Cluster manifolds and Painlevé equations

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based on joint paper with P. Gavrylenko and A. Marshakov

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Deatonomization of cluster integrable system

Newton polygon $\Delta$

$\Downarrow$

Thurston diagram

$\Downarrow$

Bipartite graph on a torus

$\Downarrow$

Quiver $Q$, $X$-cluster variety $X_Q$

$\Downarrow$

Integrable system. Casimir elements.

$\Downarrow$

Group of discrete flows $G_\Delta \subset G_Q$

$\Downarrow$

Deautomization, $q$-difference equations.
Main example

Bipartite graph on a torus (example):

All edges are oriented from black to white. To each edge $e$ we assign the weight $w(e)$. In other words we have a discrete $GL(1)$ connection on a graph.
Dimer models

Dimer configurations is the set of edges with the property that every vertex is the endpoint of a unique edge in the set.

There are 8 dimer configurations for our bipartite graph
Weights of the configurations

The weight of each dimer configuration $D$ is a product of weights of the edges

$$w(D) = \prod_{e \in D} w(e).$$

If we pick one dimer configuration $D_0$ then $D - D_0$ is cycle, $\partial (D - D_0) = 0$, for any $D$. Therefore, the weight $w(D_0)^{-1} w(D)$ is given by weights of elementary cycles — faces and $A, B$ cycles on torus

$$\prod_{e \in \partial \text{Face}_i} w(e) = x_i, \quad \prod_{e \in A\text{--cycle}} w(e) = \lambda, \quad \prod_{e \in B\text{--cycle}} w(e) = \mu$$

Note that $\lambda, \mu$ depend on concrete choice of $A, B$ cycles.

We have $\prod_i x_i = \prod_{e \in \partial \mathbb{T}^2} W(e) = 1$, since $\partial \mathbb{T}^2 = 0$. 

The conditions

\[
\prod_{e \in \partial \text{Face}_i} w(e) = x_i, \quad \prod_{e \in A-\text{cycle}} w(e) = \lambda, \quad \prod_{e \in B-\text{cycle}} w(e) = \mu
\]

can be fulfilled by

![Diagram showing the conditions with variables and arrows indicating the weights and cycles.](attachment:diagram.png)
Weights of the dimer configurations

\[ W(x) = x_1^{-1}x_2^{-1} + x_4 + \mu^{-1}x_4 + \mu x_1^{-1} \]

Partition function \( Z(\lambda, \mu) = W(D_0)^{-1} \sum W(D) \), where

\[ Z(\lambda, \mu) = x_1^{-1}x_2^{-1}\lambda + \lambda^{-1} + \mu x_1^{-1} + \mu^{-1}x_4 + H, \]

where

\[ H = 1 + x_1^{-1} + x_1^{-1}x_2^{-1} + x_4 \]
Cluster structure

Quiver:

Poisson bracket is

\[
\{x_1, x_2\} = 2x_1x_2, \quad \{x_2, x_3\} = 2x_2x_3, \quad \{x_3, x_4\} = 2x_3x_4, \quad \{x_4, x_1\} = 2x_4x_1
\]

Using the \((\mathbb{C}^\times)^3\) action \(Z(\lambda, \mu) \mapsto t_Z \cdot Z(t_{\lambda} \cdot \lambda, t_{\mu} \cdot \mu)\) one can get

\[
Z(\lambda, \mu) = \lambda + z\lambda^{-1} + \mu + \mu^{-1} + H = 0
\]

Casimirs and Hamiltonian:

\[
1 = x_1x_2x_3x_4, \quad z = x_1x_3, \quad H = \sqrt{x_1x_2} + \frac{1}{\sqrt{x_1x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}
\]
The discrete flows come from mutations of the quiver

\[
\begin{align*}
\mu_j : & \quad \epsilon_{ik} \mapsto \epsilon_{ik}, \quad \text{if } i = j \text{ or } k = j, \\
& \quad \epsilon_{ik} \mapsto \epsilon_{ik} + \epsilon_{ij} | \epsilon_{jk} | + \epsilon_{jk} | \epsilon_{ij} |, \\
\text{otherwise.}
\end{align*}
\]

\[
\begin{align*}
\mu_j : & \quad x_j \mapsto x_j - 1 \epsilon_{ij}, \\
& \quad x_i \mapsto x_i (1 + \sign \epsilon_{ij} \epsilon_{ij}), \\
& \quad \{x'_i, x'_k\} = \epsilon'_{ik} x'_i x'_k.
\end{align*}
\]
The discrete flows come from mutations of the quiver

\[ \begin{array}{c}
\text{Mutation } \mu_1
\end{array} \]
The discrete flows come from mutations of the quiver

\[
\mu_1: \begin{align*}
    x_1 &\mapsto x_1^{-1} \\
    x_i &\mapsto x_i(1 + x_{\text{sgn} \epsilon_{ij}})
\end{align*}
\]

Reverse all incoming and outgoing arrows

\[x'_1 = x_1^{-1}\]
The discrete flows come from mutations of the quiver

Mutation $\mu_1$

Reverse all incoming and outgoing arrows

$x'_1 = x_1^{-1}$

Complete cycles through mutation vertex

$x'_4 = x_4(1 + x_1)^2$

$x'_2 = x_2(1 + x_1^{-1})^{-2}$
The discrete flows come from mutations of the quiver

Mutation $\mu_1$

Reverse all incoming and outgoing arrows

$x'_1 = x_1^{-1}$

Complete cycles through mutation vertex

$x'_4 = x_4(1 + x_1)^2$
$x'_2 = x_2(1 + x_1^{-1})^{-2}$
The discrete flows come from mutations of the quiver

Mutation $\mu_1$

Reverse all incoming and outgoing arrows

$x_1' = x_1^{-1}$

Complete cycles through mutation vertex

$x_4' = x_4(1 + x_1)^2$
$x_2' = x_2(1 + x_1^{-1})^{-2}$
The discrete flows come from mutations of the quiver

Mutation $\mu_1$

Reverse all incoming and outgoing arrows

$x'_1 = x_1^{-1}$

Complete cycles through mutation vertex

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The discrete flows come from mutations of the quiver

![Diagram of quiver mutations]

**Mutation $\mu_1$**

- Reverse all incoming and outgoing arrows
- $x'_1 = x_1^{-1}$
- Complete cycles through mutation vertex
  - $x'_4 = x_4(1 + x_1)^2$
  - $x'_2 = x_2(1 + x_1^{-1})^{-2}$
The discrete flows come from mutations of the quiver

Mutation $\mu_1$

Reverse all incoming and outgoing arrows

$x'_1 = x_1^{-1}$

Complete cycles through mutation vertex

$x'_4 = x_4(1 + x_1)^2$

$x'_2 = x_2(1 + x_1^{-1})^{-2}$

Formulas:

$\mu_j : \epsilon_{ik} \mapsto -\epsilon_{ik}$, if $i = j$ or $k = j$, $\epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}\epsilon_{jk} + \epsilon_{jk}\epsilon_{ij}}{2}$ otherwise.

$\mu_j : x_j \mapsto x_j^{-1}$, $x_i \mapsto x_i \left(1 + x_j^{\text{sgn}\epsilon_{ij}}\right)^{\epsilon_{ij}}$, $i \neq j$. $\{x'_i, x'_k\} = \epsilon'_{ik} x'_i x'_k$
Cluster automorphisms (group $G_Q$)

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. Example:

```
  1  2  3  4
  |   |   |
  v   v   v
  3   1  2  4
  |   |   |
  v   v   v
  2   3  4  1
```

Rational functions with nonnegative integer coefficients. Laurent phenomenon in $\tau$ variables.

Cluster automorphisms (group $G_Q$)

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. Example:

\[
\begin{align*}
x_1 & \quad x_2 \\
1 & \rightarrow 2 \quad \rightarrow 3 \\
& \quad \rightarrow 4 \\
4 & \rightarrow 1
\end{align*}
\]

\[
\begin{align*}
x_2 & \quad (1 + x_1^{-1})^{-2} \quad x_3 \\
2 & \rightarrow 2 \quad \rightarrow 3 \\
& \quad \rightarrow 1 \\
1 & \rightarrow 4
\end{align*}
\]

Rational functions with nonnegative integer coefficients. Laurent phenomenon in $\tau$ variables.

Cluster automorphisms (group $G_Q$)

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. Example:

$$
\begin{align*}
&x_2, x_3 \\
&\begin{array}{c}
1 \\
2 \\
3 \\
4 
\end{array} \\
&x_1, x_4
\end{align*}

\quad \rightarrow \quad

\begin{align*}
&x_2(1 + x_1^{-1})^{-2}, x_3 \\
&\begin{array}{c}
1 \\
2 \\
3 \\
4 
\end{array} \\
&x_1^{-1}, x_4(1 + x_1)^2
\end{align*}

\quad \rightarrow \quad

\begin{align*}
&x_2\left(\frac{1+x_3}{1+x_1^{-1}}\right)^2, x_3^{-1} \\
&\begin{array}{c}
1 \\
2 \\
3 \\
4 
\end{array} \\
&x_1^{-1}, x_4\left(\frac{1+x_1}{1+x_3^{-1}}\right)^2
\end{align*}
Cluster automorphisms (group $G_Q$)

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. Example:

\[
x_2 \xrightarrow{\quad} x_2(1 + x_1^{-1})^{-2}\quad x_3
\]

\[
\begin{align*}
x_1 & \quad x_4 \\
1 & \quad 4
\end{align*}
\]

\[
\begin{align*}
x_1^{-1} & \quad x_4(1 + x_1)^2 \\
1 & \quad 4
\end{align*}
\]

\[
\begin{align*}
x_2 \left( \frac{1 + x_3}{1 + x_1^{-1}} \right)^2 & \quad x_3^{-1} \\
2 & \quad 3
\end{align*}
\]

\[
\begin{align*}
x_1^{-1} & \quad x_4 \left( \frac{1 + x_1^{-1}}{1 + x_3} \right)^2 \\
1 & \quad 4
\end{align*}
\]


Cluster manifolds and Painlevé equations

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Cluster automorphisms (group $G_Q$)

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. Example:

$$T: (x_1, x_2, x_3, x_4) \mapsto (x_2 (1 + x_1^{-1})^{-2}, x_3, x_1^{-1}, x_4 (1 + x_1)^2).$$

Rational functions with nonnegative integer coefficients. Laurent phenomenon in $\tau$ variables.

Cluster automorphisms (group $G_Q$)

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. Example:

$$T : (x_1, x_2, x_3, x_4) \mapsto \left( x_2 \left( \frac{1+x_3}{1+x_1^{-1}} \right)^2, x_1^{-1}, x_4 \left( \frac{1+x_1}{1+x_3^{-1}} \right)^2, x_3^{-1} \right).$$

Rational functions with nonnegative integer coefficients. Laurent phenomenon in $\tau$ variables.
Set \( x_1 x_2 x_3 x_4 = q \), (since \( x_1 x_2 x_3 x_4 \neq 1 \implies \) no integrable system.) \( z = x_1 x_3 \)

\[
T : (x_1, x_2, x_3, x_4) \mapsto \left( x_2 \left( \frac{1 + x_3}{1 + x_1^{-1}} \right)^2, x_1^{-1}, x_4 \left( \frac{1 + x_1}{1 + x_3^{-1}} \right)^2, x_3^{-1} \right)
\]

\[
T : (x_1, x_2, z, q) \mapsto \left( x_2 \left( \frac{x_1 + z}{x_1 + 1} \right)^2, x_1^{-1}, qz, q \right)
\]

Casimir \( z \) becomes “time”, so introduce \( x_i = x_i(z) \), \( T : x_i(z) \mapsto x_i(qz) \).

\[
x_1(qz)x_1(q^{-1}z) = \left( \frac{x_1(z) + z}{x_1(z) + 1} \right)^2
\]

This is \( q \)-Painlevé III\(_3\) equation, or \( P(A_7^{(1)\prime}) \).
Directions of the generalization

4 boundary points, internal points on one line

Non-autonomous discrete Hirota equations. Integrable system is relativistic Toda

One internal point

? General Newton polygons

q-difference Painlevé equations
Newton polygons with one internal point

Same area $\iff$ same quivers, except for $4_a, 4_c$ vs $4_b$. 

Quivers and their automorphism groups

$S_3 \quad Dih_4 \times W(A_1^{(1)}) \quad W(A_1^{(1)}) \quad \tilde{W}((A_1 + A_1)^{(1)}) \quad \tilde{W}((A_1 + A_2)^{(1)})$

\[ \tilde{W}(D_4^{(1)}) \quad \tilde{W}(D_5^{(1)}) \quad \tilde{W}(E_6^{(1)}) \]
\(8_{a,b,c} \) case — figures
The generators of the group $G_Q$:

$$
s_0 = (1, 2), \quad s_1 = (5, 6), \quad s_2 = (1, 5) \circ \mu_5 \circ \mu_1, \quad s_3 = (3, 7) \circ \mu_3 \circ \mu_7, \quad s_4 = (3, 4), \quad s_5 = (7, 8), \quad \pi = (1, 7, 5, 3)(2, 8, 6, 4), \quad \sigma = (1, 7)(2, 8)(3, 5)(4, 6) \circ \zeta.
$$
\( \delta_{a,b,c} \) case — \( q - PVI \)

Action on cluster coordinates

\[
s_0 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_2, x_1, x_3, x_4, x_5, x_6, x_7, x_8)
\]

\[
s_1 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_1, x_2, x_3, x_4, x_6, x_5, x_7, x_8)
\]

\[
s_2 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_5^{-1}, x_2, x_3 \frac{1+x_5}{1+x_1^{-1}}, x_4 \frac{1+x_5}{1+x_1^{-1}}, x_1^{-1}, x_6, x_7 \frac{1+x_1}{1+x_5^{-1}}, x_8 \frac{1+x_1}{1+x_5^{-1}})
\]

\[
s_3 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_1 \frac{1+x_3}{1+x_7^{-1}}, x_2 \frac{1+x_3}{1+x_7^{-1}}, x_7^{-1}, x_4, x_5 \frac{1+x_7}{1+x_3^{-1}}, x_6 \frac{1+x_7}{1+x_3^{-1}}, x_3^{-1}, x_8)
\]

\[
s_4 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)
\]

\[
s_5 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_7)
\]

This transformations belong to \( Aut(X_Q) \) and generate affine Weyl group \( W(D_5^{(1)}) \).
Phase space is a submanifold of $X_Q$ with fixed Casimirs. In this case it has dimension 2. Conjecturally it is Sakai surface: $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in 8 points. Positions of the blow up points depend on type of the quiver and values the Casimirs.

Automorphisms of quivers — automorphisms of Sakai surface that can move blown-up points.

Affine Weyl group acts on $X_Q$. Conjecturally this is full cluster mapping class group $G_Q$.

Affine Weyl group gives Painlevé equation and its Bäcklund transformations.
The generators of the group $G_Q$:

$s_1 = (2, 3)$, $s_2 = (1, 2)$, $s_4 = (4, 5)$, $s_5 = (5, 6)$, $s_6 = (7, 8)$, $s_0 = (8, 9)$,

$s_3 = (4, 7) \circ \mu_1 \circ \mu_4 \circ \mu_7 \circ \mu_1$, $\pi = (1, 4, 7)(2, 5, 8)(3, 6, 9)$, $\sigma = (1, 7)(2, 8)(3, 9) \circ \varsigma$. 

M. Bershtein based on joint paper with P. Gavrylenko and A. Marshakov: 

9 case

\[ S_1: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_3, x_2, x_4, x_5, x_6, x_7, x_8, x_9) \]

\[ S_2: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_2, x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \]

\[ S_3: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto \left( \frac{x_1}{x_4 x_7}, \frac{1+x_4+x_1^{-1}}{1+x_1+x_7^{-1}}, \frac{1+x_4+x_1^{-1}}{x_1 x_2}, \frac{1+x_4+x_1^{-1}}{1+x_1+x_7^{-1}}, \frac{1+x_4+x_1^{-1}}{x_1 x_3}, \frac{1+x_4+x_1^{-1}}{1+x_1+x_7^{-1}}, \frac{x_4}{x_1 x_7}, \frac{1+x_7+x_4^{-1}}{1+x_4+x_1^{-1}}, \frac{x_4 x_5}{x_4 x_6}, \frac{1+x_7+x_4^{-1}}{1+x_4+x_1^{-1}} \right) \]

\[ S_4: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2, x_3, x_5, x_4, x_6, x_7, x_8, x_9) \]

\[ S_5: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2, x_3, x_4, x_6, x_5, x_7, x_8, x_9) \]

\[ S_6: (x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_7, x_9) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \]

\[ S_0: (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_8) \]
Classification: $Y^{N,k}$ polygons with $0 \leq k \leq N$ (see picture below) and $L^{1,2N-1,2}$ polygons (which we omit in the talk):
Bipartite graphs and quivers for $Y^{N,k}$

The bipartite graphs can be constructed from the building blocks. $N_0$ type 0 blocks, $N_1$ type I blocks and $N_{-1}$ type $-1$ blocks on the torus with $N = N_0 + N_1 + N_{-1}$, $k = N_1 - N_{-1}$. The order of blocks can be arbitrary.

Quivers for $Y^{N,k}$ theories can be glued from blocks of three types 0, 1, -1.
Quivers and mutations

It is convenient to draw mutations on the integer lattice

Quivers should be periodic with period \((N, k)\).
There are in total \(3^N\) quivers, but only \(N + 1\) of them are inequivalent.
Action of the automorphism group

N=6, k=2


Cluster manifolds and Painlevé equations
Mutable "+"-variables labelled by the points of integer lattice: $x_{(n,m)}$. They satisfy periodicity condition and Y-system in order to be compatible with mutations:

$$\frac{x_{(n,m+1)}x_{(n,m-1)}}{x_{(n,m)}^2} = \frac{(1 + x_{(n+1,m)})(1 + x_{(n-1,m)})}{(1 + x_{(n,m)})^2}, \quad x_{(n,m)} = x_{(n+N,m+k)}$$

One can move from Y-system to T-system (from X-clusters to A-clusters):

$$x_{(n,m)} = z_0^{1/N} q^{(kn-Nm+N)/N^2} \frac{\tau_{(n-1,m-1)}\tau_{(n+1,m-1)}}{\tau_{(n,m-1)}^2}, \quad \tau_{(n,m)} = \tau_{(n+N,m+k)}$$

And after some change of labeling:

$$\tau_j(qz) \tau_j(q^{-1}z) = \tau_j(z)^2 + z^{1/N} \tau_{j+1} \left(q^{k/N}z\right) \tau_{j-1} \left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$
Quantization

- In addition to non-autonomous parameter $q$ one may add quantum deformation $p$:

$$x_i x_j = p^{-2\epsilon_{ij}} x_j x_i$$

- All groups $G_Q$ remain the same.

- There are quantum mutations

$$\mu_j : \ x_j \mapsto x_j^{-1}, \ \ x_i^{1/|\epsilon_{ij}|} \mapsto x_i^{1/|\epsilon_{ij}|} \left(1 + px_j^{\sgn \epsilon_{ij}}\right)^{\sgn \epsilon_{ij}}, \ i \neq j$$

- And so there are quantum deformations of all systems. For example, quantum $q$-Painlevé III$_3$:

$$\begin{cases} x_1(q^{-1}z)^{1/2} \ x_1(qz)^{1/2} = \frac{x_1(z) + pz}{x_1(z) + p}, \\ x_1(z)x_1(q^{-1}z) = p^4 x_1(q^{-1}z)x_1(qz). \end{cases}$$
What I did not mention

1. Solution of the autonomous equations in terms of theta functions,

2. Solution of the nonautonomous equations in terms of
   - 5d Nekrasov partition functions with $\epsilon_1 + \epsilon_2 = 0$.
   - topological string amplitues.
   - $q$-deformed conformal blocks for special integer central charge.

Thank you for your attention!
Some numerology

\( B \) is the number of boundary integer points of \( \Delta \).
\( B - 3 \) is the number of Casimirs for the Poisson bracket.
\( B - 3 \) is the rank of a free abelian group of discrete flows in \( G_Q \)

\( I \) is the number of interior integer points of \( \Delta \).
\( 2I \) is the rank of the Poisson bracket of \( X_Q \)
\( I \) is the number of commuting Hamiltonians \( H_1, \ldots, H_I \).
\( I \) is the genus of a spectral curve.

\( S = I + B/2 - 1 \) is the area of \( \Delta \)
\( 2S \) is the number of vertices in quiver \( Q \).