.

# Open Richardson varieties

The flag variety admits the *Schubert decomposition* and the *opposite Schubert decomposition*. The strata in either of them are parameterized by permutations: $\overset{\circ}{X}_w$, resp. $\overset{\circ}{X}^w$.

An **open Richardson variety** $\mathcal{R}^\circ(u,w)$ is the intersection $\overset{\circ}{X}_w \cap \overset{\circ}{X}^u$.

$R^\circ(u,w) \neq \emptyset$ if and only if $u \leq w$ in the Bruhat order.

**Theorem (Brion, Knutson-Lam-Speyer, Balan, Escobar, CGGS)**

*Let $u, w \in S_n$ be such that $u \leq w$ in Bruhat order, and $\beta(w), \beta(u^{-1}w_0) \in \mathrm{Br}_n$ positive lifts of $w, u^{-1}w_0$. Then we have an isomorphism of affine algebraic varities*

$$X(\beta(w)\beta(u^{-1}w_0); w_0) \cong \mathcal{R}^\circ(u,w).$$

# Positroids

The Grassmannian $Gr(k, n)$ admits a stratification by **open positroid varieties.** They have many different desriptions/parameterization by various combinatorial pieces of data (Postnikov, KLS):

- A cyclic rank matrix;
- A juggling pattern;
- A decorated affine permutation;
- $u, w \in S_n$ s. t. $u \leq w$ and $w$ is k-Grassmannian;
- A reduced plabic graph;
- ...

**Theorem (KLS)**

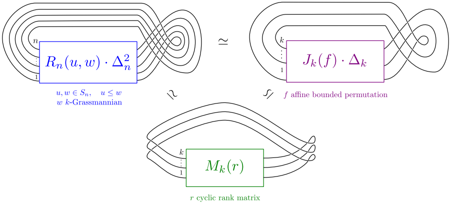
*For each $u, w \in S_n$ s. t. $u \leq w$ and $w$ is k-Grassmannian, the positroid $\Pi_{u,w}$ is isomorphic to the open Richardson variety $\mathcal{R}^\circ(u, w)$.*

**Corollary**

*Open positroid varieties are braid varieties.*

# Positroid links

In fact, we associate various Legendrian links to the combinatorial pieces of data defining positroids:




## Theorem

*Let $u, w \in S_n$ with $u \leq w$ in Bruhat order, $w$ a $k$-Grassmannian permutation, $R_n(u,w) = \beta(u)\beta(w)^{-1}$ and $f := u^{-1} t_k w$ the corresponding $k$-bounded affine permutation. Then we have*

$$\Pi_{u,w} \cong X(R_n(u,w)\Delta_n)/V \cong X(\beta(w)\beta(u^{-1}w_{0,n}); w_{0,n}) \cong$$

$$X(J_k(f); w_{0,k}) \times (\mathbb{C}^*)^{n-s-k}.$$

# Automorphisms

The braid varieties $X(s_i\beta; w_0)$ and $X(\beta s_{n-i}; w_0)$ are isomorphic (invariance by conjugation).
This easily implies that the centralizer of $\beta$ acts on $X(\beta\Delta; w_0)$ by automorphisms.
It is clear that the relations between different $\gamma$ yield the same relations between the automorphisms.
[Fraser]: There is a natural braid group action on the top positroid cell.
[Fraser-Keller, in preparation]: This generalizes to all positroids.
Expectation: this is the same action, for certain braids on the last slide.

# Toric charts

Consider the positive braid word $\beta = \beta_1 \sigma_i \beta_2$ and $\beta' = \beta_1 \beta_2$, with $\sigma_i$ on the $r-$th place in $\beta$.

## Lemma

*There exists a rational map*

$$\Omega_{\sigma_i} : X(\beta, \delta(\beta)) \dashrightarrow X(\beta', \delta(\beta)) \times \mathbb{C}^*$$

*which restricts to an isomorphism between the open locus $\{z_r \neq 0\} \subseteq X(\beta, \delta(\beta))$ and $X(\beta', \delta(\beta)) \times \mathbb{C}^*$.*

## Proposition

*Let $\beta \in Br_n^+$. For each ordering $\tau(\beta) \in S_{\ell(\beta)}$ of the crossings of $\beta$, there exists an open set $T_{\tau(\beta)} \subseteq X(\beta \cdot \Delta; w_0)$ which is isomorphic to a torus $(\mathbb{C}^*)^{\ell(\beta)}$ and stable under the $(\mathbb{C}^*)^{n-1}$-action on $X(\beta \cdot \Delta; w_0)$.*

# Toric cluster charts and stratifications

## Theorem (Gao-Shen-Weng)

*$X_0(\beta \cdot \Delta; w_0)$ is a* **cluster variety***: it has a special atlas of toric charts called* **cluster charts***. Birational transition functions have very special form of* **cluster mutations***.*

We also stratify $X_0(\beta; w_0)$ by strata described via certain planar diagrams (**weaves**). The diagrammatics resembles Soergel calculus, but takes mutations into account.

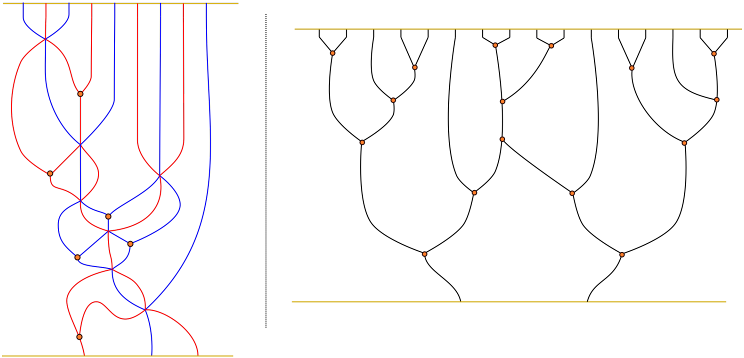
## Theorem

*The complement*

$$X_0(\beta \cdot \Delta; w_0) \setminus \left( \bigcup_{\tau(\beta) \in S_{\ell(\beta)}} T_{\tau(\beta)} \right) \subseteq X_0(\beta \cdot \Delta; w_0)$$

*has codimension at least 2. It can be stratified into $(\mathbb{C})^a \times (\mathbb{C}^*)^b$ using weaves.*

# Examples of weaves

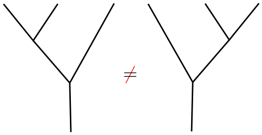


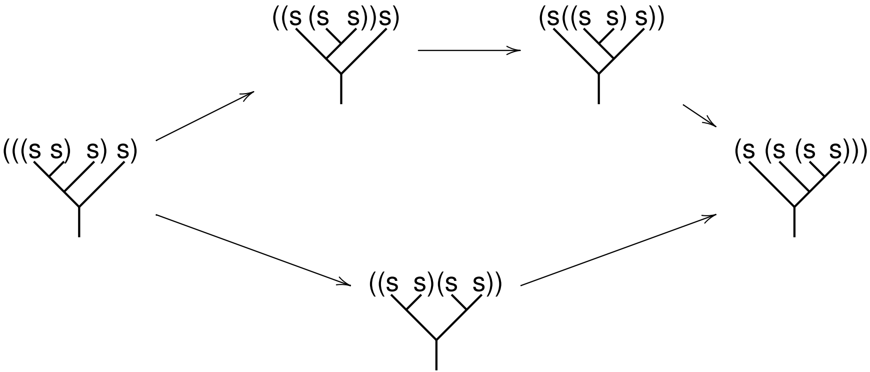
Left: A 3-weave from $\beta_2 = (\sigma_1\sigma_2)^4\sigma_1 \in \mathrm{Br}_3^+$ to $\beta_1 = \sigma_2\sigma_1\sigma_2 \in Br_3^+$. The blue color indicates a transposition label $s_1 \in S_3$ and the red color indicates the transposition label $s_2 \in S_3$.

Right: A 2-weave from $\beta_2 = \sigma_1^{16} \in \mathrm{Br}_2^+$ to $\beta_1 = \sigma_1^2 \in \mathrm{Br}_2^+$, all black edges are labeled with the unique transposition $s_1 \in S^2$.

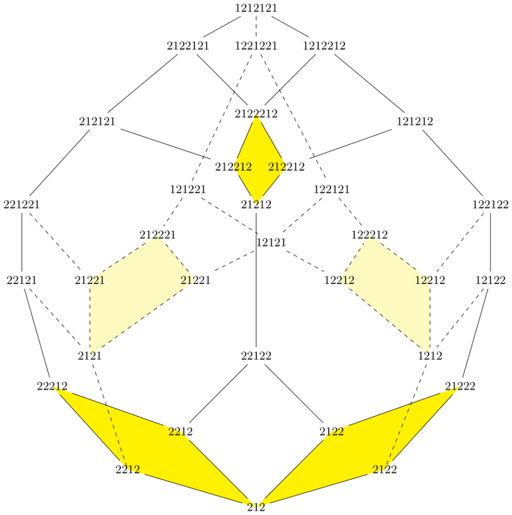
# Mutations

Our diagrammatic calculus resembles Soergel calculus of Elias, Elias-Khovanov, Elias-Williamson...
The crucial difference is that two weaves $sss \to s$ are not considered to be equivalent: the are related by a mutation:






All edges are oriented in the direction $(ss)s \to s(ss)$. This a Hasse graph of the Tamari lattice.
For trees with $(n+1)$ leaves, the graph is the $1-$skeleton of a polytope: the $(n-1)$-dimensional associahedron.




Weaves $s_1 s_2 s_1 s_2 s_1 s_2 s_2 \to s_2 s_1 s_2$ with only 6-valent vertices $s_1 s_2 s_1 \to s_2 s_1 s_2$ and 3-valent vertices $s_2 s_2 \to s_2$ allowed represent monotone paths from the top vertex to the bottom vertex. The mutation graph is a pentagon!

# Cluster charts and mutations

**Conjecture**

*$T_{\tau(\beta)}$ are the cluster charts. Mutations correspond to mutations of weaves (proved in typed D by Hughes, some evidence in finite and affine types by An-Bae-Lee).*

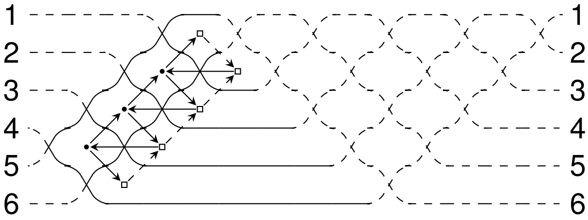
**Theorem**

*Equivalent weaves give rise to the same toric chart.*

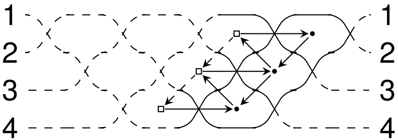
Big positroid cell in $Gr(2,5)$, up to a torus:

- $X_0(\sigma_1\sigma_1\sigma_1 \cdot \Delta, w_0)$ in $\mathrm{Br}_2$;
- $X_0(\sigma_1\sigma_2\sigma_1\sigma_2 \cdot \Delta, w_0)$ in $\mathrm{Br}_3$.

For each of these braids, the closure is the trefoil knot. The augmentation varieties depend only on the link, so they are isomorphic (up to the choice of marked points/torus actions). These varieties are both of cluster type $A_2$. The mutation graph is the pentagon, so we recover all clusters via weaves.




(a) $R_6(1, w_4)\Delta_6$ for the shuffle braid $R_6(1, w_4) = \beta(w_4) = (\sigma_4\sigma_3\sigma_2\sigma_1)(\sigma_5\sigma_4\sigma_3\sigma_2) \in \mathrm{Br}_6^+$. Here $w_4$ is the maximal 4-Grassmannian permutation in $S_6$.




(b) $J_4(f)$ for the (4, 2) torus braid $(\sigma_3\sigma_2\sigma_1)^2 \in \mathrm{Br}_4^+$.

# Cluster structure: general case

## Conjecture

- *The coordinate ring of any braid variety $X(\eta)$ admits a structure of a cluster algebra.*
- *The exchange type of the mutable part of its defining quiver is preserved under Reidemeister II moves, Reidemeister III moves and $\Delta$-conjugations of the braid word $\eta$. In addition, each such move gives rise to a quasi-cluster transformation.*
- *A positive stabilization adds one frozen vertex to the defining quiver, and a positive destabilization specializes one frozen variable to 1.*
- *The 2-forms considered by Mellit are Gekhtman-Shapiro-Vainstein forms for such cluster structures.*

Partially known for GSW varieties $X(\beta \cdot \Delta, w_0)$, open Richardson varieties, open positroid varieties. Deodhar stratifications of open Richardson varieties correspond to certain weaves.

Brick manifolds, spherical subword complexes, Soergel calculus are well-defined beyond the type A.

## Conjecture

*The coordinate ring of any open brick variety* $\mathrm{brick}^{\circ}(\eta)$ *in any type admits a structure of a cluster algebra. A version of the weave calculus can be developed for all types. Demazure weaves give cluster charts.*

Partially proved for analogues of Gao-Shen-Weng varieties (half-decorated double Bott-Samelson cells) by [Shen-Weng], for open Richardson varieties in types ADE [Leclerc, Ménard, Keller-Cao; Ingermanson].

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