Exceptional collections on Grassmannians

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based on joint work with Alexander Kuznetsov

Exceptional collections

X – smooth projective variety over $\mathbb{C}$

$D^b(X)$ – bounded derived category of coherent sheaves on X

1. An object E of $D^b(X)$ is called **exceptional** iff

$$\mathrm{Hom}(E, E) = \mathbb{C} id_E \quad \text{and} \quad \mathrm{Ext}^i(E, E) = 0 \quad \forall i \neq 0.$$

2. A sequence of exceptional objects $E_1, \ldots, E_n$ is called an **exceptional collection** iff

$$\mathrm{Ext}^k(E_i, E_j) = 0 \quad \text{for } i > j \quad \forall k.$$

3. An exceptional collection $E_1, \ldots, E_n$ is said to be **full** iff it generates $D^b(X)$ in some sense. In this case we write

$$D^b(X) = \langle E_1, \ldots, E_n \rangle.$$

More precisely, the smallest full triangulated subcategory containing all $E_1, \ldots, E_n$ should be equivalent to $D^b(X)$.

Fullness is a very important, but somewhat technical aspect of this story and we'll mostly ignore it today.

Examples of exceptional collections

1. Projective spaces $\mathbb{P}^n$ (Beilinson, ≈ 1978)

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$$

2. Grassmannians $\mathrm{G}(k, n)$ and quadrics Q^n (Kapranov, ≈ 1983)

   For $\mathrm{G}(2, 4)$, which is both a Grassmannian and a quadric, Kapranov's collection becomes

$$D^b(\mathrm{G}(2,4)) = \langle \mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*, \mathcal{O}(1), \mathcal{U}^*(1), \mathcal{O}(2) \rangle$$

3. More examples later!

Remark. In these examples checking the exceptionality of the collection can be done relatively easily. For $\mathbb{P}^n$ this is just the standard computation of cohomology of line bundles on $\mathbb{P}^n$. For $\mathrm{G}(k, n)$ one can apply Borel-Weil-Bott theorem. As is usual in this business, the difficult part is to prove fullness!

Simple consequences of having a FEC

Assume that $D^b(X)$ has a full exceptional collection

$$D^b(X) = \langle E_1, \ldots, E_n \rangle.$$

Then we have:

1. The Hodge numbers $h^{p,q}(X) = 0$ for $p \neq q$.
2. $K_0(X)$ is a free abelian group of rank n and classes $[E_1], \ldots, [E_n]$ form a basis.
3. The number of exceptional objects in any full exceptional collection in $D^b(X)$ is the same and is equal to

$$n = \operatorname{rk} K_0(X) = \dim_{\mathbb{C}} H^*(X, \mathbb{C}).$$

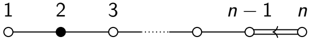
Exceptional collections on G/P: general conjecture

G is a simple simply connected algebraic group $\leftrightsquigarrow$ Dynkin diagram
$P \subset G$ is a maximal parabolic subgroup $\leftrightsquigarrow$ choice of a vertex

Example:

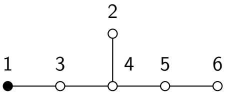
- Symplectic isotropic Grassmannian $\mathsf{IG}(2, 2n)$

$$\mathsf{IG}(2, 2n) = \mathrm{C}_n/\mathrm{P}_2 \quad \leftrightsquigarrow$$



- Cayley plane $\mathrm{E}_6/\mathrm{P}_1$

$$\mathrm{E}_6/\mathrm{P}_1 \quad \leftrightsquigarrow$$



Folklore Conjecture: For any rational homogeneous space G/P the derived category $D^b(\mathrm{G}/\mathrm{P})$ has a full exceptional collection.

Exceptional collections on G/P: methods

The main source of exceptional objects on $\mathrm{G/P}$:

G-equivariant vector bundles

There is a monoidal equivalence of categories

$$\begin{array}{ccc} \mathsf{VB}^{\mathrm{G}}(\mathrm{G/P}) & \to & \mathsf{Rep}\,\mathrm{P} \\ F & \mapsto & F_{[\mathrm{P}]} \end{array}$$

Particularly nice are **irreducible** G-equivariant vector bundles, i.e. those bundles that correspond to irreducible representations of the Levi subgroup $\mathrm{L} \subset \mathrm{P}$.

For irreducible G-equivariant vector bundles there are very efficient ways to check exceptionality (Borel-Weil-Bott theorem).

Unfortunately irreducible bundles do not suffice! And one has to work with arbitrary representations of P, which is much more complicated.

Exceptional collections on G/P: results I

The are **many known results** on $D^b(\mathrm{G}/\mathrm{P})$ available in the literature, but the picture is still **very far from being complete**.

Classical Dynkin types:

- Type A: $\mathrm{A}_n/\mathrm{P}_k = \mathsf{G}(k, n+1)$ [Kapranov, ≈ 1983]
- Type B:
  - $\mathrm{B}_n/\mathrm{P}_1 = \mathbb{P}^{2n-1}$ [Beilinson, ≈ 1978]
  - $\mathrm{B}_n/\mathrm{P}_2 = \mathsf{OG}(2, 2n+1)$ [Kuznetsov, 2005]
- Type C:
  - $\mathrm{C}_n/\mathrm{P}_1 = Q_{2n-1}$ [Kapranov, ≈ 1983]
  - $\mathrm{C}_n/\mathrm{P}_2 = \mathsf{IG}(2, 2n)$ [Kuznetsov, 2005]
  - $\mathrm{C}_n/\mathrm{P}_n = \mathsf{IG}(n, 2n)$ [Fonarev, 2019]
- Types D:
  - $\mathrm{D}_n/\mathrm{P}_1 = Q_{2n-2}$ [Kapranov, ≈ 1983]
  - $\mathrm{D}_n/\mathrm{P}_2 = \mathsf{OG}(2, 2n)$ [Kuznetsov–S., 2020]

Exceptional collections on G/P: results II

Classical Dynkin types (cont.):

Remark 1. Before I have only listed series of examples, but there are also some isolated cases that played an important role in the development of the subject:

- $\mathrm{IG}(3,6)$ [Samokhin, 2001]
- $\mathrm{IG}(2,6)$ [Samokhin, 2006]
- $\mathrm{IG}(4,8)$ and $\mathrm{IG}(5,10)$ [Samokhin–Polishchuk, 2009]
- $\mathrm{IG}(3,8)$ [Guseva, 2018]
- $\mathrm{IG}(3,10)$ [Novikov, 2020]
- ... (apologies!)

Remark 2. Major progress in this field is [Kuznetsov–Polishchuk, 2011], where they propose candidates for full exceptional collections in all classical types. Fullness of these collections is unknown in general.

Exceptional collections on G/P: results III

Exceptional Dynkin types:

- Types $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$: $\mathrm{E}_6/\mathrm{P}_1$ [Faenzi–Manivel, 2012]
- Type F_4:
  - $\mathrm{F}_4/\mathrm{P}_1$ [S., 2021]
  - $\mathrm{F}_4/\mathrm{P}_4$ [Belmans–Kuznetsov–S., 2020]
- Type G_2:
  - $\mathrm{G}_2/\mathrm{P}_1$ [Kuznetsov and Razin, 2006 and 19??]
  - $\mathrm{G}_2/\mathrm{P}_2 = Q_5$ [Kapranov, $\approx$ 1983]

Interesting directions. One can ask the same questions for G/P in positive charachteristic, or over non-closed fields, or even over Spec($\mathbb{Z}$). There are results in all these directions, but I won't be able to give a survey on them here.

Dubrovin's conjecture

X – smooth projective Fano variety over $\mathbb{C}$.

Conjecture (Dubrovin, ICM 1998).

1. $D^b(X)$ has a full exceptional collection if and only if the big quantum cohomology $\mathrm{BQH}(X)$ is generically semisimple.
2. Further conjectures relating the Gram matrix of the exceptional collection to the Stokes matrix of some differential equation given by $\mathrm{BQH}(X)$ ...

Remark. This conjecture can be motivated/explained by sufficiently optimistic formulations of the HMS.

Quantum cohomology I

X – smooth projective Fano variety over $\mathbb{C}$.

Additional assumptions: $\operatorname{Pic} X = \mathbb{Z}$ and $H^{odd}(X, \mathbb{C}) = 0$.

Then, $H^*(X, \mathbb{C})$ is a finite dimensional commutative algebra.

Genus zero Gromov-Witten invariants $\rightsquigarrow$ deformation of the classical cup-product $\cup$ $\rightsquigarrow$ **quantum product** $\star$

Definition. Fix a graded basis $\Delta_0, \ldots, \Delta_s$ in $H^*(X, \mathbb{C})$ and dual linear coordinates $t_0, \ldots, t_s$. It is customary to choose $\Delta_0 = 1$.

For cohomology classes we use the Chow grading, i.e. we divide the topological degree by two.

For variables t_i we set $\deg(t_i) = 1 - \deg(\Delta_i)$.

Let q be a formal variable of degree $\deg(q) = \operatorname{index}(X)$.

Quantum cohomology II: definition continued

The genus zero **Gromov–Witten potential** of X is a formal power series $F \in \mathbb{C}[[t_0, \ldots, t_s]]$ defined by the formula

$$F(t_0, \ldots, t_s) = \sum_{(i_0, \ldots, i_s)} \langle \Delta_0^{\otimes i_0}, \ldots, \Delta_s^{\otimes i_s} \rangle \frac{t_0^{i_0} \ldots t_s^{i_s}}{i_0! \ldots i_s!},$$

where

$$\langle \Delta_0^{\otimes i_0}, \ldots, \Delta_s^{\otimes i_s} \rangle = \sum_{d=0}^{\infty} \langle \Delta_0^{\otimes i_0}, \ldots, \Delta_s^{\otimes i_s} \rangle_d q^d,$$

and $\langle \Delta_0^{\otimes i_0}, \ldots, \Delta_s^{\otimes i_s} \rangle_d$ are rational numbers called Gromov–Witten invariants of X of degree d.

Since X is assumed to be Fano, we can put $q = 1$ in the above formulas. But I will write it to keep track of the grading.

With respect to the grading defined above F is homogeneous of degree $3 - \dim X$.

Quantum cohomology III: definition continued

The **big quantum cohomology ring** $\mathrm{BQH}(X)$ is

$$\mathrm{BQH}(X) := H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[t_0, \ldots, t_s]]$$

with a ring structure defined by

$$\Delta_a \star \Delta_b = \sum_c \frac{\partial^3 F}{\partial t_a \partial t_b \partial t_c} \Delta^c,$$

where $\Delta^0, \ldots, \Delta^s$ is the basis dual to $\Delta_0, \ldots, \Delta_s$ with respect to the Poincaré pairing.

The **small quantum cohomology** ring $\mathrm{QH}(X)$ is the quotient of $\mathrm{BQH}(X)$ with respect to the ideal $(t_0, \ldots, t_s)$. Equivalently, the small quantum cohomology $\mathrm{QH}(X) = H^*(X, \mathbb{C})$ as a vector spaces and the product is defined by

$$\Delta_a \circ \Delta_b = \sum_c \langle \Delta_a, \Delta_b, \Delta_c \rangle \Delta^c.$$

Again, we are setting $q = 1$ everywhere.

Example

Projective spaces $\mathbb{P}^n$. The presentation for the classical cohomology of $\mathbb{P}^n$ is

$$H^*(\mathbb{P}^n, \mathbb{C}) = \mathbb{C}[h]/h^{n+1}.$$

Since the relation is in degree $n+1$ and $\deg(q) = n+1$, up to a scalar there is a unique possibility to obtain a presentation for the small quantum cohomology

$$\mathrm{QH}(\mathbb{P}^n) = \mathbb{C}[h]/h^{n+1} - q,$$

where we as usual set $q = 1$.

Semisimplicity. The classical cohomology algebra $H^*(\mathbb{P}^n, \mathbb{C})$ is nilpotent, but the small quantum cohomology $\mathrm{QH}(\mathbb{P}^n)$ is semisimple, i.e. it decomposes into the direct product

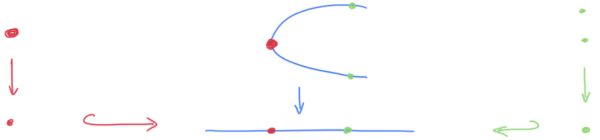
$$\mathrm{QH}(\mathbb{P}^n) = \mathbb{C}[h]/h^{n+1} - 1 = \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C},$$

or, in other words, $\mathrm{QH}(\mathbb{P}^n)$ has no nilpotent elements.

Generic semisimplicity

We think of $\mathrm{BQH}(X)$ as a formal family of finite dimensional commutative algebras (or 0-dimensional schemes), whose special fiber is the small quantum cohomology:

$$
\begin{array}{ccccc}
\mathrm{Spec}(\mathrm{QH}(X)) & \longrightarrow & \mathrm{Spec}(\mathrm{BQH}(X)) & \longleftarrow & \mathrm{Spec}(\mathrm{BQH}(X)_\eta) \\
\downarrow & & \downarrow \pi & & \downarrow \pi_\eta \\
\mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[[t_0, \dots, t_s]]) & \longleftarrow & \eta
\end{array}
$$



Definition. We say that $\mathrm{BQH}(X)$ is generically semisimple, if the generic fiber $\mathrm{BQH}(X)_\eta$ is a semisimple algebra.

Remark. If $\mathrm{QH}(X)$ is semisimple, then $\mathrm{BQH}(X)$ is generically semisimple, but the converse is false.

Back to Dubrovin's conjecture

Recall the original statement:

$$D^b(X) \text{ has a f.e.c.} \iff \mathrm{BQH}(X) \text{ is generically semisimple}$$

Where do we want to go?

1. We have little understanding about $\mathrm{BQH}(X)$, as it is usually very hard to compute in practice.
2. We understand $\mathrm{QH}(X)$ much better. There are lots of examples in the literature.
3. **Question:** Can we use the structure of $\mathrm{QH}(X)$ to make some finer conjectures about $D^b(X)$?
4. **Answer:** Lefschetz collections seem to work very well for this purpose!

Lefschetz exceptional collections

This is a special type of exceptional collections introduced by Alexander Kuznetsov (around 2006) in his work on homological projective duality.

Let X be a smooth projective variety endowed with an (ample) line bundle $\mathcal{O}(1)$.

- A **Lefschetz collection** with respect to $\mathcal{O}(1)$ is an exceptional collection, which has a block structure

  $$\underbrace{E_1, E_2, \ldots, E_{\sigma_0}}; \underbrace{E_1(1), E_2(1), \ldots, E_{\sigma_1}(1)}; \ldots; \underbrace{E_1(m), E_2(m), \ldots, E_{\sigma_m}(m)}$$

  where $\sigma = (\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m \geq 0)$ is a non-increasing sequence of non-negative integers called the **support partition** of the collection.

- If $\sigma_0 = \sigma_1 = \cdots = \sigma_m$, then the corresponding Lefschetz collection is called **rectangular**.

Examples of Lefschetz collections

1. Beilinson's collection

$$D^b(\mathbb{P}^n) = \langle \mathcal{O};\ \mathcal{O}(1);\ \dots;\ \mathcal{O}(n) \rangle$$

   is Lefschetz with the starting block $(\mathcal{O})$ and support partition $1, \dots, 1$.

2. Kapranov's collection

$$D^b(\mathsf{G}(2,4)) = \langle \mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*;\ \mathcal{O}(1), \mathcal{U}^*(1);\ \mathcal{O}(2) \rangle$$

   is Lefschetz with the starting block $(\mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*)$ and support partition $3, 2, 1$.

3. For $\mathsf{G}(2,4)$ one can make the starting block smaller by taking $(\mathcal{O}, \mathcal{U}^*)$ with the support partition $2, 2, 1, 1$

$$D^b(\mathsf{G}(2,4)) = \langle \mathcal{O}, \mathcal{U}^*;\ \mathcal{O}(1), \mathcal{U}^*(1);\ \mathcal{O}(2);\ \mathcal{O}(3) \rangle$$

   Lefschetz collections with the smallest possible starting block are called **minimal**.

Residual category of a Lefschetz collection

Let X and $\mathcal{O}(1)$ be as before, and consider a Lefschetz exceptional collection

$$E_1, E_2, \dots, E_{\sigma_0};\ E_1(1), E_2(1), \dots, E_{\sigma_1}(1);\ \dots;\ E_1(m), E_2(m), \dots, E_{\sigma_m}(m)$$

We can take its rectangular part

$$E_1, E_2, \dots, E_{\sigma_m}; \dots; E_1(m), E_2(m), \dots, E_{\sigma_m}(m).$$

and define the **residual category** of this Lefschetz collection to be the subcategory of $D^b(X)$ left orthogonal to the rectangular part:

$$\mathcal{R} = \left\langle E_1, E_2, \dots, E_{\sigma_m}; \dots; E_1(m), E_2(m), \dots, E_{\sigma_m}(m) \right\rangle^{\perp}.$$

Thus, we have a semiorthogonal decomposition

$$D^b(X) = \left\langle \mathcal{R}\ ;\ E_1, E_2, \dots, E_{\sigma_m}; \dots; E_1(m), E_2(m), \dots, E_{\sigma_m}(m) \right\rangle.$$

The residual category is zero if and only if $(E_\bullet, \sigma)$ is full and rectangular.

Residual category for G(2, 4)

Consdier the minimal Lefschetz collection on G(2, 4)

$$D^b(\mathsf{G}(2,4)) = \langle \mathcal{O}, \mathcal{U}^*; \mathcal{O}(1), \mathcal{U}^*(1); \mathcal{O}(2); \mathcal{O}(3) \rangle.$$

Objects not belonging to the rectangular part are highlighted in red. Projecting them into the residual category $\mathcal{R}$ we obtain the exceptional collection

$$D^b(\mathsf{G}(2,4)) = \langle A, B; \mathcal{O}; \mathcal{O}(1); \mathcal{O}(2); \mathcal{O}(3) \rangle \quad \text{and} \quad \mathcal{R} = \langle A, B \rangle.$$

General feature: Projecting the objects not belonging to the rectangular part into $\mathcal{R}$ gives rise to an exceptional collection in $\mathcal{R}$. Technical name for this is *mutation of exceptional collections*.

Interesting phenomenon for G(2, 4): Since A, B form an exceptional pair, we necessarily have $\mathrm{Ext}^\bullet(B, A) = 0$. **Surprisingly** we also have

$$\mathrm{Ext}^\bullet(A, B) = 0.$$

Thus, A and B are completely orthogonal!

Residual category for $\mathrm{IG}(2,6)$

The simplest interesting example of X for which $\mathrm{QH}(X)$ is not semisimple is the symplectic isotropic Grassmannians $\mathrm{IG}(2,6)$.

A minimal Lefschetz collection for $\mathrm{IG}(2,6)$ has been constructed by Alexander Kuznetsov ($\approx$ 2005).

$$D^b(\mathrm{IG}(2,6)) = \langle \mathcal{O}, \mathcal{U}^*, {\color{red}S^2\mathcal{U}^*}, \mathcal{O}(1), \mathcal{U}^*(1), {\color{red}S^2\mathcal{U}^*(1)}, \\ \mathcal{O}(2), \mathcal{U}^*(2), \mathcal{O}(3), \mathcal{U}^*(3), \mathcal{O}(4), \mathcal{U}^*(4)\rangle.$$

Mutating the red objects into the residual category we get

$$\mathcal{R} = \langle {\color{red}A}, {\color{red}B} \rangle \quad \text{and} \quad \mathrm{Ext}^i(A,B) = \begin{cases} \mathbb{C} & \text{for} \quad i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that we have $\mathcal{R} \simeq D^b(A_2)$.

This matches perfectly with the structure of $\mathrm{QH}(\mathrm{IG}(2,6))$!

Small quantum cohomology of $\mathrm{IG}(2,6)$

Theorem (Buch–Kresch–Tamvakis, 2009)**.** The small quantum cohomology of $\mathrm{IG}(2,6)$ has the following presentation

$$\mathrm{QH}(\mathrm{IG}(2,6)) = \mathbb{C}[\sigma_1,\sigma_2,\sigma_3,\sigma_4]/(\Delta_3,\Delta_4,\Sigma_4,\Sigma_6),$$

where

$$\Delta_3 = \begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & \sigma_1 & \sigma_2 \\ 0 & 1 & \sigma_1 \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ 1 & \sigma_1 & \sigma_2 & \sigma_3 \\ 0 & 1 & \sigma_1 & \sigma_2 \\ 0 & 0 & 1 & \sigma_1 \end{vmatrix},$$

$$\Sigma_4 = \sigma_2^2 - 2\sigma_3\sigma_1 + 2\sigma_4, \quad \Sigma_6 = \sigma_3^2 - 2\sigma_4\sigma_2 + q\sigma_1.$$

Here $\sigma_i = c_i(Q) \in H^*(\mathrm{IG}(2,6))$ are the so called special Schubert classes, and Q is the tautological quotient bundle.

The value of q does not play much role here as long as $q \neq 0$. So we fix $q = 1$, as usual.

Fat points of the small quantum cohomology of $\mathrm{IG}(2,6)$

- $\mathrm{QH}(\mathrm{IG}(2,6))$ is not semisimple (i.e. has nilpotent elements)
- How can we see that?
  - View $\mathrm{QH}(\mathrm{IG}(2,6)) = \mathbb{C}[\sigma_1, \sigma_2, \sigma_3, \sigma_4]/(\Delta_3, \Delta_4, \Sigma_4, \Sigma_6)$ as the algebra of functions on a finite set of (fat) points $Z \subset \mathbb{C}^4$ with coordinates $\sigma_1, \dots, \sigma_4$.
  - The origin $P = (0,0,0,0) \in \mathbb{C}^4$ is a solution of the system defining $\mathrm{QH}(\mathrm{IG}(2,6))$, i.e. $P \in Z$.
  - One computes easily the tangent space to Z at P

$$T_P Z = \{\sigma_3 = \sigma_4 = \sigma_1 = 0\} \subset \mathbb{C}^4.$$

  - Thus, we see that $\dim T_P Z = 1$, whereas $\dim Z = 0$. Therefore, P is not a smooth point of Z, i.e. there are nilpotents!
- One can show that $\mathrm{QH}(\mathrm{IG}(2,6))$ decomposes into the product

$$\mathrm{QH}(\mathrm{IG}(2,6)) \simeq \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{\text{10 copies}} \times \mathbb{C}[\varepsilon]/\varepsilon^2,$$

Some notation

Let us define the **quantum spectrum** of X as

$$\mathrm{QS}_X := \mathrm{Spec}(\mathrm{QH}(X)).$$

This finite scheme has μ_m-action, where m is the **index** of X.

The **anticanonical class** $-\mathrm{K}_X \in H^*(X)$ defines a map

$$\kappa\colon\ \mathrm{QS}_X \to \mathbb{A}^1,$$

which is μ_m-equivariant with respect to the standard action on $\mathbb{A}^1$.

Finally we define

$$\mathrm{QS}_X^\times := \kappa^{-1}(\mathbb{A}^1 \setminus \{0\}) \quad \text{and} \quad \mathrm{QS}_X^\circ := \mathrm{QS}_X \setminus \mathrm{QS}_X^\times.$$

Remark. In terms of LG models, the scheme QS_X corresponds to the critical locus of the LG potential and the map κ corresponds to the restriction of the LG potential to its critical locus.

Conjecture

Conjecture (Kuznetsov-S.). Let X be a Fano variety of index m and assume that the big quantum cohomology $\mathrm{BQH}(X)$ is generically semisimple.

1. There is an exceptional collection $E_1, \ldots, E_k$ in $D^b(X)$, where k is the length of $\mathrm{QS}_X^{\times}$ divided by m, which extends to a rectangular Lefschetz collection in $D^b(X)$.
2. The residual category $\mathcal{R}$ of this collection has a completely orthogonal decomposition

$$\mathcal{R} = \bigoplus_{\xi \in \mathrm{QS}_X^{\circ}} \mathcal{R}_{\xi}$$

   with components indexed by closed points $\xi \in \mathrm{QS}_X^{\circ}$; moreover, the component $\mathcal{R}_{\xi}$ of $\mathcal{R}$ is generated by an exceptional collection of length equal to the length of the localization $(\mathrm{QS}_X^{\circ})_{\xi}$ at ξ.

Remarks

Semisimple $\mathrm{QH}(X)$**:** in this case QS_X° is reduced and the residual category $\mathcal{R}$ should be generated by a completely orthogonal exceptional collection, whose objects are indexed by the closed points $\xi \in \mathrm{QS}_X^\circ$. This happens for $\mathrm{G}(k, n)$.

Structure of components $\mathcal{R}_\xi$**:** In general one expects $\mathcal{R}_\xi$ to be equivalent to the Fukaya-Seidel category of the corresponding critical point of the central fiber of an appropriate LG model.

Coadjoint varieties: Special class of homogeneous spaces G/P. There is exactly one varietiy for each Dynkin type. For types *BCD* we have

$$\text{type } \mathrm{B}_n : \ Q_{2n-1}, \quad \text{type } \mathrm{C}_n : \ \mathrm{IG}(2, 2n), \quad \text{type } \mathrm{D}_n : \ \mathrm{OG}(2, 2n).$$

For coadjoint varieties QS_X° has one point and we expect

$$\mathcal{R} = \mathcal{R}_\xi = D^b(\mathrm{Rep}\, Q),$$

where the quiver Q obtained by taking the **subdiagram of short roots** of the Dynkin diagram of G (for C_n this gives A_{n-1}).

Known examples

1. Cases with semisimple $\mathrm{QH}(X)$:
   1.1 $\mathrm{G}(k, n)$ — either if k, n are coprime or $k = p$ and $n = pm$. In general we expect this to be true for the minimal Lefschetz collections constructed by Fonarev for any $\mathrm{G}(k, n)$.
   1.2 Quadrics — follows from Kapranov's work.
   1.3 $\mathrm{OG}(2, 2n+1)$ — follows from Kuznetsov's work.
   1.4 Some sporadic examples:
      - $\mathrm{G}_2/\mathrm{P}_2$ by Kuznetsov
      - $\mathrm{IG}(3, 8)$ by Guseva
      - $\mathrm{IG}(3, 10)$ by Novikov
      - Caley plane $\mathrm{E}_6/\mathrm{P}_1$ is a combination of Faenzi–Manivel and Belmans–Kuznetsov–S.
      - $\mathrm{IG}(4, 8)$ and $\mathrm{IG}(5, 10)$ should follow from Polishchuk–Samokhin and Fonarev, but it is not written down.
      - $\mathrm{F}_4/\mathrm{P}_1$ by S.
2. Cases with non-semisimple $\mathrm{QH}(X)$:
   2.1 $\mathrm{IG}(2, 2n)$ by Kuznetsov
   2.2 $\mathrm{OG}(2, 2n)$ by Kuznetsov–S.
   2.3 $\mathrm{F}_4/\mathrm{P}_4$ by Belmans–Kuznetsov–S.

All these cases are examples of coadjoint varieties.

Thank you!