Symmetric representation theory of quivers & Lie theory

(jt. M. Boos)

Algebraic varieties in Lie theory with group action $\longleftrightarrow$ Rep. theory of f.d. associative algebras.

Ex:

0) conjugacy classes, $GL_n(\mathbb{C}) \curvearrowright \mathfrak{g} = \mathfrak{gl}_n$

1) 2-nilpotent B-orbits, $B \subset GL_n(\mathbb{C})$

2) E. Feigin's degenerate flag varieties.

Aim: Generalize those to the case where the group acting in $O(n)$ or $Sp(2n)$.

Tool: Algebras with self-duality.
Derksen-Weyman 2002 "symmetric quivers"

More precisely:

- Description of orbits and orbit closures.

Representation varieties of algebras

$$V = \bigoplus_{i \in Q_0} V_i \qquad V_i : \text{ f.d. } \mathbb{C}\text{- vector space}$$

$Q_0 = \{1, \dots, n\}$: finite set of "vertices".

$$R = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)$$

$Q_1 = \{\alpha : i \to j\}$: finite set of arrows.

$G = \prod_{i \in Q_0} GL(V_i)$ acts on R by "change of basis":

$$g = (g_i)_{i \in Q_0} \qquad x = (x_\alpha)_{\alpha : i \to j}$$

$$(g \cdot x)_\alpha = (g_j \, x_\alpha \, g_i^{-1})_{\alpha : i \to j}$$

$X \subset R$ affine subvarieties, closed under G,

$$X = \{ (x_\alpha) \mid x_p = 0 \}$$

$$p = \sum c_{i_1 \cdots i_k} \alpha_{i_1} \cdots \alpha_{i_k} \qquad \cdot \xrightarrow{\alpha_{i_1}} \cdot \longrightarrow \cdot \cdots \cdot \xrightarrow{\alpha_{i_k}} \cdot$$

$\underline{Ex}$: 0) $V = \mathbb{C}^n$ $\quad (Q_0 = \{1\})$

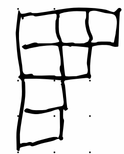
$$R = \mathrm{End}(V) \supset X = \mathcal{N}(n) = \{A \mid A^n = 0\}$$

$$Q_1 = \{\alpha : 1 \to 1\} \qquad Q : 1 \circlearrowleft \alpha$$

$G = GL_n(\mathbb{C}) \curvearrowright R$ by conjugation

$$\{G\text{-orbits}\} \longleftrightarrow \mathcal{P}(n) = \{\text{partitions of } n\}$$

$$G.\begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_h} \end{pmatrix} = A_\lambda \longleftarrow\!\shortmid \lambda = (\lambda_1 \geq \dots \geq \lambda_h)$$



$$J_{\lambda_i} = \begin{array}{c} \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \lambda_i \\ \lambda_i \end{array}$$

In $\mathcal{P}(n)$ there is $\underline{\text{dominance order}}$

$$\lambda \geq \mu \iff \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \quad \forall k$$

$$\iff \sum_{i \geq k} \hat{\lambda}_i \geq \sum_{i \geq k} \hat{\mu}_i \quad \forall k$$

$$\iff \mathrm{rk}\, A_\lambda^{k-1} \geq \mathrm{rk}\, A_\mu^{k-1}$$

<u>Theorem</u> [Gerstenhaber, Hesselink]:

$$\overline{G.A_\lambda} \ni A_\mu \iff \lambda \geq \mu$$

$$\left(G.A_\lambda = G A_\mu \iff \lambda = \mu\right)$$

1) $V = V_1 \oplus V_2$

$R = \mathrm{Hom}_{\mathbb{C}}(V_1, V_2) = X$

$G = GL(V_1) \times GL(V_2)$

$\{G\text{-orbits}\} \longleftrightarrow \{\text{ranks}\}.$

$Q: \ 1 \xrightarrow{\alpha} 2$

2) $V = V_1 \oplus V_2$

$R = \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)$

$\cup$

$X = \{ (A, B) \mid (AB)^n = 0, \ (BA)^n = 0 \} \quad n \gg 0.$

Kraft-Procesi

$Q: \ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$

Thm [Abeasis-Del Fra, Riedtmann, Bongartz, Zwara]:

Suppose that X contains finitely many G-orbits. Then $\forall x, y \in X$

$y \in \overline{G.x}$ $\iff$ $\dim \mathrm{Hom}_Q(x, z) \leq \dim \mathrm{Hom}_Q(y, z)$

where z varies in the finite set of "indecomposable" representations.

Ex: $\bullet \circlearrowleft \alpha \, / \, (\alpha^n)$ indec. $\{J_{\lambda_i}\}$

$\bullet \longrightarrow \bullet$ indec. $\mathbb{C} \xrightarrow{1} \mathbb{C}$

$\mathbb{C} \longrightarrow 0$

$0 \longrightarrow \mathbb{C}$

Ex: this theorem can be used to prove Gerstenhaber, Hesselink theorem on conjugacy. [Bongartz].

the construction of Boos-Reineke

$$F_\bullet = U_1 \overset{i_1}{\hookrightarrow} U_2 \overset{i_2}{\hookrightarrow} \cdots \hookrightarrow U_{n-1} \overset{i_{n-1}}{\hookrightarrow} U_n = \mathbb{C}^n = \mathrm{Span}(e_1, \dots, e_n)$$

$$U_i = \mathrm{Span}(e_1, \dots, e_i)$$

$$B = \mathrm{Stab}_{GL_m}(F_\bullet) = \{ g \in GL_m \mid g U_i = U_i \ \forall i = 1, \dots, m \}$$

$$X = \mathcal{N} = \{ A \in \mathfrak{gl}_m \mid A^n = 0 \}.$$

B acts on X.

$$G = \prod GL(V_i) \curvearrowright \mathcal{X} = \{ (f_1, \dots, f_{m-1}, A) \in \bigoplus \mathrm{Hom}(V_i, V_{i+1}) \times \mathcal{N} \mid A^k = 0 \}$$

$\downarrow p$ G-equivariant.

$$G \curvearrowright Y = \{ (f_1, \dots, f_{n-1}) \} \ni y_0 = (i_1, \dots, i_{n-1})$$

$$X = p^{-1}(G \cdot y_0) = \{ (f_i, A) \mid f_i \text{ is injective} \}.$$

$\downarrow \pi$

$$Y = G \cdot y_0$$

$$\left.\begin{array}{l} F = \pi^{-1}(y_0) = \mathcal{N}(k) = \{ A \mid A^k = 0 \}. \\ \mathrm{Stab}_G(y_0) \simeq B \\ (g|_{U_1}, \dots, g|_{U_n}) \leftarrow g \end{array}\right\} \Rightarrow \boxed{X \simeq G \times^B F}$$

the inclusion $F \hookrightarrow X$ induces bijection

$$\{ B\text{-orbits in } F \} \xleftrightarrow{1-1} \{ G\text{-orbits in } X \}$$

interesting object.

$\boxed{\text{preserves orbit closures.}}$

Can use Rep. theory of alg. To describe G-orbits and their closures.

Pb : If $K > 2$ There are infinitely many orbits. (Zwara's thm does not apply).

$\rightsquigarrow$ Assume $K = 2$.

Quiver with relation:

$$\mathbb{C}\left(1 \to 2 \to \cdots \to n \circlearrowleft \alpha\right) / (\alpha^2)$$

Rep. theory well-known!

Indec. $\longleftrightarrow$ connected walks avoiding the relation

Ex: $n=2$ $\qquad 1 \to 2 \circlearrowleft$

indec. $\quad 1 \qquad \begin{tikzcd}[row sep=small] 1 \arrow[d] \\ 2 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] 1 \arrow[d] \\ 2 \arrow[d] \\ 2 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] 1 \arrow[d] & \\ 2 \arrow[d] & 1 \arrow[dl] \\ 2 & \end{tikzcd}$

$2 \qquad \begin{tikzcd}[row sep=small] 2 \arrow[d] \\ 2 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] 2 \arrow[d, "\alpha"'] & 1 \arrow[dl] \\ 2 & \end{tikzcd}$

RK:

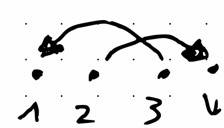
$G \cdot \begin{tikzcd}[row sep=small] 1 \arrow[d] \\ 2 \arrow[d] \\ 2 \end{tikzcd} \ni \begin{tikzcd}[row sep=small] 2 \arrow[d] & 1 \arrow[dl, "\alpha"] \\ 2 & \end{tikzcd}$

deg. ord. $\longleftrightarrow$ Zwara's theorem.

Boos-Reineke

$\{ B\text{-orbits in } N(2) \} \overset{1-1}{\longleftrightarrow} \{ \text{oriented link patterns} \}$

$$B \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow$$



Deg. order : Let x be an o.l.p. on n vertices and A_x the corresponding matrix. Define $\forall k \in \{1, .., m\}$

p_k^x = # {vertices $\leq k$ not incident with an arrow} + # {arrows of x whose target is $\leq k$}.

$q_{k,l}^x = p_k^x + \ldots$

thm [Boos-Reineke]

$\overline{B.A_x} \ni A_y \iff p_k^x \leq p_k^y \quad \forall k$

$q_{k,l}^x \leq q_{k,l}^y \quad \forall k, l.$

what about O and SP ?

Symmetric rep. varieties

$$V = \bigoplus_{i \in Q_0} V_i$$

endowed with a non-degenerate bilinear form

$$\langle , \rangle : V \times V \to \mathbb{C}$$

s.t.

1) $\langle , \rangle \big|_{V_i \times V_j} = 0$ unless $j = \sigma(i)$

where $\sigma : Q_0 \to Q_0$ is an involution.

$V_{\sigma(i)} \simeq V_i^*$ by $\langle , \rangle$.

Fix $\varepsilon = 1$ or $\varepsilon = -1$

2) $\langle v, w \rangle = \varepsilon \langle w, v \rangle \quad \forall v, w \in V$.

$$G(V, \langle , \rangle) = \{ g \in G \mid \langle gv, gw \rangle = \langle v, w \rangle \ \forall v, w \}$$

$$\simeq \prod_{i \neq \sigma(i)} GL(V_i) \times \prod_{i = \sigma(i)} \underbrace{G(V_i, \langle , \rangle|_{V_i})}_{\substack{O \text{ if } \varepsilon = 1 \\ Sp \text{ if } \varepsilon = -1.}}$$

$$R = \bigoplus_{i \to j} \mathrm{Hom}(V_i, V_j)$$

We want to consider anti-self-adjoint points of R.

$$V \xrightarrow{f} W \qquad f^{ad} : W \longrightarrow V$$

$\langle , \rangle_1 \qquad \langle , \rangle_2$

$$\langle f(v), w \rangle_2 = \langle v, f^{ad}(w) \rangle_1$$

$$V_i \xrightarrow{f} V_j \qquad V_{\sigma(j)} \xrightarrow{f^{ad}} V_{\sigma(i)}$$

Want: $\forall \alpha : i \to j \quad \exists \ \sigma(\alpha) = \beta : \sigma(j) \to \sigma(i)$

σ must induce a orientation reversing bijection on the arrows.

$$R \supset R^{\varepsilon} = \{ (f_\alpha)_{\alpha \in Q_1} \mid f_{\sigma(\alpha)} = -f_\alpha^{ad} \ \forall \alpha \}$$

$$\begin{array}{ccccc} X & \subset & R & \curvearrowleft & G \\ \cup & & \cup & & \vee \\ X^{\varepsilon} & \subset & R^{\varepsilon} & \curvearrowleft & G(V, \langle , \rangle) = G^{\varepsilon} \end{array}$$

<u>thm</u> [Magyar - Weyman - Zelevinsky, CI - Boos, Derksen - Weyman]

$\forall x \in X^{\varepsilon}$

$$G.x \cap X^{\varepsilon} = G^{\varepsilon}.x$$

<u>Ex</u>: Two symmetric $n \times n$ matrices are congruent iff they have the same rank.

<u>Thm2</u> [CI-Boos]: It's <u>not</u> True that

$$\overline{G.x} \cap X^{\varepsilon} = \overline{G^{\varepsilon}.x}$$

in general.

Counterexample :

$$M = \begin{array}{cc} \begin{array}{cc} \ell & \ell \end{array} & \\ \left(\begin{array}{c|c} & \\ \hline 1 & \\ \quad -1 & \end{array}\right) & \begin{array}{c} \ell \\ \ell \end{array} \end{array} \qquad N = \left(\begin{array}{c|c} 1 & \\ \hline & \\ & 1 \end{array}\right)$$

$M, N \in$ 2-nilpotent in Lie $(O(2n))$

then

$$N \in \overline{B.M} \qquad B \subset GL_{2n}$$

but

$$\dim B^{\varepsilon}.N = \dim B^{\varepsilon} M$$

where $B^{\varepsilon} \subset O(2n)$ borel.

<u>conj.</u> : In Type C it's True

<u>Melnikov</u> : True for B and C in the upper part.

thm 3 (CI-B00?):

"For symmetric quivers of finite type the deg. order is induced"

$Q_0 = \{1, \dots, n\}$

$\sigma(i) = n+1-i$

Q_0 form a poset s.t.

$$i < j \Rightarrow \sigma(j) < \sigma(i)$$

Put an arrow $i \leftarrow j$ if $i < j$ is a cover.

Ex:

1) $1 \leftarrow 2 \leftarrow \cdots \leftarrow n$

2)
$$\begin{tikzcd}
1 \arrow[dr] & & 3=\sigma(2) \arrow[dl] \arrow[dr] & \\
& 2 & & 4=\sigma(1)
\end{tikzcd}$$

3)
$$\begin{tikzcd}
1 \arrow[dr] & & & & \\
& 2 \arrow[dr] & & 4=\sigma(2) \arrow[dl] \arrow[dr] & \\
& & 3=\sigma(3) & & 5
\end{tikzcd}$$

proof: Auslander-Reiten theory.

$$\left(1 \to \cdots \to \ell \xrightarrow[a]{} \overset{\alpha}{\circlearrowleft} \omega \xrightarrow[b]{} \ell^* \to \cdots \to 1^* \right) \Big/ (\alpha^2, ba)$$

$$U_1 \hookrightarrow \cdots \hookrightarrow U_\ell \underset{\text{isotropic.}}{\hookrightarrow} \overset{\circlearrowright}{\mathbb{C}^{2n}} \twoheadrightarrow U_\ell^\perp / U_\ell \twoheadrightarrow \cdots \to$$