Vinberg

27 января 2022 г. 17:44

Characteristic classes of subsets of tori and their applications

Alexander Esterov

HSE

aesterov@hse.ru

Vinberg seminar 2022

Intersection numbers in homogeneous spaces

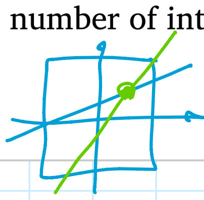
Let G be a reductive group, acting on a variety X
(e.g. $(\mathbb{C}\setminus 0)^n$ acting on itself, but not $\mathbb{C}^n$ acting on itself)

Let $X \circlearrowleft G$ be a spherical homogeneous space

Let U and $V \subset X$ have $\dim U + \dim V = \dim X$

DEFINITION The intersection number $U \circ V$ is the number of intersections of U and $gV = \{gv \mid v \in V\}$ for generic $g \in G$.

EXAMPLE Self-intersection of a line in $(\mathbb{C}\setminus 0)^2$:



$(\mathbb{C}\setminus 0)^2$

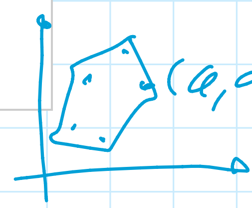
Numerical equivalence

DEFINITION $U_1 \sim U_2$ if $U_1 \circ V = U_2 \circ V$ for every $V \subset X$

EXAMPLE Curves in $(\mathbb{C}\setminus 0)^2$ are numerically equivalent if their Newton polygons are equal (up to a shift)

$C=\{f=0\}$

$x_1^{a_1}x_2^{a_2}$



(a_1,a_2)

$C_1 \sim C_2 \quad \forall V \quad C_1 \circ V = C_2 \circ V$

$MV(N_{C_1}, N_V) = MV(N_{C_2}, N_V)$

Ring of conditions

$C^k := \{\text{linear combinations of codimension } k \text{ subvarieties of } X\}/\sim$

Multiplication $C^k \times C^m \to C^{k+m}$ sends $[U],[V] \mapsto [U \cap gV]$ for generic $g \in G$

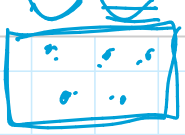
THEOREM It does not depend on the choice of g.

REMARK This fails without sphericity (e.g. in $\mathbb{C}^n$)

$(\mathbb{C}\setminus 0)^n$

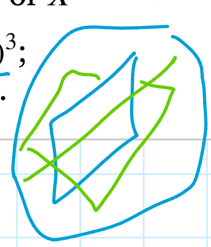
DEFINITION $C = \bigoplus_{k=0}^n C^k$ is the ring of conditions of X

EXAMPLES 1) Self-intersection of a plane in $(\mathbb{C}\setminus 0)^3$;
2) $C^k \times C^{n-k} \to C^n = \mathbb{Z}$ is the intersection number.

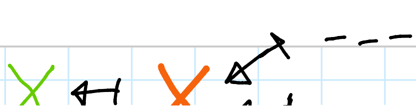


X

$\sim$



Homology of all spherical (e.g. toric) varieties at once



$$\begin{tikzcd} & X \arrow[dl] & X \arrow[l] \arrow[dl] & \cdots \arrow[l] \arrow[dl] \\ X & X \arrow[l] \arrow[ur] & X \arrow[l] \arrow[ur] & \cdots \arrow[l] \end{tikzcd}$$

$[V] \mapsto [V]$

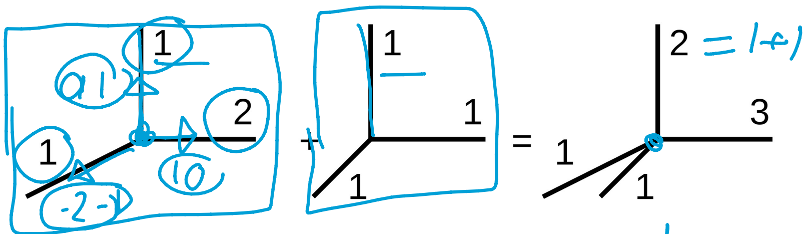
$$\begin{tikzcd} & H^\bullet(X) \arrow[r, hook] & H^\bullet(X) \arrow[r, hook] \arrow[ur] & \cdots \\ H^\bullet(X) \arrow[r, hook] \arrow[ur] & H^\bullet(X) \arrow[r, hook] \arrow[ur] & H^\bullet(X) \arrow[r, hook] & \cdots \end{tikzcd}$$

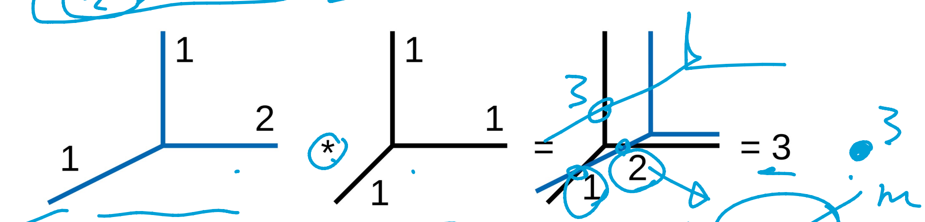
$\bigcup_X H^\bullet(X) = C$

$$\begin{tikzcd} & H_\bullet(X) \arrow[dl] & H_\bullet(X) \arrow[l] \arrow[dl] & \cdots \arrow[l] \\ H_\bullet(X) & H_\bullet(X) \arrow[l] & H_\bullet(X) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

$\lim\limits_{\leftarrow} H_\bullet = C$

How it works in the toric case: tropical Fans





3

m

$|\det(u,v)| \cdot k \cdot m$

$= 3$

K_m – vector space of COdimension m tropical fans in $\mathbb{R}^n$.

$K = \bigoplus_m K_m$ – the graded ring of tropical fans with multiplication

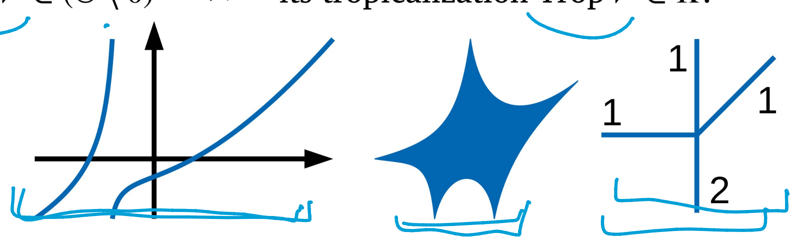
$$K_m \times K_p \to K_{m+p}.$$

$(\mathbb{C} \setminus 0)^n$

EXAMPLE $K_m \times K_{n-m} \to K_n \simeq \mathbb{Z}$ is the intersection number.

Tropicalization (FOR $C = K$)

A variety $V \subset (\mathbb{C} \setminus 0)^n \quad \mapsto \quad$ its tropicalization $\operatorname{Trop} V \in K$.



$\log|\cdot| : (\mathbb{C} \setminus 0)^n \to \mathbb{R}^n$

THEOREM Sending $[V]$ to $\operatorname{Trop} V$ is an isomorphism $C = K$

COROLLARIES

If varieties U and V are in general position, then

$\operatorname{Trop}(U \cup V) = \operatorname{Trop} U + \operatorname{Trop} V$

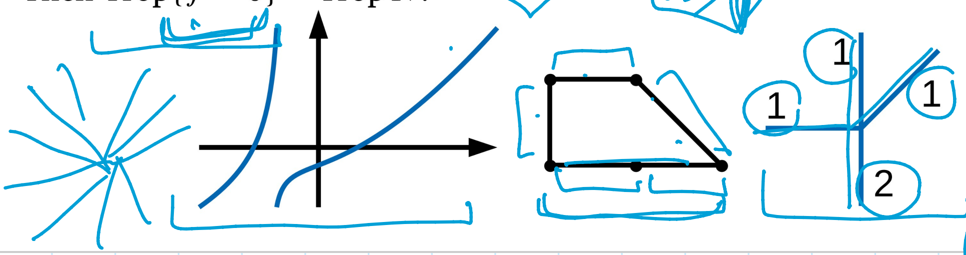
$\operatorname{Trop}(U \cap V) = \operatorname{Trop} U \star \operatorname{Trop} V$

$\dim U + \dim V = n \quad \Rightarrow \quad U \circ V \; = \; \operatorname{Trop} U \circ \operatorname{Trop} V$

Relation to Newton polytopes (Trop generalizes N_f)

Consider a Laurent polynomial $f : (\mathbb{C} \setminus 0)^n \to \mathbb{C}$,
its Newton polytope $N \subset \mathbb{R}^n$,
and its dual fan Trop N.

Then $\text{Trop}\{f = 0\} = \text{Trop}\, N$:



$\Sigma \sim X_\Sigma$ $\quad K \sim C$ $\quad H.(X_\Sigma)$

Relation to homology of toric varieties (K = C descends to H.)

For a toric variety $X_\Sigma \supset (\mathbb{C} \setminus 0)^n$, the subring $H^\bullet(X_\Sigma) \subset K$
equals {all tropical fans consisting of cones of the fan Σ} $= K_\Sigma$

Example: Fix $V \subset (\mathbb{C} \setminus 0)^n$, vary Σ

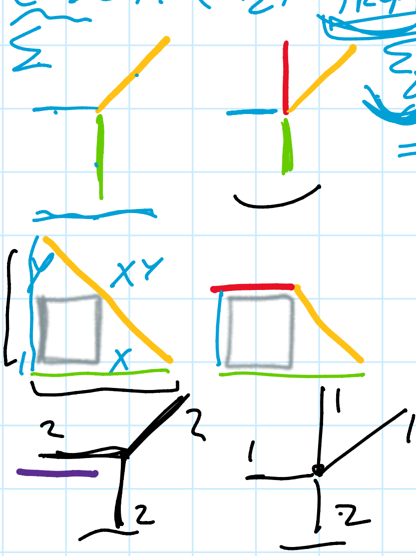
$[V] \in H^\cdot(X_\Sigma)$ $\quad$ Trop V

$[V] \in C \simeq K \ni \text{Trop}\, V$

X_{Σ_1} $\quad [V] \in H_\cdot(X_{\Sigma_1}) \simeq K_{\Sigma_1} \ni \text{Trop}\, V$

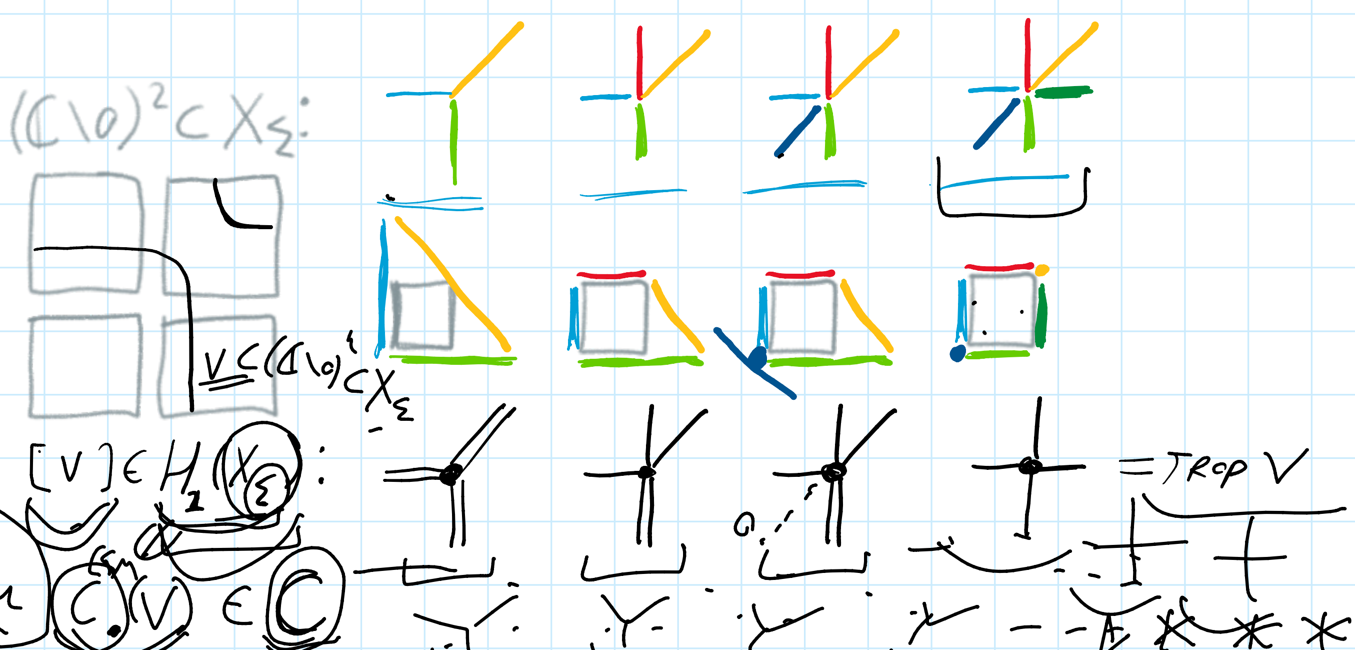
X_{Σ_2} $\quad [V] \in H_\cdot(X_{\Sigma_2}) \simeq K_{\Sigma_2} \ni \text{Trop}\, V$

X_{Σ_3} $\quad [V] \in H_\cdot(X_{\Sigma_3}) \simeq K_{\Sigma_3} \ni ?$



$(\mathbb{C}\setminus 0)^2 \subset X_\Sigma$:

$[V] \in H_2(X_\Sigma)$:



$(\mathbb{C}\setminus 0)^2 \subset X_\Sigma$:

$V \subset (\mathbb{C}\setminus 0)^2 \subset X_\Sigma$

$[V] \in H_2(X_\Sigma)$:

$= \text{Trop}\, V$

$\lim c.(V) \in C$

CSM classes and their stabilization

Chern–Schwartz–MacPherson

Observation For any $V \subset (\mathbb{C} \setminus 0)^n$, its CSM classes stabilize in the same way!
(But maybe later!)

Theorem Given smooth compact M, there exists a correspondence

(a semialgebraic set $V \subset M$) $\mapsto$ its class $c_i^{CSM}(V) \in H_i(M)$

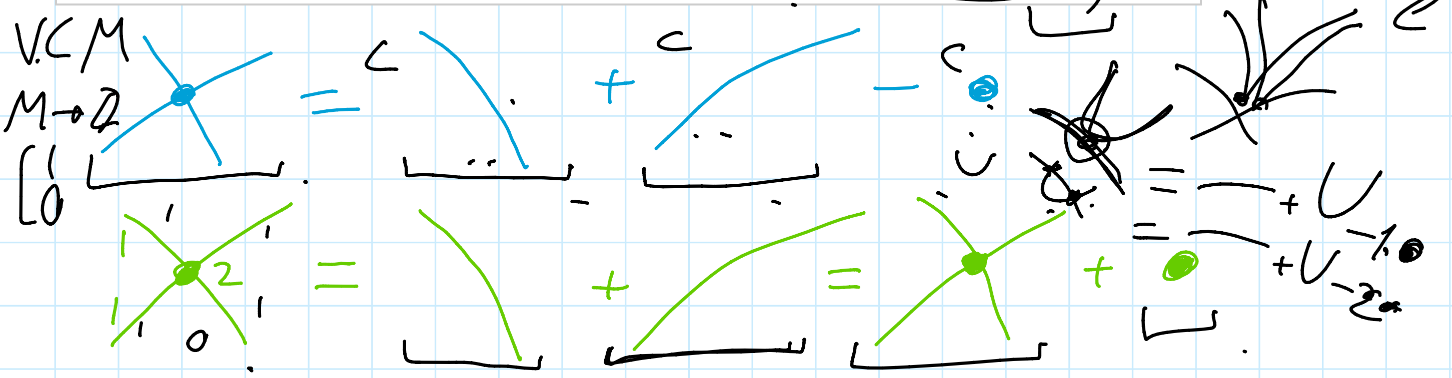
with the properties:

1) If V is closed and smooth, then $c^{CSM} =$ (Chern class V) $\in H_\cdot(V) \subset H_\cdot(M)$

with the properties:

1) If V is closed and smooth, then $c^{CSM} =$ (Chern class)(V);

2) Additivity: $c^{CSM}(U \cup V) = c^{CSM}(U) + c^{CSM}(V) - c^{CSM}(U \cap V)$;



The projection property

Remark c^{CSM} extends by additivity to constructive functions $M \to \mathbb{Z}$

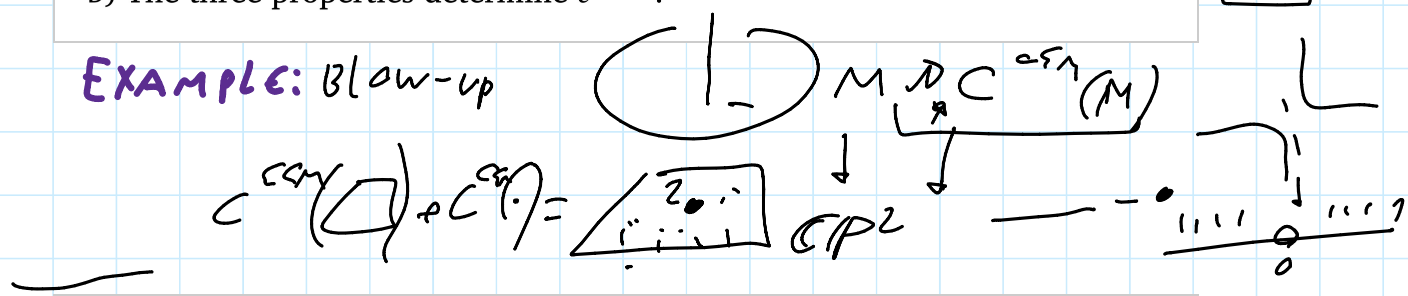
3) For a projection $\pi : M \to M'$, we have $\pi_* c^{CSM}(V) = c^{CSM}(\pi_* V)$.

Remarks

a) $\pi_* V$ is the CSM-image: $M' \to \mathbb{Z}$, $y \mapsto$ the Euler char. of $\pi^{-1}(y)$).

b) The three properties determine c^{CSM}.

Example: Blow-up



Tropical characteristic classes

Theorem ([E'13, JEMS'18])

A codimension m variety $V \subset (\mathbb{C} \setminus 0)^n \quad \mapsto \quad$ its tropical char.classes

$$\begin{array}{lcll}
\langle V \rangle_0 & = & 0 & \in K_0 \\
 & \dots & & \\
\langle V \rangle_{m-1} & = & 0 & \in K_{m-1} \\
\langle V \rangle_m & = & \text{Trop } V & \in K_m \\
\langle V \rangle_{m+1} & = & ? & \in K_{m+1} \\
 & \dots & & \\
\langle V \rangle_{n-1} & = & ? & \in K_{n-1} \\
\langle V \rangle_n & = & e(V) & \in K_n \quad = \mathbb{Z}
\end{array}$$

such that the total char.class $\langle V \rangle = \sum_j \langle V \rangle_j \in K$ satisfies the following...

Properties

1) $\langle U \rangle \times \langle V \rangle = \langle U \times V \rangle$ for all $U \subset (\mathbb{C} \setminus 0)^m$ and $V \subset (\mathbb{C} \setminus 0)^n$

2) $\langle U \rangle \star \langle V \rangle = \langle U \cap g \cdot V \rangle$ for all U and V and generic g in $(\mathbb{C} \setminus 0)^n$

3) $\langle U \rangle + \langle V \rangle = \langle U \cup V \rangle + \langle U \cap V \rangle$ for all U and $V \subset (\mathbb{C} \setminus 0)^n$

4) $\pi_* \langle V \rangle_{n-\bullet} = \langle \pi_* V \rangle_{m-\bullet}$ for $V \subset (\mathbb{C} \setminus 0)^n \xrightarrow{\pi} (\mathbb{C} \setminus 0)^m$.

Example 1

$C \subset (\mathbb{C} \setminus 0)^2$ – a curve with the Newton polygon N.

$\langle C \rangle_0 = 0$

$\langle C \rangle_1 = \text{Trop } C = \text{Trop } N$

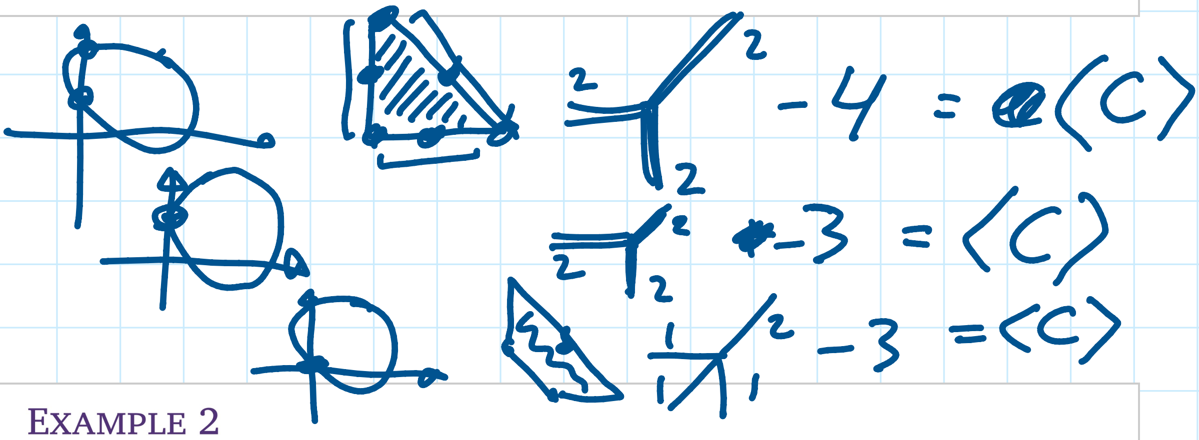
$\langle C \rangle_2 = e(C) \quad [= -\text{Vol } N = -\text{Trop } N^2$ if C is non-degenerate]

$\langle C \rangle = \text{Trop } N - \text{Trop } N^2 = \frac{\text{Trop } N}{1 + \text{Trop } N}$

If C is degenerate, $\langle C \rangle$ is not determined by Trop C!

The char.classes of V are not the char.classes of the tropicalization of V!

If C is degenerate, $\langle C\rangle$ is not determined by $\operatorname{Trop} C$!
The char.classes of V are not the char.classes of the tropicalization of V!
The case of a conic:



Example 2

$S \subset (\mathbb{C}\setminus 0)^3$ – a surface with the Newton polytope N.

$\langle S\rangle_0 = 0$

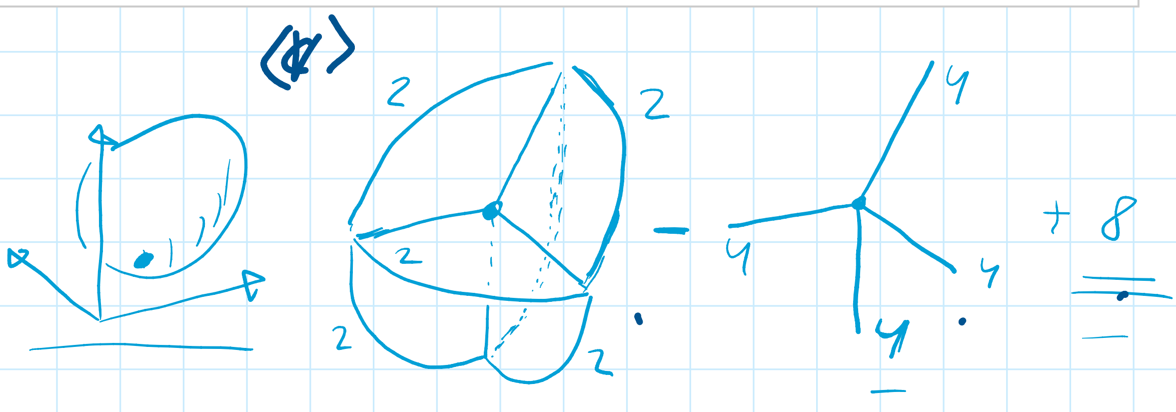
$\langle S\rangle_1 = \operatorname{Trop} S = \operatorname{Trop} N$

$\langle S\rangle_2 =$ (some 1-dim fan) $\quad [= -\operatorname{Trop} N^2$ if S is non-degenerate$]$

Exercise Deduce this from properties 1&2 of $\langle\cdot\rangle$ (the ones on $\times$ and $\star$)

$\langle S\rangle_3 = e(S) \quad [= \operatorname{Vol} N = \operatorname{Trop} N^3$ if S is non-degenerate$]$

$\langle S\rangle = \operatorname{Trop} N - \operatorname{Trop} N^2 + \operatorname{Trop} N^3 = \frac{\operatorname{Trop} N}{1+\operatorname{Trop} N}$. The case of a quadric:



Why it exists

$[V] < C$

Approach 1([A.E., arXiv:1305:3234])

$V \subset (\mathbb{C}\setminus 0)^n \mapsto$ conormal variety $NV \subset (\mathbb{C}\setminus 0)^{2n} \mapsto \langle V\rangle$
(no need for resolutions of singularities!)

$NV = \{(x,z) \mid x \in V,\ z \perp T_x V\}$

Approach 2 ([A. Gross, arXiv:1705.05719], only for smooth V so far)

$V \mapsto$ (not necessarily toric compactification $(\mathbb{C}\setminus 0)^n \supset M \supset \bar V$
and resolution $M' \to M$, $V' \to \bar V$) $\mapsto \langle V\rangle$

опр' $\langle V\rangle_i \ni$ то $\quad e(V\cap\{f=0\}) = \sum \langle V\rangle_i \cdot (\operatorname{Trop} N)^i$

$e(V\cap\{f=0\}) - e(V) = (-1)^{\dim V}\,\#\,$крит. $f|_V$ $=$

$= NV \circ ($график $\log d f) = [NV]\cdot($график $\log df] \in C$

Non-compact characteristic classes

Let X be a G-spherical variety.
Conjecture A codimension m variety $V \subset X \quad\mapsto\quad$ its char.classes

$$\left[\begin{array}{lll}\langle V\rangle_0 & = 0 & \in C^0 \\ & \dots & \end{array}\right.$$

$$\left[\begin{array}{lllll}
\langle V\rangle_0 & = & 0 & \in & C^0 \\
 & & \cdots & & \\
\langle V\rangle_{m-1} & = & 0 & \in & C^{m-1} \\
\langle V\rangle_m & = & [V] & \in & C^m \\
\langle V\rangle_{m+1} & = & ? & \in & C^{m+1} \\
 & & \cdots & & \\
\langle V\rangle_{n-1} & = & ? & \in & C^{n-1} \\
\langle V\rangle_n & = & e(V) & \in & C^n \quad = \mathbb{Z}
\end{array}\right]$$

such that the total char.class $\langle V\rangle = \sum_j \langle V\rangle_j \in C$ satisfies the following...

Properties

- 1) $\langle U\rangle \times \langle V\rangle = \langle U \times V\rangle$ for all $U \subset X_1$ and $V \subset X_2$
- 2) $\langle U\rangle \cdot \langle V\rangle / \langle X\rangle = \langle U \cap g \cdot V\rangle$ for all U and V and generic g in G
- 3) $\langle U\rangle + \langle V\rangle = \langle U \cup V\rangle + \langle U \cap V\rangle$ for all U and $V \subset X$
- 4) Given a smooth equivariant compactification $\overline{X} \supset X$, the image of $\langle V\rangle \in C$ under $C \to H_\bullet(\overline{X})$ equals the CSM class of $V \subset \overline{X}$.

Remark

The class $\langle X\rangle$ is constructed in [Kiritchenko'04], [Brion&Kausz'05], [Brion&Joshua'07], and still not computed even for $X = SL_n$ with large n.

Applications of tropical characteristic classes

to enumerative geometry (tropical correspondence theorems)

A. E. *Characteristic classes of affine varieties and Plücker formulas for affine morphisms* J. of the EMS, 20 (2018) 15–59 arXiv:1305.3234

to additive invariants of algebraic varieties (Hirzebruch genus)

A. Gross
Refined Tropicalizations for Schön Subvarieties of Tori arXiv:1705.05719

to matroid theory (g-polynomial)

L. López de Medrano, F. Rincón, K. Shaw. *Chern-Schwartz-MacPherson cycles of matroids* Proc. LMS 120 (2020) 1–27 arXiv:1707.07303

to singularity theory (the monodromy conjecture)

A. E. *Tropical nearby monodromy eigenvalues* arXiv:1807.00609

A. E., A. Lemahieu, K. Takeuchi *On the monodromy conjecture for non-degenerate hypersurfaces* to appear in J. EMS, arXiv:1309.0630

Thank you!

Example 3

$V = \{f_1 = \cdots = f_k = 0\} \subset (\mathbb{C} \setminus 0)^n$ – a generic complete intersection with the Newton polytopes $N_1, \ldots, N_k$ in $\mathbb{R}^n$

$$\langle V\rangle = \langle \textstyle\bigcap \{f_i = 0\}\rangle = \star_i \langle f_i = 0\rangle = \frac{\mathrm{Trop}\, N_1}{1+\mathrm{Trop}\, N_1} \star \cdots \star \frac{\mathrm{Trop}\, N_k}{1+\mathrm{Trop}\, N_k}$$

$$e(V) = \langle V\rangle_n = \Big(\text{the 0-dim component of } \frac{\mathrm{Trop}\, N_1}{1+\mathrm{Trop}\, N_1} \star \cdots \star \frac{\mathrm{Trop}\, N_k}{1+\mathrm{Trop}\, N_k}\Big)$$

[Khovanskii'76, Func.An&Appl] Тогда $E(X) = \prod \Delta_i (1+\Delta_i)^{-1}$.