Some open problems in Invariant Theory

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Frobenius subalgebras of simple Lie algebras

Notation. $\Bbbk = \overline{\Bbbk}$, $\operatorname{char} \Bbbk = 0$.

– $\mathfrak{q}$ is any algebraic Lie algebra and $\operatorname{ind} \mathfrak{q}$ is the *index* of $\mathfrak{q}$;

– $\mathfrak{q}$ is called a *Frobenius* Lie algebra, if $\operatorname{ind} \mathfrak{q} = 0$ (i.e., Q has a dense orbit in $\mathfrak{q}^*$ or $\det \mathcal{M} \neq 0$, where $\mathcal{M}_{ij} = [x_i, x_j]$);

– $\mathfrak{g}$ is a simple Lie algebra.

Problem 1. *Determine the maximal dimension of a Frobenius subalgebra of* $\mathfrak{g}$.

Example 1. If $\mathfrak{g} = \mathfrak{sl}_n$ (or $\mathfrak{gl}_n$), then $\max \dim(\text{frob-subalg}) = n^2 - n$.

This value is attained on a maximal parabolic subalgebra, which is also a subalgebra of $\mathfrak{sl}_n$ of maximal dimension.

Fact 1 (В.В.Морозов, 1943). If $\mathfrak{q} \subsetneq \mathfrak{g}$ is a maximal subalgebra, then $\mathfrak{q}$ is semisimple or regular, i.e., is normalised by a Cartan subalgebra (“Теорема регулярности”). He also gives lists of maximal regular subgroups.

Fact 2 (Ф.И.Карпелевич, 1951). A maximal nonsemisimple subalgebra of $\mathfrak{g}$ is parabolic.

Example 2. $\mathfrak{g} = \mathbf{G}_2$. The maximal subalgebras are:

(a) Two maximal parabolic subalgebras, where dim $= 9$ & ind $= 1$;

(b) regular: $\mathfrak{sl}_3$ or $\mathfrak{sl}_2 \dotplus \mathfrak{sl}_2$ with ind $= 2$ & the S-subalgebra $\mathfrak{sl}_2$.

Here a Borel $\mathfrak{b} = \mathfrak{b}(\mathbf{G}_2)$ is a Frobenius subalgebra of maximal dimension, $\dim \mathfrak{b} = 8$.

It is known for a long time (see e.g. В.В.Трофимов, 1979-80) that

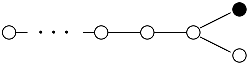
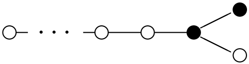
- $\operatorname{ind} \mathfrak{b} = 0$ if and only if $\mathfrak{g} \notin \{\mathbf{A}_n, \mathbf{D}_{2n+1}, \mathbf{E}_6\}$;
- $\operatorname{ind} \mathfrak{b}(\mathfrak{sl}_{n+1}) = [n/2]$, $\operatorname{ind} \mathfrak{b}(\mathfrak{so}_{4n+2}) = 1$, and $\operatorname{ind} \mathfrak{b}(\mathbf{E}_6) = 2$.

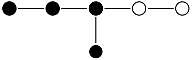
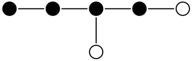
Question 1. *Suppose that* $\operatorname{ind} \mathfrak{b} = 0$. *Is it true that* $\mathfrak{b}$ *is a Frobenius subalgebra of maximal dimension?*

- *If* $\operatorname{ind} \mathfrak{b} = 0$, *then* $\operatorname{ind} \mathfrak{p} > 0$ *for any* $\mathfrak{p} \supsetneq \mathfrak{b}$. This exploits the general formula for the index of seaweed subalgebras of $\mathfrak{g}$ (Joseph, Tauvel–Yu).
- *If* $\operatorname{ind} \mathfrak{b} > 0$, *then there do exist Frobenius parabolic subalgebras* $\mathfrak{p}$.

Question 2. *Is it true that the maximum of dimension of the Frobenius subalgebras of* $\mathfrak{g}$ *is always attained on parabolic subalgebras?*

Example 3. If $\mathfrak{g}$ is of type $\mathbf{D}_{2n+1}$, then $\dim \mathfrak{b} = (2n+1)^2$ and there are Frobenius parabolics of dimension $\dim \mathfrak{b} + 1$ and $\dim \mathfrak{b} + 3$:

 and  .

Example 4. If $\mathfrak{g}$ is of type $\mathbf{E}_6$, then $\dim \mathfrak{b} = 42$ and there are Frobenius parabolics of dimension 52:  or  .

One can verify that these are Frobenius parabolics of maximal dimension.

Strange orbits

Def. (M. Raïs). The coadjoint orbit $Q{\cdot}\xi \subset \mathfrak{q}^*$ is said to be *strange*, if there is a subalgebra $\mathfrak{h} \subset \mathfrak{q}$ such that $\mathfrak{h} \oplus \mathfrak{q}^\xi = \mathfrak{q}$ (the vector space direct sum).

Lemma 1. *If* $\mathfrak{h} + \mathfrak{q}^\xi = \mathfrak{q}$*, then* $\operatorname{ind}\mathfrak{h} \leqslant \dim(\mathfrak{q}^\xi \cap \mathfrak{h})$ *(equivalently,* $\dim Q{\cdot}\xi \leqslant \dim\mathfrak{h} - \operatorname{ind}\mathfrak{h}$*.) In particular, if* $\mathfrak{h} \oplus \mathfrak{q}^\xi = \mathfrak{q}$*, then* $\mathfrak{h}$ *is Frobenius.*

Proof. Here $\mathfrak{h}^* \simeq \mathfrak{q}^*/\mathfrak{h}^\perp$ and $\bar{\xi} \in \mathfrak{q}^*/\mathfrak{h}^\perp$ yields a suitable H-orbit in $\mathfrak{h}^*$. □

Vinberg's inequality: $\operatorname{ind}\mathfrak{q}^\xi \geqslant \operatorname{ind}\mathfrak{q}$. (This can be strict!)

Question 3. *Is it true that if* $Q{\cdot}\xi$ *is strange, then* $\operatorname{ind}\mathfrak{q}^\xi = \operatorname{ind}\mathfrak{q}$*?*

For $\mathfrak{g}$, we have $\mathfrak{g} \simeq \mathfrak{g}^*$, 'coadjoint' = 'adjoint', and $\operatorname{ind}\mathfrak{g}^\xi = \operatorname{ind}\mathfrak{g}$ $\forall\xi$ (the Elashvili "conjecture" = "гипотеза" А.Г.Элашвили). By Lemma 1,

$$\max\dim(\text{strange orbit}) \leqslant \max\dim(\text{frob-subalg}) \qquad (*).$$

Question 4. *Is it true that the equality holds in* $(*)$*?*

The answer is "yes" for

- $\mathfrak{g} = \mathfrak{sl}_n$, where $\mathcal{O}_{\text{reg}} \subset \mathfrak{N}$ is strange;
- $\mathfrak{g} = \mathbf{G}_2$, where $\mathcal{O}_8 \subset \mathfrak{N}$ is strange.

It is easily seen that $\mathcal{O}_{\min} \subset \mathfrak{g}$ is always strange.

Proposition 1 (P.).
(i) *If $\mathcal{O} \subset \mathfrak{N}$ and $c_G(\mathcal{O}) \leqslant 1$, then $\mathcal{O}$ is strange. Moreover, if $c_G(\mathcal{O}) = 0$, then the complementary subalgebra $\mathfrak{h}$ can be chosen to be solvable.*
(ii) *If $e \in (\mathfrak{sl}_n)_{\mathsf{reg}} \cap \mathfrak{N}$, then $SL_n{\cdot}e^k$ is strange.*

- If $\mathcal{O} \subset \mathfrak{N}$ and $c_G(\mathcal{O}) = 1$, then $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathcal{O} \sim (3, 1, \dots, 1)$;
- If ind $\mathfrak{b} = 0$, then ("yes" in Question 2) $\Rightarrow$ ("yes" in Question 4).
- If $\mathcal{O} = G{\cdot}e$ is strange and $c_G(\mathcal{O}) > 0$, then a complementary subalgebra $\mathfrak{h}$ for $\mathfrak{g}^e$ cannot be solvable;
- there can be complementary subalgebras $\mathfrak{h}$ with different structure. See e.g. $\mathcal{O} \sim (2, \dots, 2)$ for $\mathfrak{sl}_{2n}$.

Proposition 2 (P.). *Suppose that $\mathcal{O} \subset \mathfrak{N}$ is strange. Let $\mathcal{S}_{\mathcal{O}}$ be a sheet of $\mathfrak{g}$ that contains $\mathcal{O}$. Then all G-orbits in $\mathcal{S}_{\mathcal{O}}$ are strange. Moreover, if $e \in \mathcal{O}$ and $\mathfrak{g}^e \oplus \mathfrak{h} = \mathfrak{g}$, then this $\mathfrak{h}$ is also valid for every orbit in $\mathcal{S}_{\mathcal{O}}$.*

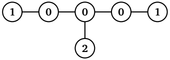
Remark. For $\mathbf{G}_2, \mathbf{F}_4$, and $\mathbf{E}_8$, all spherical nilpotent orbits are rigid. In the other series, the non-rigid spherical orbits are Richardson, i.e., $\mathcal{S}_{\mathcal{O}} \neq \mathcal{O}$.

Problem 2. *Classify the strange (nilpotent) orbits in the simple Lie algebras.*

Question 5. *Suppose that* ind $\mathfrak{b} = 0$.
Is it true that only the spherical orbits are strange? [Yes, for $\mathbf{G}_2$.]

Question 6. *Are there non-spherical strange orbits for* $\mathbf{D}_{2m+1}$ *or* $\mathbf{E}_6$ *?*

Example 5. For $\mathfrak{g} = \mathbf{E}_6$, there is a unique nilpotent orbit of dim=52.

A_3: 

. Can one prove or disprove that this orbit is strange?

- One always has $\dim \mathfrak{N}^{\mathsf{sph}} \leqslant \dim \mathfrak{b} - \text{ind}\, \mathfrak{b}$. (Lemma 1 with $\mathfrak{h} = \mathfrak{b}$.) Actually, $\dim \mathfrak{N}^{\mathsf{sph}} = \dim \mathfrak{b} - \text{ind}\, \mathfrak{b}$ (P., *A.I.F.* '99);
- if $\mathfrak{p}$ is a minimal Frobenius parabolic, then $\dim \mathfrak{p} = \dim \mathfrak{b} + \text{ind}\, \mathfrak{b}$.

Aside questions:

1. Why is $\mathfrak{N}^{\mathsf{sph}}$ irreducible?
2. Suppose that $\mathfrak{h} \oplus \mathfrak{r} = \mathfrak{g}$. Is it true that $\text{ind}\, \mathfrak{h} + \text{ind}\, \mathfrak{r} \geqslant \text{ind}\, \mathfrak{g}$? (One should assume that $\mathfrak{g}$ is semisimple.)

Философский Тезис:
Если имеется проблема/ситуация, включающая параметр, и для значения параметра n_0 всё здорово и прекрасно, то для n_0+1 кое-что тоже может быть хорошо, но при дополнительных ограничениях. А при n_0+2 всё уже нередко плохо!

1. Rational vs. unirational varieties (the Lüroth problem), dim $=1$ and 2;
2. $c_G(X)=0$ and 1, where X is a G-variety (numerous topics!);
3. $r_G(X)=0$ and 1;
4. $\dim V/\!\!/G=1$ and 2, where V is a G-module.
5. …

Recollections: Let X be irreducible and $(G:X)$. Then

- $c_G(X)=\dim X-\max_{x\in X}\dim B{\cdot}x$ is the complexity of X;
- if X is quasiaffine and $\Bbbk[X]^U=\bigoplus_{\lambda\in\Gamma}\Bbbk[X]^U_\lambda$, then $r_G(X)=\dim_{\mathbb Q}(\mathbb Q\Gamma)$ is the rank of X. Here $\Gamma=\Gamma(X)$ is a monoid of dominant weights.

Factorisations of a simple algebraic group G

H_1 and H_2 are connected reductive subgroups of G.

Def. (А.Л.Онищик). The triple (G, H_1, H_2) is a *factorisation* (of G) if

$$H_1 \text{ acts transitively on } G/H_2. \qquad (\Diamond)$$

Then any $g \in G$ can be written as $g = h_1 h_2$. The factorisations of simple algebraic groups have been studied (and classified) by А.Л.Онищик (Труды ММО, т.**11**, 1962). If $(\Diamond)$ holds, then a generic stabiliser for $(H_1 : G/H_2)$ equals $S = H_1 \cap H_2$ and $G/H_2 \simeq H_1/S$.

- It is clear that $\dim G + \dim S = \dim H_1 + \dim H_2$;
- condition $(\Diamond) \Leftrightarrow \mathfrak{h}_1 + \mathfrak{h}_2 = \mathfrak{g}$ (Онищик, 1969).

Example 1. $G = SL_{2n}$, $H_1 = Sp_{2n}$, and $H_2 = SL_{2n-1}$. Then $S = Sp_{2n-2}$.

Let $\mathcal{P}(G; z)$ denote the *Poincaré polynomial* of (a compact real form of G). If $m_i = d_i - 1$ ($i = 1, \dots, l = \operatorname{rk} G$) are the *exponents* of G, then

$$\mathcal{P}(G; z) = \prod_{i=1}^{l} (1 + z^{2m_i+1}).$$

Note that $\deg \mathcal{P}(G; z) = \dim G$. Let $\exp(G) := \{m_1, \dots, m_l\}$

Set $$\mathcal{Q}(z) = \frac{\mathcal{P}(G;z)\cdot\mathcal{P}(S;z)}{\mathcal{P}(H_1;z)\cdot\mathcal{P}(H_2;z)}.$$

Using cohomological methods, А.Л.Онищик proved that $\mathcal{Q}(z) \equiv 1$ for the factorisations. This readily implies that

1. $\mathrm{rk}\, G + \mathrm{rk}\, S = \mathrm{rk}\, H_1 + \mathrm{rk}\, H_2$;
2. $\exp(H_1) \cup \exp(H_2) \underset{multi}{=} \exp(G) \cup \exp(S)$;
3. either H_1 or H_2 is a subgroup of **maximal** exponent in G.

Problem 1. *Find another (more algebraic? invariant-theoretic?) proof.*

Fact 1. In Onishchik's list, at least one of the subgroups H_i is spherical.

Problem 2. *Prove/explain this.*

Actually, if $H \subset G$ is a subgroup of maximal exponent (e.g. $\mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$ or $\mathbf{F}_4 \subset \mathbf{E}_6$), then H appears to be spherical. *Why?*

- If (G, H_1, H_2) is factorisation, then $\mathfrak{s} \neq 0$;
- There is an application of factorisations to classifying the spherical homogeneous spaces of G (И.В. Микитюк, *Матем. Сб.,* 1986).

Quasi-factorisations of G

Def. (P., 1992) The triple (G, H_1, H_2) is called a *quasi-factorisation* (of G) if a generic H_1-orbit in G/H_2 is of codimension 1.

Then $\dim\big((G/H_2)/\!\!/H_1\big) = 1$, hence $\Bbbk[G/H_2]^{H_1} = \Bbbk[f]$ for some polynomial f. Let S be a generic stabiliser for $(H_1 : G/H_2)$.

- If $\{H_2\} \in G/H_2$ is a generic point, then $S = H_1 \cap H_2$;
- in general, $S = H_1 \cap g{\cdot}H_2{\cdot}g^{-1}$ for a **suitable** $g \in G$.
- Here $\dim G + \dim S = \dim H_1 + \dim H_2 + 1$.

More suggestive (symmetric) notation: $H_1\backslash\!\!\backslash G/\!\!/H_2 = \operatorname{Spec}\big({}^{H_1}\Bbbk[G]^{H_2}\big)$.

My observations. For all known examples of quasi-factorisations, one has

1. $\mathrm{rk}\, G + \mathrm{rk}\, S = \mathrm{rk}\, H_1 + \mathrm{rk}\, H_2 \pm 1$ (2 possibilities);
2. At least one homogeneous space G/H_i is of complexity $\leqslant 1$;
3. either H_1 or H_2 is a subgroup of **submaximal** exponent in G.

Problem 3. *Prove all/some of this and explain the rôle of* ± 1.

Problem 4. *Classify all quasi-factorisations of simple algebraic groups.*

Lemma 1 (P. 1992). *(G, H, H) is a quasi-factorisation if and only if G/H is a spherical homogeneous space of rank 1. Then $\mathsf{rk}\, G = \mathsf{rk}\, S - 1$ and either $\mathsf{rk}\, H = \mathsf{rk}\, G$* (the (-1)-case), *or $\mathsf{rk}\, H = \mathsf{rk}\, S$* (the $(+1)$-case).

Proof. $\dim(H \backslash\!\backslash G /\!\!/ H) = 2c_G(G/H) + r_G(G/H)$ and $r_G(G/H) = \mathsf{rk}\, G - \mathsf{rk}\, S$. □

Example 2. $G = SO_n$ and $H_1 = H_2 = SO_{n-1}$. Then $S = SO_{n-2}$ and the '+1' -case occurs if and only if n is even.

For the quasi-factorisations, $\mathcal{Q}$ is a rational function in z of degree 1.

- In all examples, we have $Q(1) = 2$ or $1/2$.

Question 1. Is there a geometric meaning of $\mathcal{Q}$ for quasi-factorisations?

Example 3. Some quasi-factorisations $\mathfrak{g} \supset (\mathfrak{h}_1, \mathfrak{h}_2) \supset \mathfrak{s}$:

- ◇ (-1) $\mathbf{B}_n \supset (\mathbf{D}_n, \mathbf{D}_n) \supset \mathbf{B}_{n-1}$, $\mathcal{Q} = (1 + z^{4n-1})/(1 + z^{2n-1})^2$;
- ◇ (-1) $\mathbf{E}_6 \supset (\mathbf{F}_4, \mathbf{D}_5 \dotplus \mathfrak{t}_1) \supset \mathbf{B}_3$, $\mathcal{Q} = (1 + z^{17})/(1 + z)(1 + z^{15})$;
- ◇ $(+1)$ $\mathbf{D}_4 \supset (\mathbf{B}_3, \mathbf{G}_2) \supset \mathbf{A}_2$, $\mathcal{Q} = (1 + z^5)(1 + z^7)/(1 + z^{11})$.

It is important to keep track of the embeddings $H_i \hookrightarrow G$.

$$\mathfrak{so}_8 \supset (\mathfrak{so}_7, \mathfrak{spin}(7)) \supset \mathbf{G}_2 \quad \text{vs.} \quad \mathfrak{so}_8 \supset (\mathfrak{so}_7, \mathfrak{so}_7) \supset \mathfrak{so}_6$$

factorisation (А.Л.Он.) quasi-factorisation (P.).

A generalisation of Example 2:

$$\mathfrak{so}_n \supset (\mathfrak{so}_{n-1},\ \mathfrak{so}_k \dotplus \mathfrak{so}_{n-k}) \supset \mathfrak{so}_{k-1} \dotplus \mathfrak{so}_{n-k-1},$$

where $1 \leqslant k \leqslant n-k$. Here the '+1'-case occurs if and only if $n+k$ is odd.

- It can happen that $\mathfrak{s} = \{0\}$, e.g. $\mathbf{B}_3 \supset (\mathbf{G}_2, \mathbf{A}_1 \dotplus \mathbf{A}_1) \supset \{0\}$.

Two related exotic cases:

1. $\mathbf{D}_8 \supset (\mathbf{B}_7, \mathbf{B}_4) \supset \mathbf{B}_3$ – factorisation (Онищик);
2. $\mathbf{D}_8 \supset (\mathbf{D}_7, \mathbf{B}_4) \supset \mathbf{A}_2$ – quasi-factorisation (P.).

Here the embedding $\mathbf{B}_4 \hookrightarrow \mathfrak{so}_{16}$ is given by the spinor representation and $SO_{16}/Spin(9)$ is an isotropy irreducible homogeneous space.

⊛ Note that $c(\mathbf{D}_8/\mathbf{B}_7) = 0$, $c(\mathbf{D}_8/\mathbf{D}_7) = 1$, and $c(\mathbf{D}_8/\mathbf{B}_4) = 20$.

▶ There is an application of (certain!) quasi-factorisations to classifying the homogeneous spaces of G of complexity 1 (Panyushev, 1992).

References for Part 2

- А.Л. Онищик. Отношения включения между транзитивными компактными группами преобразований, *Труды Моск. Матем. Об-ва*, т.**11** (1962), с.199–242.
- А.Л. Онищик. Разложения редуктивных групп Ли, *Матем. сб.*, т. **80**(122):4(12) (1969), с.553–599.
- D. Panyushev. Complexity of quasiaffine homogeneous varieties, t-decompositions and affine homogeneous spaces of complexity 1, in: E.B. Vinberg (Ed.), "*Lie groups, their discrete subgroups and Invariant Theory*" (Adv. Sov. Math., vol. **8**, pp.151–166) Providence: AMS 1992. [§ 3]
- D. Panyushev. Complexity and rank of actions in invariant theory, *J. Math. Sci.* (New York), **95** (1999), 1925–1985. [Chap. 3]

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