Noncommutative Poisson structures, Hochschild type complexes and Groebner-Shirshov bases theory

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Murray Gerstenhaber, Alexander Voronov 'Higher operations on Hochschild complex':

A recent explosion of algebraic structures derived in quantum field theory and in theory of vertex operator algebras has led to renaissance of operads and algebras with several operations. The physicist' vision of the Universe revealed evidence of a number of new mathematical structures now being intensively studied by mathematicians.'

[Getzler, Jones, Ginzburg, Kapranov, Kontsevich, Kimura, Stasheff, Gerstenhaber, Voronov]

I. Pre-Calabi-Yau algebras and double Poisson structures

We establish a clear connection between these two structures in a constructive way.

We show that double Poisson structures are in 1-1 correspondence with a particular part of the pre-Calabi-Yau structures and describe this correspondence explicitly.

This justifies the role of pre-Calabi-Yau algebras as a noncommutative Poisson structures, via result of [Van den Bergh, 2008] on double Poisson algebras.

Double Poisson algebras

The double Poisson structure on associative algebra A is a linear map

$$\{\{,\}\} : A \otimes A \to A \otimes A,$$

which satisfy the following axioms.

Anti-symmetry:

$$\{\{a, b\}\} = -\{\{b, a\}\}^{op}$$

Here $\{\{b, a\}\}^{op}$ means the twist in the tensor product, i.e. if $\{\{b, a\}\} = \sum_i b_i \otimes c_i$, then $\{\{b, a\}\}^{op} = \sum_i c_i \otimes b_i$.

Double Leibniz:

$$\{\{a, bc\}\} = b\{\{a, c\}\} + \{\{a, b\}\}c$$

and double Jacobi identity:

$$\{\{a,\{\{b,c\}\}\}\}_L+\tau_{(123)}\{\{b\{\{c,a\}\}\}\}_L+\tau_{(132)}\{\{c\{\{a,b\}\}\}\}_L = 0$$

Here for $\alpha \in A \otimes A \otimes A,$ and $\sigma \in S_3$

$$\tau_\sigma(\alpha) = \alpha_{\sigma^{-1}} \otimes \alpha_{\sigma^{-1}} \otimes \alpha_{\sigma^{-1}} \; .$$

The $\{\{\ \}\}_L$ is defined as

$$\{\{b, a_1 \otimes ... \otimes a_n\}\}_L = \{\{b, a_1\}\} \otimes a \otimes ... \otimes a_n$$

Pre-Calabi-Yau structures

It is a version of the notion of the A_∞-structure or *strong homotopy associative algebra* introduced by [J.Stasheff, 1963].

Namely, one can impose an A_∞-structure on $A \oplus A^*$, which is cyclic invariant w.r.t. the natural inner form on $A \oplus A^*$.

This idea appeared in string theory to describe the TQFT in case of open strings. The notion was introduced in [T.Tradler, M.Zeinalian, 2006; P.Seidel, 2007; M.Kontsevich, Y.Vlassopoulos, 2006].

This structure is present in many examples from algebraic geometry, symplectic geometry, physics, etc.

Definition of an A_∞-algebra (strong homotopy associative algebra).

The main bit of data in the definition of A_∞-algebra is a graded vector space $A = \bigoplus_{k\in\mathbb{Z}} A_k$,

with a collection of $n-$ary operations (multilinear maps)

$$m_n : A \times A \times ... \times A \to A, \quad n = 1, 2, ...,$$

satisfying certain conditions.

Conventions.

We will deal with shifts of gradings, so it is important for further definitions which *convention* on the relation between *grading* and *degrees of operations* we choose.

Conventions differ by a shift in numeration of graded components.

The main two conventions are as follows.

Conv.0 (naive)

'Multiplication' - binary operation m_2 has degree 0:

$deg(m_2(a_1, a_2)) = deg(a_1) + deg(a_2)$.

Then $deg(m_n) = 2 - n$.

Conv.1 (respecting Koszul rule)

Each operation m_n has degree 1:

$$deg(m_n(a_1, ..., a_n)) =$$

$$deg(a_1) + ... + deg(a_n) + 1, \forall n.$$

Note that if the degree of element x in *Conv.0* is $deg\, x = |x|$, then after the shift of grading on A: $A^{sh} = A[1]$, $deg^{sh} x = |x|' = |x| - 1$ we fall into *Conv.1*

In our notations $|x|'$ stands for grading satisfying *Conv.1* and $|x|$ for the one satisfying *Conv.0*.

In formulae in precise definitions (signs there depend on the convention) we mainly use *Conv.1*.

Hochschild cochains

Let A be a $\mathbb{Z}$ graded vector space $A = \bigoplus\limits_{n \in \mathbb{Z}} A_n$.

Let $C^l(A, A)$ be Hochschild cochains

$$C^l(A, A) = \underline{Hom}(A[1]^{\otimes l}, A[1]), l \geqslant 0,$$

$$C^\bullet(A, A) = \prod_{k \geqslant 1} C^l(A, A).$$

On $C^\bullet(A,A)[1]$ there is a natural structure of graded pre-Lie algebra, defined via composition:

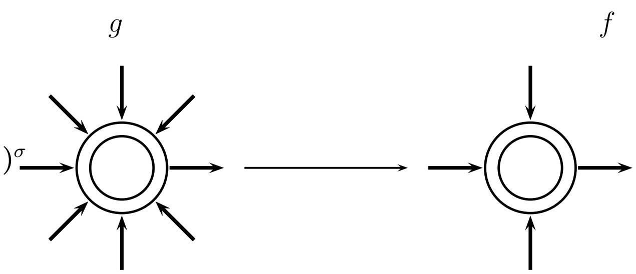
$$\circ : C^{l_1}(A,A) \otimes C^{l_2}(A,A) \to C^{l_1+l_2-1}(A,A):$$

$$f \circ g(a_1 \otimes \ldots \otimes a_{l_1+l_2-1}) =$$

$$\sum (-1)^\sigma f(a_1 \otimes \ldots \otimes a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+l_2+1}) \otimes \ldots \otimes a_{l_1+l_2-1})$$

Graphically:

$$f \circ g = \sum_{\text{inputs of } f} (-1)^\sigma$$



The operation $\circ$ defined in this way does satisfy the graded right-symmetric identity:

$$(f, g, h) = (-1)^{\sigma}(f, h, g)$$

where

$$(f, g, h) = (f \circ g) \circ h - f \circ (g \circ h).$$

Analogously to the non-graded case, the graded commutator on a graded pre-Lie algebra defines a graded Lie algebra structure.

Thus the Gerstenhaber bracket $[-,-]_G$:

$$[f,g]_G = f \circ g - (-1)^\sigma g \circ f$$

makes $C^\bullet(A)[1]$ into a graded Lie algebra.

Equipped with the derivation $d = \mathrm{ad}\ m_2$, $(C^\bullet(A), \mathrm{ad}\ m_2)$ becomes a DGLA.

With respect to the Gerstenhaber bracket $[-,-]_G$ we have the Maurer-Cartan equation in this DGLA:

$$[m^{(1)}, m^{(1)}]_G = \sum_{p+q=k+1} \sum_{i=1}^{p-1} (-1)^{\varepsilon}$$

$$m_p(x_1, \ldots, x_{i-1}, m_q(x_i, \ldots, x_{i+q-1}), \ldots, x_k) = 0,$$

where

$$\varepsilon = |x_1|' + \ldots + |x_{i-1}|', \qquad |x_i|' = |x_i| - 1 = \deg x_i - 1$$

Definition.

An element $\gamma = m^{(1)} \in C^{\bullet}(A,A)[1]$ which satisfies the Maurer-Cartan equation

$$[\gamma,\gamma]_G = 0$$

with respect to the Gerstenhaber bracket $[-,-]_G$ is called an A_∞-structure on A.

Equivalently, A_∞-structure can be defined in a more compact way as a coderivation on the coalgebra of the bar complex of A.

Let us consider the bar complex $BC = \bigoplus_{k \geqslant 1} T_k(A[1])$ of algebra A. It has a structure of coalgebra in a natural way: $\Delta : BC \to BC \otimes BC$:

$$\Delta(v_1, ..., v_n) = \sum_{i=1}^{n} (v_1 \otimes ... \otimes v_i) \otimes (v_{i+1} \otimes ... \otimes v_n)$$

The coderivation on BC can be defined extending the given bunch of k-ary maps:

$m_k : T_k(A[1]) \to A[1]$

as follows:

$$\bar{m}_k(x_1, ..., x_n) = \sum_{i=1}^{n-k+1} (-1)^{|x_1|' + ... + |x_{i-1}|'}$$

$$x_1 \otimes ... \otimes m_k(x_i \otimes ... \otimes x_{i+k-1}) \otimes ... \otimes x_n$$

and $\bar{m}_k(x_1, ..., x_n) = 0$ for $k > n$

The fact that for $\bar{d} = \sum_{k=0}^{\infty} \bar{m}_k$, $\bar{d} \circ \bar{d} = 0$ is equivalent to the fact the bunch of maps m_n do satisfy the Maurer-Cartan equation.

Thus this can serve as another definition of an A_∞-algebra.

Simplest example.

An associative algebra with zero derivation $A = (A, m = m_2^{(1)}, d = 0)$ is an example of A_∞-algebra. The component of the Maurer-Cartan equation of arity 3, MC_3 will say that the binary operation of this structure, the multiplication m_2 is associative:

$$(ab)c - a(bc) = dm_3(a,b,c) +$$

$$(-1)^\sigma m_3(da,b,c) + (-1)^\sigma m_3(a,db,c) + (-1)^\sigma m_3(a,b,dc)$$

We can give now definition of pre-Calabi-Yau structure (in $Conv.1$).

Definition.

A d-pre-Calabi-Yau structure on an A_∞-algebra A is

(I). an A_∞-structure on $A \oplus A^*[1-d]$,

(II). cyclic invariant with respect to natural non-degenerate inner form on $A \oplus A^*[1-d]$, meaning:

$$\langle m_n(\alpha_1, ..., \alpha_n), \alpha_{n+1}\rangle = (-1)^{|\alpha_1|'(|\alpha_2|'+...+|\alpha_{n+1}|')}$$
$$\langle m_n(\alpha_2, ...\alpha_{n+1}), \alpha_1)\rangle$$

for all $\alpha_i \in A$ or $A^*[1-d]$.

where the inner form $\langle , \rangle$ on $A \oplus A^*$ is defined naturally as

$$\langle (a,f),(b,g) \rangle = f(b) + (-1)^{|g|'|a|'} g(a), a,b \in A, f,g \in A^*$$

(III) and such that A is an A_∞-subalgebra in

$$A \oplus A^*[1-d].$$

Example. The most simple example of pre-Calabi-Yau structure demonstrates that this structure does exist on any associative algebra.

Namely, the structure of associative algebra on A can be extended to the associative structure on $A \oplus A^*[1-d]$ in such a way, that the natural inner form is (graded)cyclic with respect to multiplication.

This amounts to the following fact: for any A-bimodule M the associative multiplication on $A \oplus M$ is given by $(a+f)(b+g) = ab+ag+fb$.

In this simplest situation both structures on A and on $A \oplus A^*$ are in fact associative algebras.

More examples of pre-Calabi-Yau structures one can find in [Iyudu, IHES preprints 2017; Rubtsov, arxiv:1208.2935; Brav, Dyckerhoff, arxiv:1606.00619]

Maurer-Cartan equation on $A \oplus A^*$

The general Maurer-Cartan equations on $R = A \oplus A^*$ for the operations $m_n : R^n \to R$ have the shape

$$\sum_{p+q=k+1} \sum_{i=1}^{p-1} (-1)^{\varepsilon}$$

$$m_p(x_1, \dots, x_{i-1}, m_q(x_i, \dots, x_{i+q-1}), \dots, x_k),$$

where

$$\varepsilon = |x_1|' + \dots + |x_{i-1}|', \qquad |x_s|' = \mathrm{deg} x_s - 1$$

where $x_i \in A$ or A^*.

What we have from the Maurer-Cartan in arities *four* and *five*, is exactly what is relevant for comparison with the double Leibniz and Jacobi identities for the double bracket.

In arity 4, Maurer-Cartan equation, MC_4 reads:

$$m_3(x_1x_2, x_3, x_4) + (-1)^{|x_1|'} m_3(x_1, x_2x_3, x_4) + \\ (-1)^{|x_1|'+|x_2|'} m_3(x_1, x_2, x_3x_4) + \\ (-1)^{|x_1|'} m_2(x_1, m_3(x_2, x_3, x_4)) + m_2(m_3(x_1, x_2, x_3), x_4) = 0$$

In arity 5, Maurer-Cartan equation, MC_5 reads:

$$m_3(m_3(x_1, x_2, x_3), x_4, x_5)+$$
$$(-1)^{|x_1|'} m_3(x_1, m_3(x_2, x_3, x_4), x_5)+$$
$$(-1)^{|x_1|'+|x_2|'} m_3(x_1, x_2, m_3(x_3, x_4, x_5)) = 0$$

Operations of arity 4 are absent due to our imposed condition $m_4 = 0$ in A_∞-structure on $A \oplus A^*$.

Construct a double bracket from a pre-Calabi-Yau structure

Now we are going to construct from cyclic invariant operations w.r.t. inner form on $R = A \oplus A^*$, $m_n : R^n \to R$ a double bracket

$$\{\{,\}\} : A \otimes A \to A \otimes A,$$

in such a way, that corresponding components of MC (MC_4 and MC_5) ensure the axioms of double Poisson bracket.

Definition

The double bracket we define as:

$$\langle g \otimes f, \{\{b, a\}\}\rangle := \langle m_3(a, f, b), g\rangle,$$

where $a, b \in A$, $f, g \in A^*$ and

$m_3(a, f, b) = c \in A$ corresponds to the component of m_3: $A \times A^* \times A \to A$.

By choosing this definition we set up a one-to-one correspondence between appropriate part of pre-Calabi-Yau structures

(of 'type B': meaning the operations
$m_3 : A \times A^* \times A \to A$, corresponding to the tenzor $A^* \otimes A \otimes A^* \otimes A$;
plus with $m_i = 0, i \geqslant 4$)

and double Poisson brackets.

We will check that double bracket defined in this way satisfies all axioms of the double Poisson bracket.

Theorem. *Let we have A_∞-structure on $(A \oplus A^*, m = \sum_{i=2, i\neq 4}^{\infty} m_i^{(1)})$. Define the bracket by the formula*

$$\langle g \otimes f, \{\{b, a\}\} \rangle := \langle m_3(a, f, b), g \rangle,$$

where $a, b \in A$, $f, g \in A^*$

and $m_3(a, f, b) = c \in A$ corresponds to the component of type B of the solution to the Maurer-Cartan

(i.e. the component m_3: $A \times A^* \times A \to A$).

Then this bracket does satisfy all axioms of the double Poisson algebra.

Moreover, pre-Calabi-Yau structures of type B, with $m_i = 0, i \geqslant 4$ are in one-to-one correspondence with the double Poisson brackets $\{\{\cdot,\cdot\}\} : A \otimes A \to A \otimes A$ for an arbitrary associative algebra A.

The second part of the statement is proved by ensuring no other identities apart from (double) Leibniz and Jacobi appear from the Maurer-Cartan equation.

II. Higher Hochschild complex

First, we should define *higher cyclic Hochschild cochains* and *generalised necklace bracket*.

For $N \geqslant 1$ the space of *N-higher cyclic Hochschild cochains* is defined as

$$C^{(N)}_{cycl}(A) := \prod_{r_1,\ldots,r_N \geqslant 0} Hom_K(\bigotimes_{i=1}^{N} A^{\otimes r_i}, A^{\otimes N}_{cycl})^{\mathbb{Z}_N},$$

Note that for $N = 1$ we have the usual Hochschild complex.

Differential is coming from the dualised bar complex of $A^{\otimes N}$-bimodules:

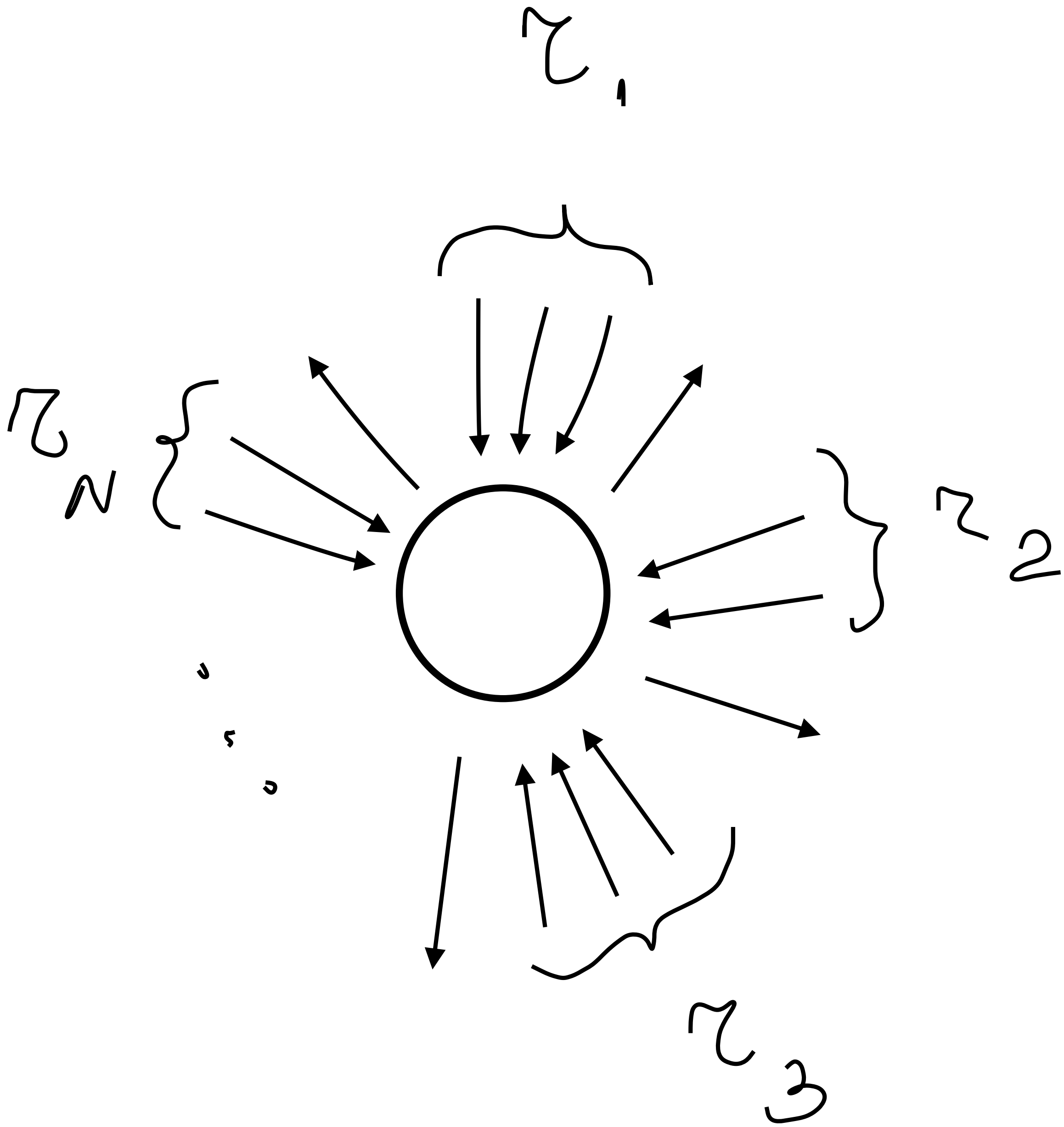
$$Hom_{A^{\otimes N}-mod-A^{\otimes N}}(Bar^{\otimes N, A^{\otimes N}_{cycl}}$$

The $A^{\otimes N}$-bimodule structure on $A^{\otimes N}_{cycl}$ is defined as follows:

for any $x_1 \otimes ... \otimes x_N \in A^{\otimes N}_{cycl}$ and elements $a_1 \otimes ... \otimes a_N, b_1 \otimes ... \otimes b_N \in A^{\otimes N}$,

$$(a_1 \otimes ... \otimes a_N) \bullet (x_1 \otimes ... \otimes x_N) \bullet (b_1 \otimes ... \otimes b_N) =$$

$$a_1 x_1 b_2 \otimes ... \otimes a_N x_N b_1.$$



τ_1
τ_2
τ_3
τ_N
...

The symbol $Hom(A^{\otimes r}, A^{\otimes N})^{\mathbb{Z}_N}$ means that we take only elements of Hom which are 'invariant' with respect to obvious $\mathbb{Z}_N$-action.

Denote by $C^{(\bullet)}_{cycl}(A) = \prod_{N \geqslant 1} C^{(N)}_{cycl}(A)$ the space of all higher cyclic Hochschild cochains.

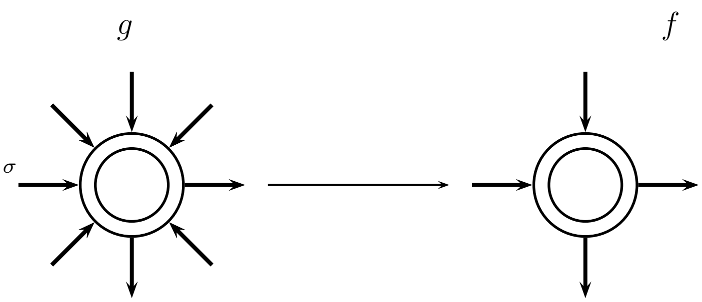
The *generalized necklace bracket* between two elements $f, g \in C^\bullet(A)$ is given by

$$[f, g]_{g.n} = f \circ g - (-1)^\sigma g \circ f,$$

where composition $f \circ g$ consists of inserting all outputs of g to all inputs from f with signs assigned according to the Koszul rule.

Graphically:

$$f \circ g = \sum_{\substack{\text{f-inputs,} \\ \text{g-outputs}}} (-1)^\sigma$$



Def. (Pre-Calabi-Yau structure in terms of HCH complex)

Pre-Calabi-Yau structure is an element

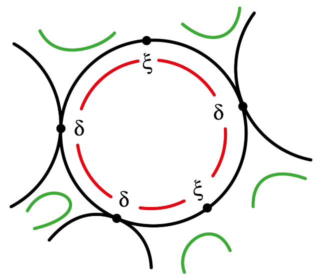
$$m = \sum m^{(n)} \in HCH(A)$$

which is a solution of the Maurer-Cartan equation $[m, m]_{g.n} = 0$.

To deal with higher cyclic Hochschild complex $C^\bullet(A)$ we choose a small subcomplex.

In case of free algebra A we specify a particular embedding of the subcomplex $\zeta^\bullet$ into $C^\bullet(A)$ by choosing a basis of $\xi\partial$-monomials and describing the operation from $C^\bullet(A)$ corresponding to a given $\xi\partial$-monomial.

The operation is the following.

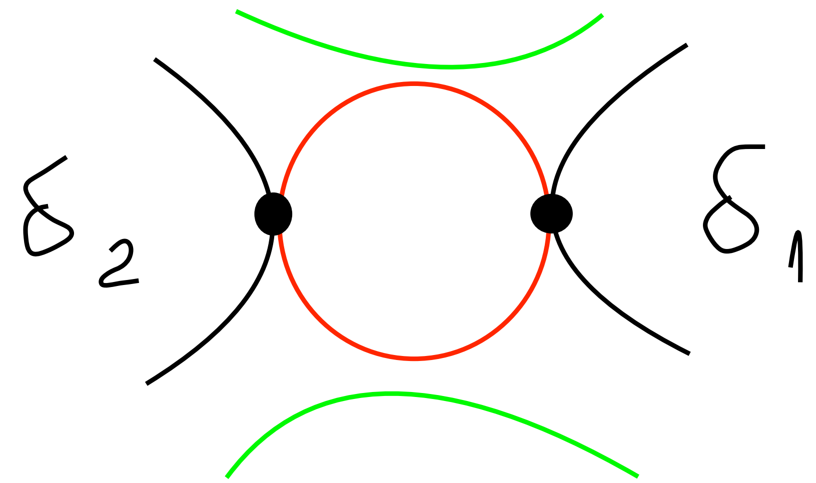


In this particular picture we see the $\xi\partial$-monomial which encodes an operation $\Phi : A^{\otimes 3} \to A^{\otimes 5}$.

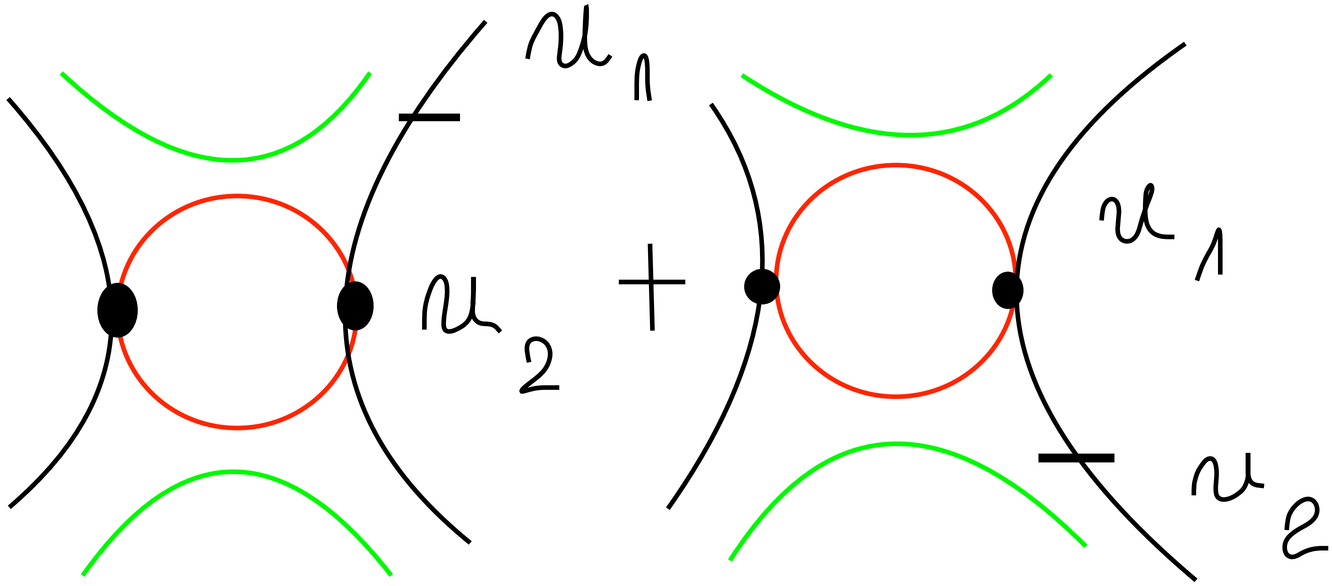
Example.

$$\{\{,\}\} : A^{\otimes 2} \to A^{\otimes 2}$$

is an element from HCH - $Hom(A^{\otimes 2}, A^{\otimes 2}_{cycl})^{\mathbb{Z}_2}$



$$\{\{v, u_1u_2\}\} = u_1\{\{v, u_2\}\} + \{\{v, u_1\}\}u_2$$



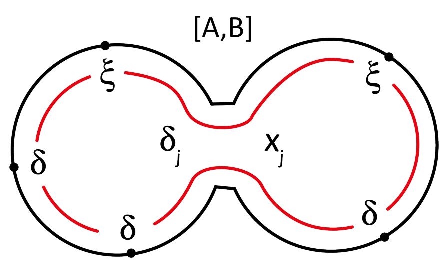
We show how generalised necklace bracket works in terms of $\xi\partial$-monomials.

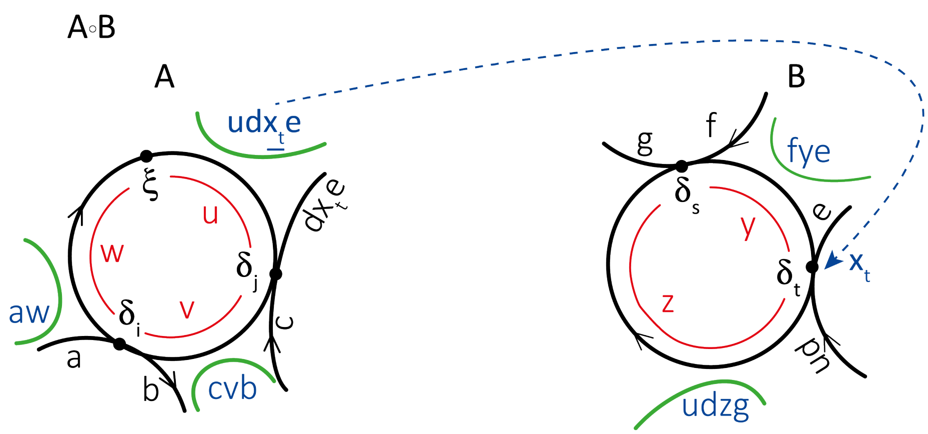
By this we not only prove that small subcomplex $\zeta_A^{(\bullet)}$ is a Lie subalgebra in

$$\mathbf{g} = (C_A^{(\bullet)}(A), [,]_{g.n}),$$

but also give a concrete combinatorial formula for this bracket via $\xi\partial$-monomials.

The bracket $[A, B]$ of two $\xi\partial$-monomials A and B is a linear combination of $\xi\partial$-monomials obtained from the initial ones as follows.





Remark on connection to anti-symmetricc solutions of CYBE Yang-Baxter equation

$$\boxed{\begin{array}{c} R12 = -R21 \\ \\ [R12, R13] - [R23, R12] + [R13, R23] = 0 \end{array}}$$

Pre-Calabi-Yau structures provide solutions of CYBE:

As noticed in [Schedler, arxiv0612493] double Poisson bracket provides a map from square of the vector space to itself, wich is a solution of CYBE due to the Jacobi identity.

III. Homology calculation for higher cyclic Hochschild complex

We reduce computations to the small subcomplex $\zeta^\bullet \subset C^{(\bullet)}(A)$, and show that homologies are sitting in the last place w.r.t cohomological grading by ξ-degree:

$$
\begin{array}{ccccccccc}
 & & \partial & & \partial\partial & & \partial\partial\partial & & \partial\partial\partial\partial \\
 & \nearrow & | & \nearrow & | & \nearrow & & \nearrow & \\
\xi & & \xi\partial & & \xi\partial\partial & & \xi\partial\partial\partial & & \\
 & \nearrow & | & \nearrow & | & \nearrow & & & \\
\xi\xi & & \xi\xi\partial & & \xi\xi\partial\partial & & & & \\
 & \nearrow & | & \nearrow & | & & & & \\
\xi\xi\xi & & \xi\xi\xi\partial & & & & & & \\
 & \nearrow & | & & & & & & \\
\xi\xi\xi\xi & & & & & & & &
\end{array}
$$

Theorem. The higher cyclic Hochschild complex over $A = \mathbb{K}\langle X\rangle$ is pure.

Example of the argument using Gröbner-Shirshov bases theory.

We reduce the computation of the homologies for the complex (ζ, d_ζ) where differential (d_ζ) is:

$$d(u_1\xi u_2\xi\dots u_n) = \sum(-1)^{g(u_1\xi u_2\xi\dots u_i)}u_1\xi u_2\xi\dots u_i\Delta u_{i+1}\dots u_n$$

if $u_1 \neq \varnothing$ (u_1 starting with ∂_i),

here $\Delta = \sum\limits_{i=1}^{r}\partial_i x_i - x_i\partial_i$,

and

$$d(\xi u_1\xi u_2\xi\dots u_n) = \xi d(u_1\xi u_2\xi\dots u_n)+$$

$$\sum_{i=1}^{r}[\partial_i x_i u_1\xi u_2\xi\dots u_n - \partial_i u_1\xi u_2\xi\dots u_n x_i]$$

if the monomial starts with ξ.

to the computation of the homologies for the complex $(\hat{\zeta}, d_{\hat{\zeta}})$ where differential $d_{\hat{\zeta}}$ is:

$$d_{\widehat{\zeta}}(u_1\xi u_2\xi \dots u_n) =$$

$$\sum(-1)^{g(u_1\xi u_2\xi\dots u_i)} u_1\xi u_2\xi \dots u_i \Delta u_{i+1} \dots u_n,$$

where $\Delta = \sum\limits_{i=1}^{r} \partial_i x_i - x_i \partial_i$.

Lemma. Consider the place in $(\widehat{\zeta}_m, d_{\widehat{\zeta}})$, ($m = \deg_{\xi,\partial}$), where $\deg_\xi u = 1$, $u \in \widehat{\zeta}_m$ (one but last place in the complexes $\widehat{\zeta}_m$). Then the homology in this place is trivial.

m

P r o o f. Let $d_{\widehat{\zeta}}(u) = 0$ for $u \in \widehat{\zeta}$ with $\deg_\xi u = 1$. We show that $u \in \operatorname{Im} d_{\widehat{\zeta}}$. Since $\deg_\xi u = 1$, u has the shape

$$u = \sum a_i \xi b_i, \quad a_i, b_i \in \mathbb{K}\langle x_1 \ldots, x_r, \partial_1 \ldots \partial_r \rangle.$$

Then

$$d_{\widehat{\zeta}} u = \sum (-1)^{g(a_j)} a_j \Delta b_j = 0.$$

Consider the ideal I in $\mathbb{K}\langle x_1 \ldots, x_r, \partial_1 \ldots \partial_r \rangle$ generated by Δ: $I = \operatorname{Id}(\Delta)$.

We use the following lemma to describe when this above equality might happen.

Lemma.(Version of Diamond Lemma) *Let* $A = \mathbb{K}\langle y_1 \dots, y_n\rangle/\mathrm{Id}(r_1, \dots, r_m)$*. Let* M *be the syzygy module of the relations* $r_1, \dots, r_m$*, that is* M *is the submodule of the free* $\mathbb{K}\langle y_1 \dots, y_n\rangle$*-bimodule generated by the symbols* $\widehat{r_1}, \dots \widehat{r_m}$ *consisting of* $\sum f_i \widehat{r_{s_i}} g_i$ *such that* $\sum f_i r_{s_i} g_i = 0$*.*

Then M *is generated by trivial syzygies* $\widehat{r_i} u r_j - r_i u \widehat{r_j}$ *and the syzygies obtained by resolutions of ambiguities between highest terms of relations (with respect to some ordering).*

Let us fix the ordering $\partial_1 > \partial_2 > \cdots > x_1 > x_2 >$

Then the leading term of the polynomial Δ is $\partial_1 x_1$.

It does not produce any ambiguities.

Hence by Lemma the corresponding syzygy module M is generated by trivial syzygies.

$$(*) \qquad \sum a_j \widehat{\Delta} b_j = \sum u_k(\widehat{\Delta} v_k \Delta - \Delta v_k \widehat{\Delta}) w_k$$

After we know this we can construct an element

$$g = \sum \gamma_k u_k \xi v_k \xi w_k$$

such that

$$d_{\widehat{\zeta}}(g) = \sum (u_k \xi v_k \Delta w_k - u_k \Delta v_k \xi w_k) = \sum a_j \xi b_j = u.$$

From the *purity* in homology we deduce *formality*.

The complex (C, d) is formal if it is quasi-isomorphic to its cohomologies $(H^\bullet C, 0)$, considered with zero differential, as L_∞-algebra.

Theorem 1. *The higher cyclic Hochschild complex* $C = \prod_N C^{(N)}_{cycl}(A)$ *is formal.*

[preprint IHES M-19-14, Iyudu, Kontsevich]

Similar techniques related to Gröbner - Shirshov theory was developed and applied before in many concrete situations:

- **for Sklyanin algebras** [J.Algebra 2017, Iyudu, Shkarin]
- **for contraction algebras (introduced by Wemyss in connection to MMP)** [in IMRN 2018, Iyudu, Smoktunowicz]
- **for semigroup algebras** [Monatshefte für Mathematik, 2012, Iyudu, Shkarin]
- **for Witt and Virasoro algebras** [arxiv:1905.07507, Iyudu, Sierra]
- **for homology of moduli spaces of pointed curves given by Keel relations** [Iyudu, arxiv:1304.6343]

etc.