Skew Howe duality and limit shape of Young diagrams for classical Lie groups

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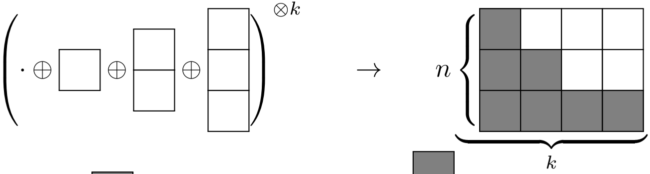
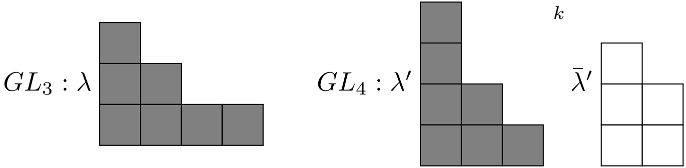
(GL_n, GL_k)-skew Howe duality

Consider $\bigwedge\left(\mathbb{C}^n \otimes \mathbb{C}^k\right)$ with the action of $GL_n \times GL_k$:

$$\bigwedge\left(\mathbb{C}^n \otimes \mathbb{C}^k\right) = \left(\bigwedge\left(\mathbb{C}^n\right)\right)^{\otimes k} \simeq \bigoplus_{\lambda} V_{GL_n}(\lambda) \otimes V_{GL_k}(\bar{\lambda}'),$$

Sum of one-column diagrams raised to k-th tensor power





The exterior algebra $\bigwedge\left(\mathbb{C}^n \otimes \mathbb{C}^k\right) = \bigoplus_{p=0}^{nk} \bigwedge^p\left(\mathbb{C}^n \otimes \mathbb{C}^k\right)$ is a graded space and $\bigwedge^p\left(\mathbb{C}^n \otimes \mathbb{C}^k\right) \simeq \bigoplus_{|\lambda|=p} V_{GL_n}(\lambda) \otimes V_{GL_k}(\bar{\lambda}')$, the limit $n, k, p \to \infty, n/k \to \mathrm{const}, p/(nk) \to \mathrm{const}$ was considered by P. Sniady and G. Panova '18.

Skew Howe duality for Sp and SO

- Sp_{2l}. Let $V = \mathbb{C}^{2l} = V_{Sp_{2l}}(\Lambda_1) = V_+ \oplus V_-, \dim V = 2l$. Then

$$\bigwedge(\mathbb{C}^{2l} \otimes \mathbb{C}^k) \simeq \left(\bigwedge V\right)^{\otimes k} \simeq \left(\bigwedge V_-\right)^{\otimes 2k} \simeq \bigoplus_\lambda V_{Sp_{2l}}(\lambda) \otimes V_{Sp_{2k}}(\bar{\lambda}').$$

- SO_{2l+1}. If $V = \mathbb{C}^{2l+1} = V_{SO_{2l+1}}(\Lambda_1)$, $\dim V = 2l+1$. Then

$$\bigwedge(\mathbb{C}^{2l+1} \otimes \mathbb{C}^k) \simeq \bigwedge(V \otimes \mathbb{C}^k) \simeq \bigoplus_\lambda V_{SO_{2l+1}}(\lambda) \otimes V_{Pin_{2k}}(\bar{\lambda}').$$

  On the other hand, $V = V_+ \oplus V_0 \oplus V_-$, $\dim V_0 = 1$ and $\bigwedge V = \bigwedge V_+ \otimes \bigwedge V_0 \otimes \bigwedge V_- \simeq 2\left(V_{SO_{2l+1}}(\Lambda_l)\right)^{\otimes 2}$. Then

$$\left(V_{SO_{2l+1}}(\Lambda_l)\right)^{\otimes 2k} \simeq \bigoplus_\lambda 2^{1-k} \dim(V_{SO_{2k}}(\bar{\lambda}')) V_{SO_{2l+1}}(\lambda).$$

- SO_{2l}. If $V = \mathbb{C}^{2l} = V_{SO_{2l}}(\Lambda_1)$ and $\bigwedge V_- = V(\Lambda_{l-1}) \oplus V(\Lambda_l)$.

$$\bigwedge(\mathbb{C}^{2l} \otimes \mathbb{C}^k) \simeq \bigwedge(V \otimes \mathbb{C}^k) \simeq \bigoplus_\lambda V_{SO_{2l}}(\lambda) \otimes V_{O_{2k}}(\bar{\lambda}'),$$

$$(V_{SO_{2l}}(\Lambda_{l-1}) \oplus V_{SO_{2l}}(\Lambda_l))^{\otimes 2k} = \bigoplus_\lambda 2 \dim(V_{SO_{2k}}(\bar{\lambda}')) V_{SO_{2l}}(\lambda).$$

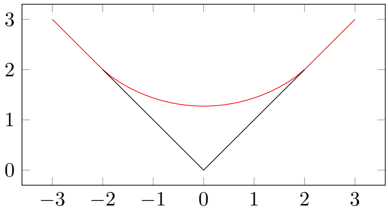
Tensor power decompositon and limit shapes of Young diagrams

Kerov '86 (Schur-Weyl duality):

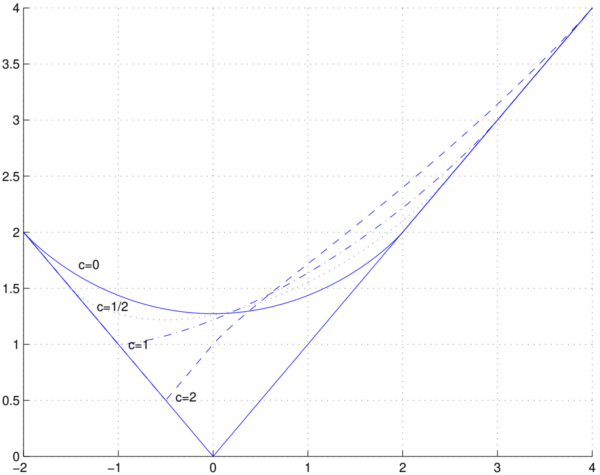
$$(\mathbb{C}^n)^{\otimes k} = (V_{GL_n}(\Lambda_1))^{\otimes k} \simeq \bigoplus_\lambda V_{GL_n}(\lambda) \otimes V_{S_k}(\lambda)$$

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \dim V_{S_k}(\lambda)}{n^k} = \\ = \frac{1}{n^k} \cdot \frac{\prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_{m=0}^{n-1} m!} \cdot \frac{k! \prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{n}(\lambda_i + n - i)!}$$

If $n, k \to \infty$, $k \sim n$ get Vershik-Kerov-Logan-Shepp limit shape:



Young diagram of k boxes, n rows. P. Biane '00: if $c = k/n^2$



This result is related to the RSK algorithm:

$$V_{GL_n}(\Lambda_1): \quad e_i \leftrightarrow \boxed{\;i\;},$$

$$V_{GL_n}(\Lambda_1)^{\otimes k} \longleftrightarrow (P, Q), \quad P = SSYT(\lambda \vdash k, n), Q = SYT(\lambda, k)$$

Can we generalize it?

Dual RSK algorithm

Basis in $\mathbb{C}^n \otimes \mathbb{C}^k$ is $\{e_{ij} = e_i \otimes e_j\}_{i=1,j=1}^{n,k}$, basis in $\bigwedge(\mathbb{C}^n \otimes \mathbb{C}^k)$: $e_{i_1 j_1} \wedge e_{i_2 j_2} \wedge \ldots$ corresponds to $n \times k$ matrices of $0, 1$:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 2 & 4 & 1 & 4 \end{pmatrix} \longrightarrow \text{dual RSK}$$

We bump equal boxes down and write upper row in the recording table Q.

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 1 & 4 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}, \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

This is a pair of $(SSYT(\lambda', k), SSYT(\lambda, n))$. Uniform measure on $n \times k$ matrices of zeros and ones, i.e. on numbers from 0 to $2^{nk} - 1$, after applying the dual RSK leads to the measure on Young diagrams $\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \cdot \dim V_{GL_k}(\lambda')}{2^{nk}}$.

Probability measure on Young diagrams

Consider the space $\bigwedge\left(\mathbb{C}^n \otimes \mathbb{C}^k\right)$ and the action the group $GL_n \times GL_k$ on it. Assuming that k is even, introduce the action of the Clifford algebra and consider the invariant subspace $\bigwedge\left(\mathbb{C}^n \otimes \mathbb{C}^{k/2}\right)$ with the actions of $SO_{2l+1} \times Pin_k$ for $n = 2l+1$, $SO_{2l} \times O_k$ for $n = 2l$, and $Sp_{2l} \times Sp_k$ for $n = 2l$ on it. Introduce the probability measures on the diagrams as

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \cdot \dim V_{GL_k}(\bar{\lambda}')}{2^{nk}},$$

and

$$\mu_{n,k/2}(\lambda) = \frac{\dim V_{G_1}(\lambda) \cdot \dim V_{G_2}(\bar{\lambda}')}{2^{nk/2}},$$

for the actions of $SO_{2l+1} \times Pin_k$ for $n = 2l+1$, $SO_{2l} \times O_k$ for $n = 2l$, and $Sp_{2l} \times Sp_k$ for $n = 2l$.

To derive the asymptotics of these measures we need the explicit formulas for $\dim V_{G_2}(\bar{\lambda}')$ in terms of row lengths $\{\lambda_i\}$

Multiplicity, Young tableaux and Gelfand-Tsetlin patterns

We need formula for $\dim V_{G_2}(\bar{\lambda}')$ in terms of row lengths $\{\lambda_i\}$.

$$\dim V_{GL_k}(\bar{\lambda}') = \#\mathrm{SSYT}(\bar{\lambda}', k)$$

Semistandard Young tableaux $\longleftrightarrow$ Gelfand–Tsetlin patterns:

1	1	1	4	4
2	2	4	6	
3	3			
4	4			
6				

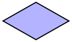
$\longleftrightarrow$

$$\begin{array}{ccccccccccc}
5 & & 4 & & 2 & & 2 & & 1 & & 0 \\
 & 5 & & 3 & & 2 & & 2 & & 0 & \\
 & & 5 & & 3 & & 2 & & 2 & & \\
 & & & 3 & & 2 & & 2 & & & \\
 & & & & 3 & & 2 & & & & \\
 & & & & & 3 & & & & &
\end{array}$$

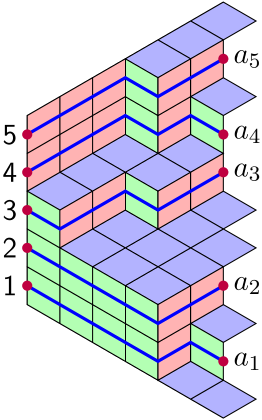
$$\begin{array}{ccccccc}
b_1^{(k)} & & b_2^{(k)} & & \dots & & b_k^{(k)} \\
 & \ddots & & \dots & & \iddots & \\
 & & b_1^{(2)} & & b_2^{(2)} & & \\
 & & & b_1^{(1)} & & &
\end{array}$$

$b_i^{(j)}$ – number of boxes with value $\leq$ j in i-th row of the diagram

Gelfand-Tsetlin patterns and lozenge tilings

Let $\tilde{b}_i^{(j)} = b_i^{(j)} + j - i$, these numbers can be seen as the positions from the bottom of  in j column from the left in the tiling.
Let $a_i = \lambda_i + n - i$, where λ_i is row length of GL_n-diagram. Then coordinates in the rightmost column are $\bar{a}'_i = \tilde{b}_i^{(k)} = \bar{\lambda}'_i + k - i$, where $\{\bar{\lambda}'_i\}$ are the row lengths of the complement conjugate GL_k-diagram $\bar{\lambda}'$.

$$
\begin{array}{ccccccccccc}
10 & & 8 & & 5 & & 4 & & 2 & & 0 \\
 & 9 & & 6 & & 4 & & 3 & & 0 & \\
 & & 8 & & 5 & & 3 & & 2 & & \\
 & & & 5 & & 3 & & 2 & & & \\
 & & & & 4 & & 2 & & & & \\
 & & & & & 3 & & & & &
\end{array}
\quad \longleftrightarrow
$$

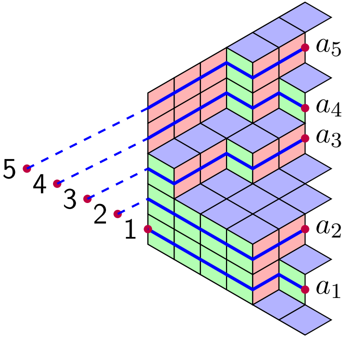


Determinant formula for multiplicity

$$\dim V_{GL_k}(\bar{\lambda}') = \#\text{Lozenge tilings of trapezoid}(k,n,k,n+k) =$$
$$= \#\text{configurations of } n \text{ non-intersecting paths } (i \to a_i) \text{ of length } k$$

Apply Lindström–Gessel–Viennot lemma:

$$\dim V_{GL_k}(\bar{\lambda}') = \det\left[\#\text{of paths } (i \to a_j)\right]_{i,j=1}^{n} =$$



$$= \det\left[\binom{k+i-1}{a_j}\right]_{i,j=1}^{n}.$$

From determinants to products

For subsets $A, B \subseteq [n]$, let M_A^B denote the submatrix of M with columns A and rows B removed. Use Desnanot–Jacobi identity

$$\det M \cdot \det M_{1,n}^{1,n} = \det M_1^1 \cdot \det M_n^n - \det M_1^n \cdot \det M_n^1.$$

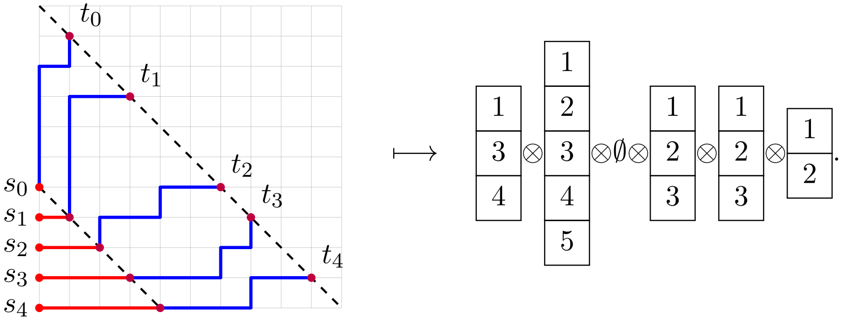
to prove that

$$\dim V_{GL_k}(\bar{\lambda}') = \det \left[\binom{k+i-1}{a_j} \right]_{i,j=1}^{n} =$$

$$= \frac{\prod_{m=0}^{n-1} (k+m)!}{\prod_{i=1}^{n} a_i! \cdot (k+n-1-a_i)!} \times \prod_{1 \le i < j \le n} (a_i - a_j).$$

Relation of paths to crystals

Paths depict signature rule. Non-intersection condition $\sim$ highest weight condition



These paths are dual to the paths that correspond to Young tableaux by row bijection, where number of j boxes in row i is the number of steps along the line j in path number i.

Weighted paths and q-multiplicity theorem for GL

Weight vertical steps in the paths by $q^{\text{column number}}$.

Theorem

Let $V = \bigwedge V(\Lambda_1)$ of GL_n. For a diagram λ contained in an $n \times k$ rectangle, define $M_q^A(\lambda) = \det \left[\begin{bmatrix} k+i \\ j+\lambda_{n-j} \end{bmatrix}_q \right]_{i,j=0}^{n-1}$.

Let $a_i = \lambda_i + n - i$. Then we have

$$M_q^A(\lambda) = q^{\|\overline{\lambda}\|} \frac{\prod_{m=0}^{n-1} [k+m]_q! \times \prod_{1 \le i < j \le n} [a_i - a_j]_q}{\prod_{i=1}^{n} [a_i]_q! [k+n-1-a_i]_q!} =$$

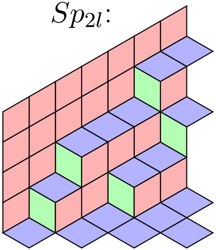
$$= q^{\|\overline{\lambda}\|} \dim_q(\overline{\lambda}') = q^{\|\overline{\lambda}\|} \dim_q(\lambda') \in \mathbb{Z}_{\ge 0}[q],$$

where $\dim_q(\nu) = \prod_{\alpha \in \Phi^+} \frac{1-q^{\langle \nu+\rho, \alpha^\vee \rangle}}{1-q^{\langle \rho, \alpha^\vee \rangle}}$ is the q-dimension of $V(\nu)$ for GL_k and $\|\lambda\| = \sum_i (i-1)\lambda_i$. Moreover, $M_1^A(\lambda)$ is equal to the multiplicity of $V(\lambda)$ in $V^{\otimes k}$.

Lozenge tilings for SO and Sp

Tilings that are strict for Sp and semi-strict for SO symmetry conditions, related to Proctor patterns and King tableaux.

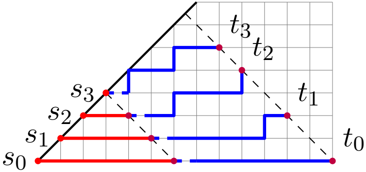
Sp_{2l}:



$\leftrightarrow$

$$\begin{matrix} 3 & & 2 & & 0 & & \\ & 3 & & 1 & & 0 & \\ & & 2 & & 1 & & \\ & & & 2 & & 0 & \\ & & & & 2 & & \\ & & & & & 1 & \end{matrix}$$

$\leftrightarrow$



SO_{2l+1}:lozenge tilings of the half hexagon that are *almost symmetric* up to the middle row of hexagons, which are then forced to be either  or 

SO_{2l}: Symmetry in the blue tiles except for the middle blue tile.

q-multiplicities for SO and Sp.

Let q-multiplicity be $M_q^{BC}(\lambda + p\Lambda_n) := \det\left[\mathcal{C}_{(a(i,j),b(i,j))}(q)\right]_{i,j=1}^{n}$,
where $a(i,j) = 2n - i - j + k + p + \lambda_j$, $b(i,j) = j - i + k - \lambda_j$
and $\mathcal{C}_{n,k}(q) = \frac{[n+k]_q![n-k+1]_q}{[k]_q![n+1]_q!}$.

Theorem

Let λ be a partition inside an $n \times k$ rectangle. Then for

$p = 0$: *We have $M_q^{BC}(\lambda)\prod_{a=1}^{k-1}(q^a + 1) = q^{\|\overline{\lambda}\|}\dim_q(\overline{\lambda}' + \omega_k)$, where $\omega_k = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_k)$ for type D_k and $\dim_q(\overline{\lambda}' + \omega_k)$ is the q-dimension of $V(\overline{\lambda}' + \omega_k)$ in type D_k. Furthermore, $M_1^{BC}(\lambda)$ equals the multiplicity of $V(\lambda)$ in $V(\Lambda_n)^{\otimes 2k}$ for type B_n and $M_q^{BC}(\lambda) \in \mathbb{Z}_{\geq 0}[q]$.*

$p = 1$: *We have $M_q^{BC}(\lambda + \Lambda_n) = q^{\|\overline{\lambda}\|}\dim_q(\overline{\lambda}') \in \mathbb{Z}_{\geq 0}[q]$, where $\dim_q(\overline{\lambda}')$ is the q-dimension of $V(\overline{\lambda}')$ in type C_k. Furthermore, $M_1^{BC}(\lambda + \Lambda_n)$ equals the multiplicity of $V(\lambda + \Lambda_n)$ in $V(\Lambda_n)^{\otimes 2k+1}$ for type B_n and $V(\lambda)$ in $V^{\otimes k}$ for $V = \bigwedge V(\Lambda_1)$ in type C_n.*

Product formula for series B and C.

Theorem

Fix positive integers k and n. Let λ be a partition contained inside of a $n \times k$ rectangle. Let $a_i = \lambda_i + (n-i) + \frac{1-p}{2}$. Then we have

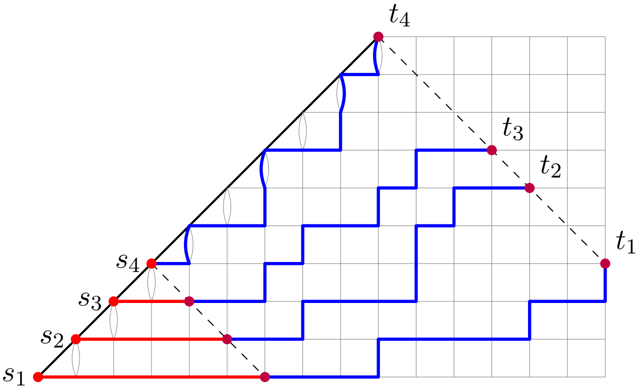
$$M_q^{BC}(\lambda + p\Lambda_n) = q^{\|\bar{\lambda}\|} \frac{\prod_{i=1}^{n} [2k+p+2i-2]_q! \, [2a_i]_q \times \prod_{1 \le i < j \le n} [a_i - a_j]_q [a_i + a_j]_q}{\prod_{i=1}^{n} \left[k+n-a_i - \frac{1-p}{2}\right]_q! \left[k+n+a_i-\frac{1-p}{2}\right]_q!}.$$

Paths for the series D.

We need to take into account the sign of the last coordinate, we do it by allowing two kinds of vertical steps on the last path near the anti-diagonal.

$$\begin{array}{|c|}\hline - \\ \hline + \\ \hline + \\ \hline + \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline + \\ \hline + \\ \hline - \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline - \\ \hline - \\ \hline - \\ \hline \end{array} \otimes \begin{array}{|c|}\hline - \\ \hline + \\ \hline + \\ \hline - \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline - \\ \hline - \\ \hline + \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline - \\ \hline + \\ \hline + \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline + \\ \hline + \\ \hline + \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline + \\ \hline - \\ \hline + \\ \hline \end{array} \otimes \begin{array}{|c|}\hline - \\ \hline + \\ \hline + \\ \hline - \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline - \\ \hline - \\ \hline - \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline + \\ \hline + \\ \hline + \\ \hline \end{array} \otimes \begin{array}{|c|}\hline + \\ \hline + \\ \hline + \\ \hline - \\ \hline \end{array} \longleftrightarrow$$

$\longleftrightarrow$



q-multiplicity formula for series D.

Theorem

Let $\mathfrak{g} = \mathfrak{so}_{2n}$ *and let* $V = V(\Lambda_{n-1}) \oplus V(\Lambda_n)$. *Let* $p = 0, 1$. *Define* $M_q^D(\lambda + p\Lambda_n) := \det \left[\begin{bmatrix} 2(k+i)+p \\ k+i-j-|\lambda_{n-j}| \end{bmatrix}_q \right]_{i,j=0}^{n-1}$. *Then the multiplicity of* $V(\lambda + p\Lambda_{n-1})$ *and* $V(\lambda + p\Lambda_n)$ *in* $V^{\otimes 2k+p}$ *is* $M_1^D(\lambda + p\Lambda_n)$. *Furthermore, we have*

$$M_q^D(\lambda) = q^{\|\overline{\lambda}\|} \frac{\prod_{i=1}^{n} [2k + 2n - 2i + p]_q! \times \prod_{1 \le i < j \le n} [a_i - a_j]_q [a_i + a_j]_q}{\prod_{i=1}^{n} \left[k + n - 1 - a_i + \frac{p}{2}\right]_q! \left[k + n - 1 + a_i + \frac{p}{2}\right]_q!},$$

where $a_i = \lambda_i + n - i + \frac{p}{2}$ *and* $M_q^D(\lambda) \in \mathbb{Z}_{\ge 0}[q]$. *We also have*

$$M_q^D(\lambda + \Lambda_n) = q^{\|\overline{\lambda}\|} \dim_q(\overline{\lambda}'),$$

where $\dim_q(\overline{\lambda}')$ *be the* q*-dimension of* $V(\overline{\lambda}')$ *in type* B_k.

Multiplicity formulas for $\bigwedge\left(\mathbb{C}^n \otimes \mathbb{C}^k\right)$ and $\bigwedge\left(\mathbb{C}^n \otimes \mathbb{C}^{k/2}\right)$

$GL_n : a_i = \lambda_i + n - i,\ Sp_{2l} : a_i = \lambda_i + l + 1 - i,$
$SO_{2l} : a_i = 2\lambda_i + 2(l-i), SO_{2l+1} : a_i = 2\lambda_i + 2(l-i) + 1$

$$M_{GL_n}(\lambda) = \frac{\prod_{m=0}^{n-1}(k+m)!}{\prod_{i=1}^{n} a_i! \cdot (k+n-1-a_i)!} \times \prod_{1\le i<j\le n}(a_i - a_j),$$

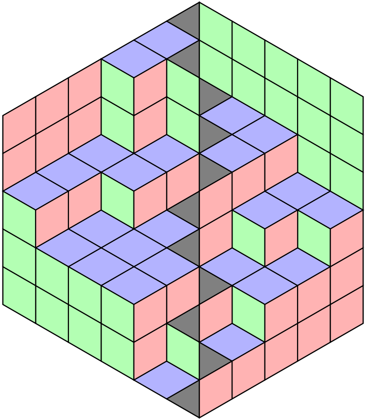
$$M_{SO_{2l+1}}(\lambda) = \prod_{m=1}^{l} \frac{(k+2m-2)!}{2^{2m-2}\left(\frac{k+a_m+2l-1}{2}\right)!\left(\frac{k-a_m+2l-1}{2}\right)!} \prod_{s=1}^{l} a_s \prod_{i<j}\left(a_i^2 - a_j^2\right)$$

$$M_{Sp_{2l}}(\lambda) = 2^l \prod_{i=1}^{l} \frac{(k-1+2i)!}{(k/2+l+a_i)!(k/2+l-a_i)!} \times \prod_{s=1}^{l} a_s \cdot \prod_{i<j}(a_i^2 - a_j^2),$$

$$M_{SO_{2l}}(\lambda) = 2^{-l(l-1)} \frac{\prod_{i=1}^{l}(2k+2l-2i)! \times \prod_{1\le i<j\le l}(a_i^2 - a_j^2)}{\prod_{i=1}^{l}\left(\frac{2k+2l-2-a_i}{2}\right)!\left(\frac{2k+2l-2+a_i}{2}\right)!}.$$

Lozenge tilings and probability measure

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \cdot \dim V_{GL_k}(\bar{\lambda}')}{2^{nk}}$$



Complimentary tilings of trapezoids $(k, n, k, n+k)$ and $(n+k, n, k, n)$.

Insertion algorithm and sampling of Sp_{2l}-diagrams

Sp_{2l}-**tableau:**
fill with $1, \bar{1}, \ldots, l, \bar{l}$ in semi-standard way in this order.
Extra condition: no numbers $\leq \bar{i}$ below row i.
We want to multiply by $\bigwedge V_-(\Lambda_1)$ and

$$\dim \bigwedge V_-(\Lambda_1) = 2^l,$$

basis element corresponds to a full column with numbers $i_1, \ldots, i_l$, where $i_k = k$ or $\bar{k}$. Insert numbers one-by-one from bottom to top, if condition is broken erase box and shift it to the boundary as in jeu de taquin (Berele insertion):

$$\begin{array}{|c|c|c|}\hline \bar{1} & 2 & \bar{3} \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \bar{3} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline \bar{2} \\ \hline 3 \\ \hline \end{array} \xrightarrow{3} \begin{array}{|c|c|c|}\hline \bar{1} & 2 & 3 \\ \hline 2 & 3 & \bar{3} \\ \hline \bar{3} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{\bar{2}} \begin{array}{|c|c|c|}\hline \bar{1} & 2 & \bar{2} \\ \hline 2 & 3 & 3 \\ \hline \bar{3} & \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \xrightarrow{1} \begin{array}{c|c|c|}\cline{2-3} & 2 & \bar{2} \\ \hline \multicolumn{1}{|c|}{2} & 3 & 3 \\ \hline \multicolumn{1}{|c|}{\bar{3}} & \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \rightarrow \begin{array}{|c|c|c|}\hline 2 & 2 & \bar{2} \\ \hline 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \bar{3} & \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

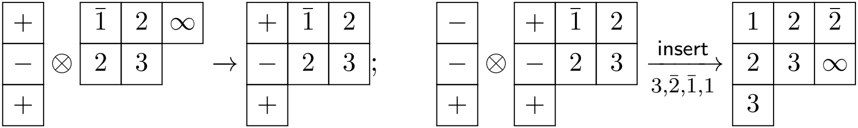
Insertion algorithm and sampling of SO_{2l+1}-diagrams

SO_{2l+1} **Sundaram tableau:**
first column can have width $\frac{1}{2}$, then it's full. Each of other columns can have at most one ∞-box.

Insertion: (Benkart-Stroomer algorithm) if there is no half-column, erase all ∞ boxes, adjoin half-column. Otherwise produce a sequence of boxes from two half-columns starting from the bottom by the rules

$$(k,k) \to k, \quad (\bar{k},\bar{k}) \to \bar{k}, \quad (k,\bar{k}) \to \emptyset, \quad (\bar{k},k) \to \bar{k},k;$$

insert them by Berele insertion, but fill the empty boxes at the edge by ∞:



Example of sampling of random diagrams

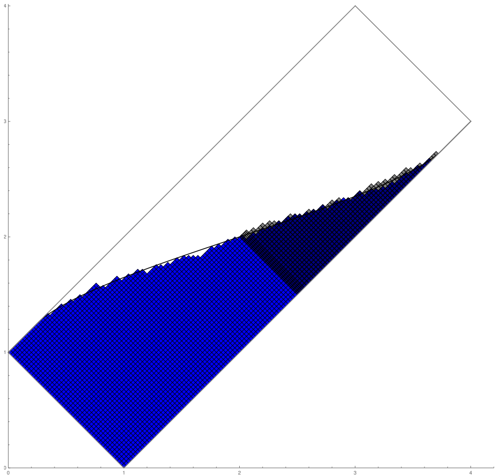


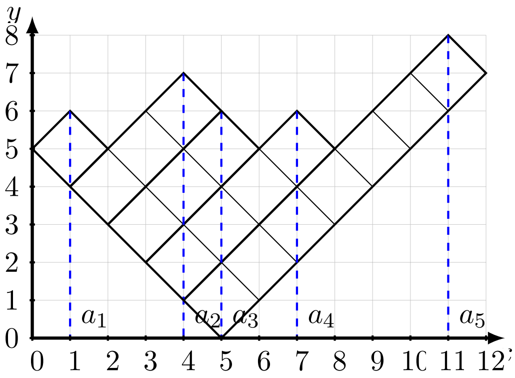
Figure: *Blue:* Random Young diagram sampled using dual RSK algorithm for GL_{50} and $k = 150$ with the limit shape for $k = 3$.*Shaded:* Random Young diagram sampled using Benkart and Stroomer algorithm for SO_{51} and $k = 150$.

Young diagrams as a determinantal point process

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \cdot \dim V_{GL_k}(\bar{\lambda}')}{2^{nk}} =$$

$$= \prod_{m=0}^{n-1} \frac{(k+m)!}{2^k \cdot m!(k+n-1)!} \times \prod_{1 \le i < j \le n} (a_i - a_j)^2 \times \prod_{i=1}^{n} \frac{(k+n-1)!}{a_i!(k+n-1-a_i)!}.$$

We have the Krawtchouk polynomial ensemble.



Convergence of the diagrams to the limit shape

Theorem

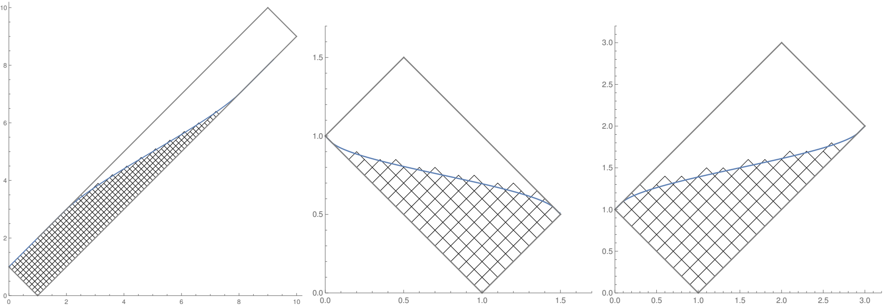
As $n \to \infty,\ k \to \infty,\ c = \lim_{n,k\to\infty} \frac{k}{n} = \mathrm{const}$, *the upper boundary* f_n *of a Young diagram in a decomposition, rotated and scaled by* $\frac{1}{n}$, *converges in probability with respect to the probability measure* $\mu_{n,k}(\lambda)$ *in the supremum norm* $\|\cdot\|_\infty$ *to the limiting shape given by the formula*

$$f(x) = 1 + \int_0^x (1 - 2\rho(t))\,\mathrm{dt} \text{ for } c > 1,\ f(x) = 1 + \int_0^x (2\rho(t) - 1)\,\mathrm{dt},\ \text{for } c < 1,$$

where the limit density $\rho(x)$ *is written explicitly as*

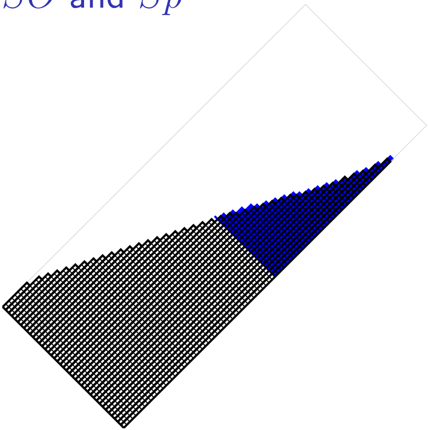
$$\rho(x) = \frac{\theta\left(\sqrt{c} - \left|x - \frac{c+1}{2}\right|\right)}{2\pi}\left[\arctan\left(\frac{-(c+1)\left(x - \frac{c+1}{2}\right) + 2c}{(c-1)\sqrt{c - \left(x - \frac{c+1}{2}\right)^2}}\right) + \arctan\left(\frac{(c+1)\left(x - \frac{c+1}{2}\right) + 2c}{(c-1)\sqrt{c - \left(x - \frac{c+1}{2}\right)^2}}\right)\right].$$

Limit shape of Young diagrams for GL



The most probable diagram for $n = 10, k = 90$ and the limit shape for $c = 9$; $n = 20, k = 10, c = 0.5$; $n = 10, k = 20, c = 2$.

Limit shape for SO and Sp



One of the most probable Young diagrams for GL_{40} and $k = 100$ and for SO_{40}, and tensor power 100.

For the groups $SO_{2l+1} \times \mathrm{Pin}_k$, $SO_{2l} \times O_k$, and $Sp_{2l} \times Sp_k$ limit shape is described by the same density $\rho(x)$ with a shifted argument $\rho\left(x + \frac{c+1}{2}\right)$ such that $x \in [0, (c+1)/2]$.

Derivation of the limit density (arXiv:2010.16383)

Denote by $f_n(x)$ the upper boundary of the scaled rotated diagram, $x \in [0, c+1]$. Let $\rho_n(x) = \frac{1}{2}(1 - f_n'(x))$, it is equal to zero on an interval of the length $\frac{1}{n}$ if there is no particle in the left boundary of the interval and is equal to 1 if there is a particle.

$$\mu_{n,k}(\{x_i\}) = \frac{1}{Z_n} \exp\left(-n^2 J[\rho_n] + \mathcal{O}(n \ln n)\right),$$

$$J[\rho_n] = \int_0^{c+1}\int_0^{c+1} \rho_n(x)\rho_n(y) \ln|x-y|^{-1}\,\mathrm{dx}\ \mathrm{dy} + \int_0^{c+1} \rho_n(x)\, V(x)\,\mathrm{dx},$$

$$V(x) = x\ln x + (c+1-x)\ln(c+1-x).$$

Shift the argument $\widetilde{x} = x - \frac{c+1}{2}$, $\widetilde{\rho}_n(\widetilde{x}) = \rho_n(x)$, $\widetilde{V}(\widetilde{x}) = \frac{1}{2}V(x)$ is even function. Assume $\mathrm{supp}\tilde{\rho} = [-a, a]$, then Euler-Lagrange equation with the normalization condition are

$$\int_{-a}^{a} \ln|x-y|^{-1}\widetilde{\rho}(y)\,\mathrm{dy} + \widetilde{V}(x) = \mathrm{const}, \qquad \int_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \widetilde{\rho}(x)\,\mathrm{dx} = 1.$$

$$-\int_{-a}^{a} \frac{\widetilde{\rho}(y)\,\mathrm{dy}}{y-x} + \widetilde{V}'(x) = 0.$$

$$G(z) := -i\int_{-a}^{a} \frac{\widetilde{\rho}(y)}{y-z}\,\mathrm{dy} \quad \text{Hilbert transform of } \tilde{\rho},$$

$G(z)$ is analytic on $\mathbb{C}\setminus[-a,a]$ and with limit values given by

$$G_{\pm}(x) = \lim_{\varepsilon\to 0}\frac{1}{i}\int \frac{\widetilde{\rho}(y)\,\mathrm{dy}}{y-(x\pm i\varepsilon)} \quad = -i\,\mathrm{p.\,v.}\int \frac{\widetilde{\rho}(y)\,\mathrm{dy}}{y-x} \pm \pi\widetilde{\rho}(x).$$

$$G_{\pm}(x) = \pm\pi\widetilde{\rho}(x) + i\widetilde{V}'(x), \quad \tilde{\rho}(x) = \frac{1}{\pi}\Re[G_{+}(x)]$$

$G(z)$ is the solution of a non-standard Riemann-Hilbert problem:

$$\begin{aligned}
&G_{+}(x) + G_{-}(x) = 2i\widetilde{V}'(x), \quad x\in[-a,a],\\
&G_{+}(x) - G_{-}(x) = 0, \quad x\notin[-a,a],\\
&G(z)\to 0,\ z\to\infty.
\end{aligned}$$

Introduce $\widetilde{G}(z) = \frac{G(z)}{\sqrt{z^2-a^2}}$, it is a solution of the standard Riemann–Hilbert problem and is given by the Plemelj formula

$$\widetilde{G}(z) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{2i\widetilde{V}'(s)\mathrm{ds}}{\left(\sqrt{s^2-a^2}\right)_{+}(s-z)}.$$

$a=\sqrt{c}$ is then computed from the asymptotic of $G(z)$ for $z\to\infty$. Then

$$\widetilde{\rho}(x)=\frac{1}{\pi^2}\Re\left[\sqrt{x^2-c}\int_{-\sqrt{c}}^{\sqrt{c}}\frac{\frac{1}{2}\left(\ln\left(\frac{c+1}{2}+s\right)-\ln\left(\frac{c+1}{2}-s\right)\right)}{\left(\sqrt{s^2-c}\right)_+(s-x)}\mathrm{ds}\right].$$

To compute the integral notice that the function $\frac{1}{\pi}\ln\left|\frac{s-(c+1)/2}{s+(c+1)/2}\right|$ is the Hilbert transform of the indicator function $\mathbf{1}_{[-(c+1)/2,(c+1)/2]}$ and use the following well-known relation:

$$\int_{-\infty}^{\infty}f(s)\widetilde{g}(s)\ \mathrm{ds}=-\int_{-\infty}^{\infty}\widetilde{f}(s)g(s)\ \mathrm{ds},$$

where $\widetilde{f}$ is a Hilbert transform of f and $f\in L^p(\mathbb{R})$, $g\in L^q(\mathbb{R})$ with $\frac{1}{p}+\frac{1}{q}=1$.

Variational problem for SO_{2l+1}.

$$\mu_{n,k}(\lambda) = \frac{2^{-l^2+2l-lk}l!}{(2l)!(2l-2)!\dots 2!} \times \prod_{m=1}^{l} \frac{(2k+2m-2)!}{2^{2m-2}\left(\frac{2k+a_m+2l-1}{2}\right)!\left(\frac{2k-a_m+2l-1}{2}\right)!} \times \prod_{s=1}^{l} a_s^2 \times \prod_{i<j}\left(a_i^2 - a_j^2\right)^2.$$

Consider the limit $n, k \to \infty$ s.t. $\lim \frac{2k}{n} = c$. $\{a_i\}$ are taking integer values in $[0, n(c+1)]$.

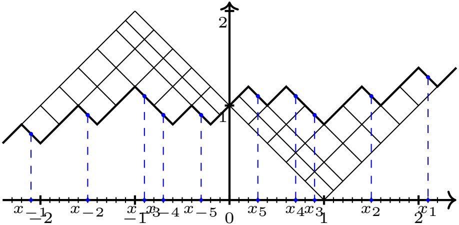


Figure: Rotated and scaled diagram for SO_{2l+1} with $l = 5$ and its continuation to negative values of coordinate x. The function $f_l(x)$ is shown in solid black, the points $x_i = \frac{a_i}{2l}$ are the midpoints of intervals, where $f_l'(x) = -1$.

Let a_{2l+1-i} be the "mirror image" of a_i: $a_{2l+1-i} \equiv -a_i$. Then we get the same variational problem, but need only half-interval. $V(u)$ is the same as in GL case:

$$\mu_{n,k}(\{a_i\}_{i=1}^{2l}) = \frac{1}{Z_l} \prod_{\substack{i \neq j \\ i,j=1}}^{2l} |a_i - a_j| \cdot \prod_{s=1}^{2l} \exp\left[-(4l)V\left(\frac{a_s}{4l}\right) - e_l(a_s)\right],$$

Proof of the convergence to the limit shape

Write J in terms of the (shifted) upper boundary $\tilde{f}_n$ as

$$J[\tilde{f}_n] = Q[\tilde{f}_n] + C, \quad Q[\tilde{f}_n] = \frac{1}{2} \iint\limits_0^{(c+1)/2} \tilde{f}_n'(x)\tilde{f}_n'(y) \ln|x-y|^{-1} \,\mathrm{dx}\ \mathrm{dy}.$$

Q is positive-definite on compactly-supported Lipschitz functions. Introduce a norm

$$\|f\|_Q = Q[f]^{1/2}.$$

Introduce a metric d_Q on a space of 1-Lipschitz functions f_1, f_2, such that $f_{1,2}'(x) = \operatorname{sgn} x$ for $|x| > \frac{c+1}{2}$:

$$d_Q(f_1, f_2) = \|f_1 - f_2\|_Q.$$

Then $\|f\|_\infty = \sup_x |f(x)| \le C_1 Q[f]^{1/4}$, where C_1 is constant.
The probability of the diagram that differs (by d_Q) from the limit shape by ε is bounded by $\mu_n(\lambda) \le C_2 e^{-n^2\varepsilon^2 + \mathcal{O}(n\ln n)}$.
Total number of diagrams in the $n \times k$ box is estimated as $C_3 e^{C_4 n}$ by Hardy-Ramanujan formula. The probability that $d_Q(\tilde{f}_n, \tilde{f})_Q > \varepsilon$ is bounded by $e^{-n^2\varepsilon^2 + \mathcal{O}(n\ln n)}$, that is

$$\mathbb{P}\left(\|\tilde{f}_n - \tilde{f}\|_Q > \varepsilon\right) \xrightarrow[n\to\infty]{} 0.$$

Limit shape as a level line of rectangular Young tableaux

P. Sniady and G. Panova considered the decomposition $\bigwedge^m \left(\mathbb{C}^n \otimes \mathbb{C}^k\right) = \bigoplus_{|\lambda|=m} V_{GL_n}(\lambda) \otimes V_{GL_k}(\lambda')$ and proved

$$\mu_{n,k}^{\langle m \rangle}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \dim V_{GL_k}(\lambda')}{\dim \bigwedge^m \left(\mathbb{C}^n \otimes \mathbb{C}^k\right)} = \frac{f^{\lambda} f^{\bar{\lambda}}}{f^{n^k}},$$

where f^{λ} is the dimension S_m-irrep, n^k – rectangular Young diagram with n rows and k columns. Then λ has the same distribution as diagram of boxes with entries $< m$ of a uniformly random rectangular $n \times k$ Young tableau. The limit shape is the same as the level lines of the limit shape for plane partitions by Romik and Pittel. Since $\bigwedge \left(\mathbb{C}^n \otimes \mathbb{C}^k\right) = \bigoplus_{m=0}^{nk} \bigwedge^m \left(\mathbb{C}^n \otimes \mathbb{C}^k\right)$, we have

$$\mu_{n,k}(\lambda) = \sum_{p=0}^{nk} \frac{\mu_{n,k}^{\langle m \rangle}(\lambda) \binom{nk}{m}}{2^{nk}}.$$

In the limit $n, k \to \infty$, the binomial distribution concentrates on the point $m = \frac{1}{2} nk$. Therefore, the limit shape for (GL_n, GL_k) coincides with the limit shape for $\mu_{n,k}^{\langle \frac{1}{2} nk \rangle}(\lambda)$ and is the same as the corresponding level line of the plane partitions in the box. Borodin-Olshanski 07: lhs is Krawtchouk ensemble. Is there such a relation for SO, Sp?

Principal specialization of dual Cauchy identity

For (GL_n, GL_k) we have dual Cauchy identity

$$\sum_{\lambda \subseteq k^n} s_\lambda(x_1, \dots, x_n) s_{\lambda'}(y_1, \dots, y_k) = \prod_{i=1}^{n} \prod_{j=1}^{k} (1 + x_i y_j).$$

Use $\mathrm{ch}(V(\overline{\lambda}'))(y_1, \dots, y_k) = \prod_{j=1}^{k} y_i^{\overline{\lambda}'_1 - n} \mathrm{ch}(V(\lambda')^*)(y_1, \dots, y_k) = \prod_{j=1}^{k} y_i^n \mathrm{ch}(V(\lambda'))(y_1^{-1}, \dots, y_k^{-1})$ and substitute $y_i \mapsto y_i^{-1}$ to account for the $\lambda' \to \overline{\lambda}'$ change, and then multiply by $y_1^n \cdots y_k^n$ to obtain

$$\sum_{\lambda \subseteq k^n} s_\lambda(x_1, \dots, x_n) s_{\bar{\lambda}'}(y_1, \dots, y_k) = \prod_{i=1}^{n} \prod_{j=1}^{k} (x_i + y_j).$$

The measure can be introduced as

$$\mu_{n,k}(\lambda | \{x\}, \{y\}) = \frac{s_\lambda(x_1, \dots, x_n) s_{\bar{\lambda}'}(y_1, \dots, y_k)}{\prod_{i=1}^{n} \prod_{j=1}^{k} (x_i + y_j)}.$$

In particular, we are interested in principal specialization $x_i = y_i = q^{i-1}$, where $s_\lambda(1, q, \dots, q^{n-1}) = q^{\|\lambda\|} \dim_q(V_{GL_n}(\lambda))$

q-deformation of limit shape

Take $x_m = y_m = q^{m-1}$, consider the limit $n, k \to \infty$, $q \to 1$, s.t. $k/n \to c$, $q \sim 1 - b/n$. From numerics limit shapes depend on c, b.

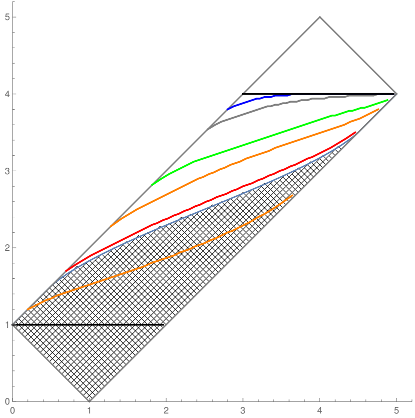


Figure: Most probable Young diagram from the measure $\mu_{n,k}(\lambda|q)$ for GL_{50}, GL_{150} and $k = 150$ for $b = -0.5, 0.1, 0.5, 2, 10, 20$. For $b = \pm\infty, q = \pm\text{const}$ we get horizontal lines.

q-deformation of limit shape and q-Krawtchouk ensemble

Use principal specialization in dual Cauchy for (GL_n, GL_k) identity

$$\mu_{n,k}(\lambda; q) = \frac{q^{\|\lambda\|}\dim_q\big(V_{GL_n}(\lambda)\big)\cdot q^{\|\overline{\lambda}'\|}\dim_q\big(V_{GL_n}(\overline{\lambda}')\big)}{N^A_{n,k}(q)},$$

$$N^A_{k,n}(q) = q^{P_{k-1}+(n-k)\binom{k}{2}}2^k\prod_{i=1}^{k-1}(q^i+1)^{2(k-i)}\cdot\prod_{j=k+1}^{n}\prod_{i=1}^{k}(q^{j-i}+1)$$

with $P_k = \frac{k(k+1)(2k+1)}{6}$. Then

$$\mu_{n,k}(\lambda; q) = \frac{1}{Z(q)}\cdot\frac{q^{\|\lambda\|+\|\lambda'\|}}{\prod_{i=1}^n[a_i]_q![k+n-1-a_i]_q!}\cdot\prod_{i<j}[a_i-a_j]_q^2$$

q-Krawtchouk polynomials are defined by the weight $\frac{(q^{-N};q)_x}{(q;q)_x}(-p)^{-x}$ on the lattice q^{-x}, where q-Pochhammer symbols are $(\alpha;q)_k = \prod_{i=1}^k(1-\alpha q^{i-1})$. If we take $N = k+n-1$, $x = a_i$, $p = \frac{1}{q^{2n-1}}$, we recover the weight above.

Using approach of Borodin and Olshansky, we were able to derive the limit shape, the proof of convergence is not complete yet.

Limit shape for q-Krawtchouk ensemble

Use difference equation on q-Krawtchouk polynomials, write it as a difference operator, demonstrate that it is a spectral projection and compute the spectral density in the limit. This would be $\rho(x)$.

$$\rho(x;b,c) = \frac{1}{\pi}\arccos\left(\frac{1}{2}e^{b(c-1-x)/2}\left(\frac{e^{b(c+1)}-e^{2b}}{e^{b(c+1)}-e^{bx}}\right)\sqrt{\frac{1-e^{-b(c+1-x)}}{1-e^{-bx}}}\right)$$

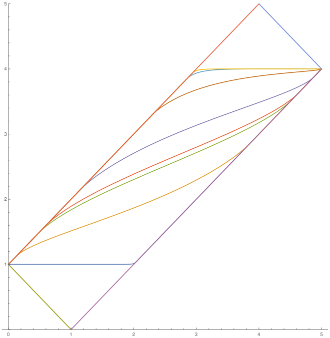


Figure: Limit shape for $c = 4$ and $b = -50, -0.5, 0.01, 0.1, 0.5, 2, 10, 25$.

Limit shape for q, q^{-1}-specialization

Principal specialization in q for GL_n and in q^{-1} for GL_k, $x_m = q^{m-1}$, $y_m = q^{1-m}$, use $[m]_{1/q} = q^{1-m}[m]_q$, the measure is

$$\mu_{n,k}(\lambda; q) = \frac{1}{Z_{n,k}} q^{\sum_{i=1}^{n} \binom{a_i}{2} + (n-1)a_i} \prod_{i<j} \left(q^{-a_i} - q^{-a_j}\right)^2 \cdot \prod_{i=1}^{n} \begin{bmatrix} n+k-1 \\ a_i \end{bmatrix}_q$$

We again see q-Krawtchouk ensemble, limit shape is given by

$$\rho(x; b, c) = \frac{1}{\pi} \arccos\left(\frac{1}{2} \frac{e^{-b/2}\left(-e^{b(x-c)} + e^{bx} - e^{b} + 1\right)}{\sqrt{(1-e^{bx})\left(e^{b(x-(c+1))} - 1\right)}} \right)$$

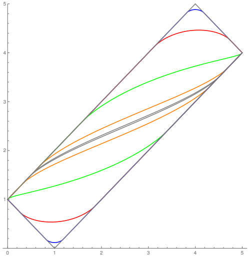


Figure: $c = 4$, $b = -10, -2, -0.5, -0.1, -0.01, 0.01, 0.1, 0.5, 2, 10.$

Possible further developments

- Skew Howe duality for Lie superalgebras
- We can prove central limit theorem for global fluctuations using orthogonal polynomials. We have Krawtchouk polynomial ensemble. Breuer-Duits '16, Johannson '02
- Prove the convergence for q-deformed case
- q-dimensions are principal specialization of the characters. For (GL_n, GL_k) consider dual Cauchy identity and the measure:

$$\sum_{\lambda \subseteq k^n} s_\lambda(x_1, \ldots, x_n) s_{\bar{\lambda}'}(y_1, \ldots, y_k) = \prod_{i=1}^{n} \prod_{j=1}^{k} (x_i + y_j).$$

$$\mu_{n,k}(\lambda | \{x\}, \{y\}) = \frac{s_\lambda(x_1, \ldots, x_n) s_{\bar{\lambda}'}(y_1, \ldots, y_k)}{\prod_{i=1}^{n} \prod_{j=1}^{k} (x_i + y_j)}.$$

The limit shape for $n, k \to \infty$ and $x_i = e^{\varphi(i/n)}, y_j = e^{\psi(j/n)}$ with smooth φ, ψ is supposedly described by Burgers equation, as in lozenge tilings.

Thank you for your attention!