Cox Rings

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CHAPTER I

Basic concepts

In this chapter we introduce the Cox ring and, more generally, the *Cox sheaf* and its geometric counterpart, the *characteristic space*. Moreover, algebraic and geometric aspects are discussed. Section 1 is devoted to commutative algebras graded by monoids. In Section 2, we recall the correspondence between actions of quasitori (also called diagonalizable groups) on affine varieties and affine algebras graded by abelian groups, and we provide the necessary background on good quotients. Section 3 is a first step towards Cox rings. Given a normal variety X and a finitely generated subgroup $K \subseteq \mathrm{WDiv}(X)$ of the group of Weil divisors, we consider the associated *sheaf of divisorial algebras*

$$\mathcal{S} \quad = \quad \bigoplus_{D \in K} \mathcal{O}_X(D).$$

We present criteria for local finite generation and consider the relative spectrum. A first result says that $\Gamma(X, \mathcal{S})$ is a unique factorization domain if K generates the divisor class group $\mathrm{Cl}(X)$. Moreover, we characterize divisibility in the ring $\Gamma(X, \mathcal{S})$ in terms of divisors on X. In Section 4, the Cox sheaf of a normal variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ is introduced; roughly speaking it is given as

$$\mathcal{R} \quad = \quad \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{O}_X(D).$$

The Cox ring then is the corresponding ring of global sections. In the case of a free divisor class group well-definiteness is straightforward. The case of torsion needs some effort, the precise way to define $\mathcal{R}$ then is to take the quotient of an appropriate sheaf of divisorial algebras by a certain ideal sheaf. Basic algebraic properties and divisibility theory of the Cox ring are investigated in Section 5. Finally, in Section 6, we study the characteristic space, i.e., the relative spectrum $\widehat{X} = \mathrm{Spec}_X \mathcal{R}$ of the Cox sheaf. It comes with an action of the *characteristic quasitorus* $H = \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$ and a good quotient $\widehat{X} \to X$. We relate geometric properties of X to properties of this action and give a characterization of the characteristic space in terms of Geometric Invariant Theory.

1. Graded algebras

1.1. Monoid graded algebras. We recall basic notions on algebras graded by abelian monoids. In this subsection, R denotes a commutative ring with unit element.

Definition 1.1.1. Let K be an abelian monoid. A *K-graded R-algebra* is an associative, commutative R-algebra A with unit that comes with a direct sum decomposition

$$A \quad = \quad \bigoplus_{w \in K} A_w$$

into R-submodules $A_w \subseteq A$ such that $A_w \cdot A_{w'} \subseteq A_{w+w'}$ holds for any two elements $w, w' \in K$.

We will also speak of a K-graded R-algebra as a monoid graded algebra or just as a graded algebra. In order to compare R-algebras A and A', which are graded by different abelian monoids K and K', we work with the following notion of a morphism.

Definition 1.1.2. A *morphism* from a K-graded algebra A to a K'-graded algebra A' is a pair $(\psi, \widetilde{\psi})$, where $\psi\colon A \to A'$ is a homomorphism of R-algebras, $\widetilde{\psi}\colon K \to K'$ is a homomorphism of abelian monoids and

$$\psi(A_w) \quad \subseteq \quad A_{\widetilde{\psi}(w)}$$

holds for every $w \in K$. In the case $K = K'$ and $\widetilde{\psi} = \mathrm{id}_K$, we denote a morphism of graded algebras just by $\psi\colon A \to A'$ and also refer to it as a *(K-)graded homomorphism.*

Example 1.1.3. Given an abelian monoid K and $w_1, \ldots, w_r \in K$, the polynomial ring $R[T_1, \ldots, T_r]$ can be made into a K-graded R-algebra by setting

$$R[T_1, \ldots, T_r]_w \quad := \quad \left\{ \sum_{\nu \in \mathbb{Z}^r_{\geq 0}} a_\nu T^\nu;\; a_\nu \in R,\, \nu_1 w_1 + \ldots + \nu_r w_r = w \right\}.$$

This K-grading is determined by $\deg(T_i) = w_i$ for $1 \leq i \leq r$. Moreover, $R[T_1, \ldots, T_r]$ comes with the natural $\mathbb{Z}^r_{\geq 0}$-grading given by

$$R[T_1, \ldots, T_r]_\nu \quad := \quad R \cdot T^\nu,$$

and we have a canonical morphism $(\psi, \widetilde{\psi})$ from $R[T_1, \ldots, T_r]$ to itself, where $\psi = \mathrm{id}$ and $\widetilde{\psi}\colon \mathbb{Z}^r_{\geq 0} \to K$ sends ν to $\nu_1 w_1 + \ldots + \nu_r w_r$.

For any abelian monoid K, we denote by $K^\pm$ the associated group of differences and by $K_\mathbb{Q} := K^\pm \otimes_\mathbb{Z} \mathbb{Q}$ the associated rational vector space. Note that we have canonical maps $K \to K^\pm \to K_\mathbb{Q}$, where the first one is injective if K admits cancellation and the second one is injective if $K^\pm$ is free. By an integral R-algebra, we mean an R-algebra A without zero-divisors.

Definition 1.1.4. Let A be an integral K-graded R-algebra. The *weight monoid* of A is the submonoid

$$S(A) \;:=\; \{w \in K;\; A_w \neq 0\} \;\subseteq\; K.$$

The *weight group* of A is the subgroup $K(A) \subseteq K^\pm$ generated by $S(A) \subseteq K$. The *weight cone* of A is the convex cone $\omega(A) \subseteq K_\mathbb{Q}$ generated by $S(A) \subseteq K$.

We recall the construction of the algebra associated to an abelian monoid; it defines a covariant functor from the category of abelian monoids to the category of monoid graded algebras.

Construction 1.1.5. Let K be an abelian monoid. As an R-module, the associated *monoid algebra* over R is given by

$$R[K] \quad := \quad \bigoplus_{w \in K} R \cdot \chi^w$$

and its multiplication is defined by $\chi^w \cdot \chi^{w'} := \chi^{w+w'}$. If K' is a further abelian monoid and $\widetilde{\psi}\colon K \to K'$ is a homomorphism, then we have a homomorphism

$$\psi := R[\widetilde{\psi}]\colon R[K] \;\to\; R[K'], \qquad \chi^w \;\mapsto\; \chi^{\widetilde{\psi}(w)}.$$

The pair $(\psi, \widetilde{\psi})$ is a morphism from the K-graded algebra $R[K]$ to the K'-graded algebra $R[K']$, and this assignment is functorial.

Note that the monoid algebra $R[K]$ has K as its weight monoid, and $R[K]$ is finitely generated over R if and only if the monoid K is finitely generated. In general, if a K-graded algebra A is finitely generated over R, then its weight monoid is finitely generated and its weight cone is polyhedral.

CONSTRUCTION 1.1.6 (Trivial extension). Let $K \subseteq K'$ be an inclusion of abelian monoids and A a K-graded R-algebra. Then we obtain an K'-graded R-algebra A' by setting

$$A' := \bigoplus_{u \in K'} A'_u, \qquad A'_u := \begin{cases} A_u & \text{if } u \in K, \\ \{0\} & \text{else.} \end{cases}$$

CONSTRUCTION 1.1.7 (Lifting). Let $G\colon \widetilde{K} \to K$ be a homomorphism of abelian monoids and A a K-graded R-algebra. Then we obtain a $\widetilde{K}$-graded R-algebra

$$\widetilde{A} := \bigoplus_{u \in \widetilde{K}} \tilde{A}_u, \qquad \widetilde{A}_u := A_{G(u)}.$$

DEFINITION 1.1.8. Let A be a K-graded R-algebra. An ideal $I \subseteq A$ is called *(K-)homogeneous* if it is generated by (K-)homogeneous elements.

An ideal $I \subseteq A$ of a K-graded R-algebra A is homogeneous if and only if it has a direct sum decomposition

$$I = \bigoplus_{w \in K} I_w, \qquad I_w := I \cap A_w.$$

CONSTRUCTION 1.1.9. Let A be a K-graded R-algebra and $I \subseteq A$ a homogeneous ideal. Then the factor algebra A/I is K-graded by

$$A/I = \bigoplus_{w \in K} (A/I)_w \qquad (A/I)_w := A_w + I.$$

Moreover, for each homogeneous component $(A/I)_w \subseteq A/I$, one has a canonical isomorphism of R-modules

$$A_w/I_w \to (A/I)_w, \qquad f + I_w \mapsto f + I.$$

CONSTRUCTION 1.1.10. Let A be a K-graded R-algebra, and $\widetilde{\psi}\colon K \to K'$ be a homomorphism of abelian monoids. Then one may consider A as a K'-graded algebra with respect to the *coarsened grading*

$$A = \bigoplus_{u \in K'} A_u, \qquad A_u := \bigoplus_{\widetilde{\psi}(w)=u} A_w.$$

EXAMPLE 1.1.11. Let $K = \mathbb{Z}^2$ and consider the K-grading of $R[T_1, \ldots, T_5]$ given by $\deg(T_i) = w_i$, where

$$w_1 = (-1, 2), \quad w_2 = (1, 0), \quad w_3 = (0, 1), \quad w_4 = (2, -1), \quad w_5 = (-2, 3).$$

Then the polynomial $T_1T_2 + T_3^2 + T_4T_5$ is K-homogeneous of degree $(0, 2)$, and thus we have a K-graded factor algebra

$$A = R[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2 + T_4T_5 \rangle.$$

The standard $\mathbb{Z}$-grading of the algebra A with $\deg(T_1) = \ldots = \deg(T_5) = 1$ may be obtained by coarsening via the homomorphism $\widetilde{\psi}\colon \mathbb{Z}^2 \to \mathbb{Z}$, $(a, b) \mapsto a + b$.

PROPOSITION 1.1.12. *Let A be a $\mathbb{Z}^r$-graded R-algebra satisfying $ff' \neq 0$ for any two non-zero homogeneous $f, f' \in A$. Then the following statements hold.*

(i) *The algebra A is integral.*
(ii) *If gg' is homogeneous for $0 \neq g, g' \in A$, then g and g' are homogeneous.*
(iii) *Every unit $f \in A^*$ is homogeneous.*

Proof. Fix a lexicographic ordering on $\mathbb{Z}^r$. Given two non-zero $g, g' \in A$, write $g = \sum f_u$ and $g' = \sum f'_u$ with homogeneous f_u and f'_u. Then the maximal (minimal) component of gg' is $f_w f'_{w'} \neq 0$, where f_w and $f'_{w'}$ are the maximal (minimal) components of f and f' respectively. The first two assertions follow. For the third one observe that $1 \in A$ is homogeneous (of degree zero). □

1.2. Veronese subalgebras. We introduce Veronese subalgebras of monoid graded algebras and present statements relating finite generation of the algebra to finite generation of a given Veronese subalgebra and vice versa. Again, R is a commutative ring with unit element.

Definition 1.2.1. Given an abelian monoid K admitting cancellation, a K-graded R-algebra A and a submonoid $L \subseteq K$, one defines the associated *Veronese subalgebra* to be

$$A(L) \ := \ \bigoplus_{w \in L} A_w \ \subseteq \ \bigoplus_{w \in K} A_w \ = \ A.$$

Proposition 1.2.2. *Let K be an abelian monoid admitting cancellation and A a finitely generated K-graded R-algebra. Let $L \subseteq K$ be a finitely generated submonoid. Then the associated Veronese subalgebra $A(L)$ is finitely generated over R.*

Lemma 1.2.3. *Let K be an abelian monoid admitting cancellation and let $L, M \subseteq K$ be finitely generated submonoids. Then $L \cap M$ is finitely generated.*

Proof. Consider the embedding $K \subseteq K^{\pm}$ into the group of differences and define a homomorphism $\alpha\colon \mathbb{Z}^r \to K^{\pm}$ with $L, M \subseteq \alpha(\mathbb{Z}^r)$. Then $\alpha^{-1}(L)$ and $\alpha^{-1}(M)$ are finitely generated monoids; indeed, if $w_i = \alpha(v_i)$, where $1 \le i \le k$, generate L and $u_1, \ldots, u_s$ is a basis for $\ker(\alpha)$, then $\alpha^{-1}(L)$ is generated by $v_1, \ldots, v_k$ and $\pm u_1, \ldots, \pm u_s$.

To prove the assertion, it suffices to show that the intersection $\alpha^{-1}(L) \cap \alpha^{-1}(M)$ is finitely generated. In other words, we may assume that $K = \mathbb{Z}^r$ holds. Then L and M generate convex polyhedral cones τ and σ in $\mathbb{Q}^r$, respectively. Consider $\omega := \tau \cap \sigma$ and the tower of algebras

$$\mathbb{Q} \ \subseteq \ \mathbb{Q}[L \cap M] \ \subseteq \ \mathbb{Q}[\omega \cap \mathbb{Z}^r].$$

Gordan's Lemma [**112**, Theorem 7.6] shows that $\mathbb{Q}[\omega \cap \mathbb{Z}^r]$ is finitely generated over $\mathbb{Q}$. Moreover, for every $v \in \tau \cap \sigma$, some positive integral multiple $k \cdot v$ belongs to $L \cap M$. Thus, $\mathbb{Q}[\omega \cap \mathbb{Z}^r]$ is integral and hence finite over $\mathbb{Q}[L \cap M]$. The Artin-Tate Lemma [**66**, page 144] tells us that $\mathbb{Q}[L \cap M]$ is finitely generated over $\mathbb{Q}$. Consequently, the weight monoid $L \cap M$ of $\mathbb{Q}[L \cap M]$ is finitely generated. □

Proof of Proposition 1.2.2. According to Lemma 1.2.3, we may assume that L is contained in the weight monoid $S(A)$. Moreover, replacing K with its group of differences, we may assume that K is a group. Fix homogeneous generators $f_1, \ldots, f_r$ for A and set $w_i := \deg(f_i)$. Then we have an epimorphism

$$\alpha\colon \mathbb{Z}^r \ \to \ K, \qquad e_i \ \mapsto \ w_i.$$

Moreover, set $B := R[T_1, \ldots, T_r]$ and endow it with the natural $\mathbb{Z}^r$-grading. Then we obtain a morphism (π, α) of graded R-algebras from B to A, where π is the epimorphism defined by

$$\pi\colon B \ \to \ A, \qquad T_i \ \mapsto \ f_i.$$

The inverse image $\alpha^{-1}(L) \subseteq \mathbb{Z}^r$ is a finitely generated monoid. By Lemma 1.2.3, the intersection $M := \alpha^{-1}(L) \cap \mathbb{Z}^r_{\ge 0}$ is finitely generated and hence generates a polyhedral convex cone $\sigma = \mathrm{cone}(M)$ in $\mathbb{Q}^r$. Consider the tower of R-algebras

$$R \ \subseteq \ R[M] \ \subseteq \ R[\sigma \cap \mathbb{Z}^r].$$

The R-algebra $R[\sigma \cap \mathbb{Z}^r]$ is finitely generated by Gordan's Lemma [**112**, Theorem 7.6], and it is integral and thus finite over $R[M]$. The Artin-Tate Lemma [**66**, page 144] then shows that $R[M]$ is finitely generated over R. By construction, $\pi\colon B \to A$ maps $R[M] \subseteq B$ onto $A(L) \subseteq A$. This implies finite generation of $A(L)$. $\square$

Proposition 1.2.4. *Suppose that R is noetherian. Let K be a finitely generated abelian group, A a K-graded integral R-algebra and $L \subseteq K$ be a submonoid such that for every $w \in S(A)$ there exists an $n \in \mathbb{Z}_{\geq 1}$ with $nw \in L$. If the Veronese subalgebra $A(L)$ is finitely generated over R, then also A is finitely generated over R.*

Proof. We may assume that K is generated by $S(A)$. A first step is to show that the quotient field of A is a finite extension of that of $A(L)$. Fix generators $u_1, \ldots, u_r$ for K. Then we may write $u_i = u_i^+ - u_i^-$ with $u_i^\pm \in S(A)$. Choose nontrivial elements $g_i^\pm \in A_{u_i^\pm}$. With $f_i := g_i^+/g_i^-$ we have

$$\mathrm{Quot}(A) \quad = \quad \mathrm{Quot}(A(L))(f_1, \ldots, f_r).$$

By our assumption, A is contained in the integral closure of $A(L)$ in $\mathrm{Quot}(A)$. Applying [**20**, Proposition 5.17] we obtain that A is a submodule of some finitely generated $A(L)$-module. Since R and hence $A(L)$ is noetherian, A is finitely generated as a module over $A(L)$ and thus as an algebra over R. $\square$

Putting Propositions 1.2.2 and 1.2.4 together, we obtain the following well known statement on gradings by abelian groups.

Corollary 1.2.5. *Let R be noetherian, K a finitely generated abelian group, A an integral K-graded R-algebra and $L \subseteq K$ a subgroup of finite index. Then the following statements are equivalent.*

(i) *The algebra A is finitely generated over R.*
(ii) *The Veronese subalgebra $A(L)$ is finitely generated over R.*

Proposition 1.2.6. *Suppose that R is noetherian. Let L, K be abelian monoids admitting cancellation and (φ, F) be a morphism from an L-graded R-algebra B to an integral K-graded R-algebra A. Assume that the weight monoid of B is finitely generated and $\varphi\colon B_u \to A_{F(u)}$ is an isomorphism for every $u \in L$. Then finite generation of A implies finite generation of B.*

Proof. We may assume that K is an abelian group. In a first step we treat the case $L = \mathbb{Z}^r$ without making use of finite generation of $S(B)$. Since A is integral, there are no $\mathbb{Z}^r$-homogeneous zero divisors in B and thus B is integral as well, see Proposition 1.1.12. Set $L_0 := \ker(F)$. By the elementary divisors theorem there is a basis $u_1, \ldots, u_r$ for $\mathbb{Z}^r$ and $a_1, \ldots, a_s \in \mathbb{Z}_{\geq 1}$ such that $a_1 u_1, \ldots, a_s u_s$ is a basis for L_0. Let $L_1 \subseteq \mathbb{Z}^r$ be the sublattice spanned by $u_{s+1}, \ldots, u_r$. This gives Veronese subalgebras

$$B_0 \ := \ \bigoplus_{u \in L_0} B_u, \qquad B_1 \ := \ \bigoplus_{u \in L_1} B_u, \qquad C \ := \ \bigoplus_{u \in L_0 \oplus L_1} B_u.$$

Note that φ maps B_1 isomorphically onto the Veronese subalgebra of A defined by $F(L_1)$. In particular, B_1 is finitely generated. Moreover, C is generated by B_1 and the (unique, invertible) elements $f_i^{\pm 1} \in B_{\pm a_i u_i}$ mapping to $1 \in A_0$. Thus, the Veronese subalgebra $C \subseteq B$ is finitely generated. Since $L_0 \oplus L_1$ is of finite index in $\mathbb{Z}^r$, also B is finitely generated, see Corollary 1.2.5.

We turn to the general case. Let $u_1, \ldots, u_r \in L$ generate the weight monoid of B. Consider the homomorphism $G\colon \mathbb{Z}^r \to L^\pm$ to the group of differences sending the i-th canonical basis vector $e_i \in \mathbb{Z}^r$ to $u_i \in L$ and the composition $G' := F^\pm \circ G$, where $F^\pm\colon L^\pm \to K$ extends $F\colon L \to K$. Regarding B as $L^\pm$-graded, G and G'

define us lifted algebras $\widetilde{B}$ and $\widetilde{A}$, see Construction 1.1.7, fitting into a commutative diagram of canonical morphisms

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ B & \underset{\varphi}{\longrightarrow} & A \end{array}$$

The canonical morphism from $\widetilde{A}$ to A is as required in the first step and thus $\widetilde{A}$ is finitely generated. The weight monoid M of $\widetilde{B}$ is generated by the kernel of G and preimages of generators of the weight monoid of B; in particular, M is finitely generated. Moreover, $\widetilde{B}$ maps isomorphically onto the Veronese subalgebra of $\widetilde{A}$ defined by $M \subseteq \mathbb{Z}^r$; here we use that $\varphi\colon B_u \to A_{F(u)}$ is an isomorphism for every $u \in L$. By Proposition 1.2.2, the algebra $\widetilde{B}$ is finitely generated. Finally, $\widetilde{B}$ maps onto B which gives finite generation of B. □

2. Gradings and quasitorus actions

2.1. Quasitori. We recall the functorial correspondence between finitely generated abelian groups and quasitori (also called diagonalizable groups). Details can be found in the standard textbooks on algebraic groups, see for example [**39**, Section 8], [**88**, Section 16], [**125**, Section 3.2.3] or [**142**, Section 2.5].

We work in the category of algebraic varieties defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. Recall that an *(affine) algebraic group* is an (affine) variety G together with a group structure such that the group laws

$$G \times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2, \qquad\qquad G \to G, \quad g \mapsto g^{-1}$$

are morphisms of varieties. A *morphism* of two algebraic groups G and G' is a morphism $G \to G'$ of the underlying varieties which moreover is a homomorphism of groups.

A *character* of an algebraic group G is a morphism $\chi\colon G \to \mathbb{K}^*$ of algebraic groups, where $\mathbb{K}^*$ is the multiplicative group of the ground field $\mathbb{K}$. The *character group* of G is the set $\mathbb{X}(G)$ of all characters of G together with pointwise multiplication. Note that $\mathbb{X}(G)$ is an abelian group, and, given any morphism $\varphi\colon G \to G'$ of algebraic groups, one has a pullback homomorphism

$$\varphi^*\colon \mathbb{X}(G') \ \to \ \mathbb{X}(G), \qquad \chi' \ \mapsto \ \chi' \circ \varphi.$$

Definition 2.1.1. A *quasitorus* is an affine algebraic group H whose algebra of regular functions $\Gamma(H, \mathcal{O})$ is generated as a $\mathbb{K}$-vector space by the characters $\chi \in \mathbb{X}(H)$. A *torus* is a connected quasitorus.

Example 2.1.2. The *standard n-torus* $\mathbb{T}^n := (\mathbb{K}^*)^n$ is a torus in the sense of 2.1.1. Its characters are precisely the Laurent monomials $T^\nu = T_1^{\nu_1} \cdots T_n^{\nu_n}$, where $\nu \in \mathbb{Z}^n$, and its algebra of regular functions is the Laurent polynomial algebra

$$\Gamma(\mathbb{T}^n, \mathcal{O}) \ = \ \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] \ = \ \bigoplus_{\nu \in \mathbb{Z}^n} \mathbb{K} \cdot T^\nu \ = \ \mathbb{K}[\mathbb{Z}^n].$$

We now associate to any finitely generated abelian group K in a functorial way a quasitorus, namely $H := \operatorname{Spec} \mathbb{K}[K]$; the construction will show that H is the direct product of a standard torus and a finite abelian group.

Construction 2.1.3. Let K be any finitely generated abelian group. Fix generators $w_1, \ldots, w_r$ of K such that the epimorphism $\pi\colon \mathbb{Z}^r \to K$, $e_i \mapsto w_i$ has the kernel

$$\ker(\pi) \quad = \quad \mathbb{Z} a_1 e_1 \oplus \ldots \oplus \mathbb{Z} a_s e_s$$

with $a_1, \ldots, a_s \in \mathbb{Z}_{\geq 1}$. Then we have the following exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}^s \xrightarrow{e_i \mapsto a_i e_i} \mathbb{Z}^r \xrightarrow{e_i \mapsto w_i} K \longrightarrow 0$$

Passing to the respective spectra of group algebras we obtain with $H := \operatorname{Spec} \mathbb{K}[K]$ the following sequence of morphisms

$$1 \longleftarrow \mathbb{T}^s \xleftarrow{(t_1^{a_1}, \ldots, t_s^{a_s}) \leftarrow\!\shortmid t} \mathbb{T}^r \xleftarrow{\imath} H \longleftarrow 1$$

The ideal of $H \subseteq \mathbb{T}^r$ is generated by $T_i^{a_i} - 1$, where $1 \leq i \leq s$. Thus H is a closed subgroup of $\mathbb{T}^r$ and the sequence is an exact sequence of quasitori; note that

$$H \;\cong\; C(a_1) \times \ldots \times C(a_s) \times \mathbb{T}^{r-s}, \qquad C(a_i) \;:=\; \{\zeta \in \mathbb{K}^*;\; \zeta^{a_i} = 1\}.$$

The group structure on $H = \operatorname{Spec} \mathbb{K}[K]$ does not depend on the choices made: the multiplication map is given by its comorphism

$$\mathbb{K}[K] \;\to\; \mathbb{K}[K] \otimes_{\mathbb{K}} \mathbb{K}[K], \qquad \chi^w \;\mapsto \chi^w \otimes \chi^w,$$

and the neutral element of $H = \operatorname{Spec} \mathbb{K}[K]$ is the ideal $\langle \chi^w - 1;\; w \in K \rangle$. Moreover, every homomorphism $\psi\colon K \to K'$ defines a morphism

$$\operatorname{Spec} \mathbb{K}[\psi] \colon \operatorname{Spec} \mathbb{K}[K'] \;\to\; \operatorname{Spec} \mathbb{K}[K].$$

THEOREM 2.1.4. *We have contravariant exact functors being essentially inverse to each other:*

$$\begin{array}{rcl} \{\textit{finitely generated abelian groups}\} & \longleftrightarrow & \{\textit{quasitori}\} \\ K & \mapsto & \operatorname{Spec} \mathbb{K}[K] \\ \psi & \mapsto & \operatorname{Spec} \mathbb{K}[\psi], \\ \mathbb{X}(H) & \leftarrow\!\shortmid & H, \\ \varphi^* & \leftarrow\!\shortmid & \varphi. \end{array}$$

Under these equivalences, the free finitely generated abelian groups correspond to the tori.

This statement includes in particular the observation that closed subgroups as well as homomorphic images of quasitori are again quasitori. Note that homomorphic images of tori are again tori, but every quasitorus occurs as a closed subgroup of a torus.

Recall that a *rational representation* of an affine algebraic group G is a morphism $\varrho\colon G \to \mathrm{GL}(V)$ to the group $\mathrm{GL}(V)$ of linear automorphisms of a finite dimensional $\mathbb{K}$-vector space V. In terms of representations, one has the following characterization of quasitori, see e.g. [**142**, Theorem 2.5.2].

PROPOSITION 2.1.5. *An affine algebraic group G is a quasitorus if and only if any rational representation of G splits into one-dimensional subrepresentations.*

2.2. Affine quasitorus actions. Again we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Recall that one has contravariant equivalences between affine algebras, i.e. finitely generated $\mathbb{K}$-algebras without nilpotent elements, and affine varieties:

$$A \;\mapsto\; \operatorname{Spec} A, \qquad X \;\mapsto\; \Gamma(X, \mathcal{O}).$$

We first specialize these correspondences to graded affine algebras and affine varieties with quasitorus action; here "graded" means graded by a finitely generated abelian group. Then we look at basic concepts such as orbits and isotropy groups from both sides.

A *G-variety* is a variety X together with a morphical action $G \times X \to X$ of an affine algebraic group G. A *morphism* from a G-variety X to G'-variety X' is a pair

$(\varphi, \widetilde{\varphi})$, where $\varphi\colon X \to X'$ is a morphism of varieties and $\widetilde{\varphi}\colon G \to G'$ is a morphism of algebraic groups such that we have

$$\varphi(g{\cdot}x) \;=\; \widetilde{\varphi}(g){\cdot}\varphi(x) \qquad \text{for all } (g,x) \;\in\; G \times X.$$

If G' equals G and $\widetilde{\varphi}$ is the identity, then we refer to this situation by calling $\varphi\colon X \to X'$ a *G-equivariant* morphism.

Example 2.2.1. Let H be a quasitorus. Any choice of characters $\chi_1, \ldots, \chi_r \in \mathbb{X}(H)$ defines a *diagonal H-action* on $\mathbb{K}^r$ by

$$h{\cdot}z \;\;:=\;\; (\chi_1(h)z_1, \ldots, \chi_r(h)z_r).$$

We now associate in functorial manner to every affine algebra graded by a finitely generated abelian group an affine variety with a quasitorus action.

Construction 2.2.2. Let K be a finitely generated abelian group and A a K-graded affine algebra. Set $X = \operatorname{Spec} A$. If $f_i \in A_{w_i}$, $i = 1, \ldots, r$, generate A, then we have a closed embedding

$$X \;\to\; \mathbb{K}^r, \qquad x \;\mapsto\; (f_1(x), \ldots, f_r(x)),$$

and $X \subseteq \mathbb{K}^r$ is invariant under the diagonal action of $H = \operatorname{Spec} \mathbb{K}[K]$ given by the characters $\chi^{w_1}, \ldots, \chi^{w_r}$. Note that for any $f \in A$ homogeneity is characterized by

$$f \in A_w \quad\Longleftrightarrow\quad f(h{\cdot}x) = \chi^w(h)f(x) \text{ for all } h \in H,\, x \in X.$$

This shows that the induced H-action on X does not depend on the embedding into $\mathbb{K}^r$: its comorphism is given by

$$A \;\to\; \mathbb{K}[K] \otimes_{\mathbb{K}} A, \qquad A_w \ni f_w \;\mapsto\; \chi^w \otimes f_w \in \mathbb{K}[K]_w \otimes_{\mathbb{K}} A_w.$$

This construction is functorial: given a morphism $(\psi, \widetilde{\psi})$ from a K-graded affine algebra A to K'-graded affine algebra A', we have a morphism $(\varphi, \widetilde{\varphi})$ from the associated H'-variety X' to the H-variety X, where $\varphi = \operatorname{Spec} \psi$ and $\widetilde{\varphi} = \operatorname{Spec} \mathbb{K}[\widetilde{\psi}]$.

For the other way round, i.e., from affine varieties X with action of a quasitorus H to graded affine algebras, the construction relies on the fact that the representation of H on $\Gamma(X, \mathcal{O})$ is *rational*, i.e., a union of finite dimensional rational subrepresentations, see [**142**, Proposition 2.3.4] and [**95**, Lemma 2.5] for non-affine X. Proposition 2.1.5 then shows that it splits into one-dimensional subrepresentations.

Construction 2.2.3. Let a quasitorus H act on a not necessarily affine variety X. Then $\Gamma(X, \mathcal{O})$ becomes a rational H-module by

$$(h{\cdot}f)(x) \;\;:=\;\; f(h{\cdot}x).$$

The decomposition of $\Gamma(X, \mathcal{O})$ into one-dimensional subrepresentations makes it into a $\mathbb{X}(H)$-graded algebra:

$$\Gamma(X, \mathcal{O}) \;=\; \bigoplus_{\chi \in \mathbb{X}(H)} \Gamma(X, \mathcal{O})_\chi, \qquad \Gamma(X, \mathcal{O})_\chi \;:=\; \{f \in \Gamma(X, \mathcal{O});\; f(h{\cdot}x) = \chi(h)f(x)\}.$$

Again this construction is functorial. If $(\varphi, \widetilde{\varphi})$ is a morphism from an H-variety X to an H'-variety X', then $(\varphi^*, \widetilde{\varphi}^*)$ is a morphism of the associated graded algebras.

THEOREM 2.2.4. *We have contravariant functors being essentially inverse to each other:*

$$\{\textit{graded affine algebras}\} \quad \longleftrightarrow \quad \{\textit{affine varieties with quasitorus action}\}$$

$$\begin{array}{rcl} A & \mapsto & \operatorname{Spec} A, \\ (\psi, \widetilde{\psi}) & \mapsto & (\operatorname{Spec}\psi, \operatorname{Spec}\mathbb{K}[\widetilde{\psi}]) \\ & & \\ \Gamma(X, \mathcal{O}) & \leftarrow\!\shortmid & X, \\ (\varphi^*, \widetilde{\varphi}^*) & \leftarrow\!\shortmid & (\varphi, \widetilde{\varphi}). \end{array}$$

Under these equivalences the graded homomorphisms correspond to the equivariant morphisms.

We use this equivalence of categories to describe some geometry of a quasitorus action in algebraic terms. The first basic observation is the following.

PROPOSITION 2.2.5. *Let A be a K-graded affine algebra and consider the action of $H = \operatorname{Spec}\mathbb{K}[K]$ on $X = \operatorname{Spec} A$. Then for any closed subvariety $Y \subseteq X$ and its vanishing ideal $I \subseteq A$, the following statements are equivalent.*

(i) *The variety Y is H-invariant.*
(ii) *The ideal I is homogeneous.*

Moreover, if one of these equivalences holds, then one has a commutative diagram of K-graded homomorphisms

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}) & \xleftrightarrow{\;\cong\;} & A \\ {\scriptstyle f \mapsto f_{|Y}} \downarrow & & \downarrow {\scriptstyle f \mapsto f + I} \\ \Gamma(Y, \mathcal{O}) & \xleftrightarrow[\;\cong\;]{} & A/I \end{array}$$

We turn to orbits and isotropy groups. First recall the following fact on general algebraic group actions, see e.g. [**88**, Section II.8.3].

PROPOSITION 2.2.6. *Let G be an algebraic group, X a G-variety, and let $x \in X$. Then the isotropy group $G_x \subseteq G$ is closed, the orbit $G{\cdot}x \subseteq X$ is locally closed, and one has a commutative diagram of equivariant morphisms of G-varieties*

$$\begin{array}{ccccc} & & G & & \\ & {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle g \mapsto g \cdot x} & \\ G/G_x & & \xrightarrow[g G_x \mapsto g \cdot x]{\cong} & & G \cdot x \end{array}$$

Moreover, the orbit closure $\overline{G{\cdot}x}$ is the union of $G{\cdot}x$ and orbits of strictly lower dimension and it contains a closed orbit.

DEFINITION 2.2.7. Let A be a K-graded affine algebra and consider the action of $H = \operatorname{Spec}\mathbb{K}[K]$ on $X = \operatorname{Spec} A$.

(i) The *orbit monoid* of $x \in X$ is the submonoid $S_x \subseteq K$ generated by all $w \in K$ that admit a function $f \in A_w$ with $f(x) \neq 0$.
(ii) The *orbit group* of $x \in X$ is the subgroup $K_x \subseteq K$ generated by the orbit monoid $S_x \subseteq K$.

Proposition 2.2.8. *Let A be a K-graded affine algebra, consider the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $X = \operatorname{Spec} A$ and let $x \in X$. Then there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_x & \longrightarrow & K & \longrightarrow & K/K_x & \longrightarrow & 0 \\ & & \downarrow{\cong} & & {\scriptstyle w \mapsto \chi^w}\downarrow{\cong} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & \mathbb{X}(H/H_x) & \xrightarrow[\pi^*]{} & \mathbb{X}(H) & \xrightarrow[\imath^*]{} & \mathbb{X}(H_x) & \longrightarrow & 0 \end{array}$$

where $\imath\colon H_x \to H$ denotes the inclusion of the isotropy group and $\pi\colon H \to H/H_x$ the projection. In particular, we obtain $H_x \cong \operatorname{Spec} \mathbb{K}[K/K_x]$.

Proof. Replacing X with $\overline{H \cdot x}$ does not change K_x. Moreover, take a homogeneous $f \in A$ vanishing along $\overline{H \cdot x} \setminus H \cdot x$ but not at x. Then replacing X with X_f does not affect K_x. Thus, we may assume that $X = H \cdot x$ holds. Then the weight monoid of the H-variety $H \cdot x$ is K_x and by the commutative diagram

$$\begin{array}{ccc} & H & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle h \mapsto h \cdot x} \\ H/H_x & \xrightarrow[\cong]{} & H \cdot x \end{array}$$

we see that $\pi^*(\mathbb{X}(H/H_x))$ consists precisely of the characters χ^w with $w \in K_x$, which gives the desired diagram. □

Proposition 2.2.9. *Let A be a K-graded affine algebra, consider the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $X = \operatorname{Spec} A$ and let $x \in X$. Then the orbit closure $\overline{H \cdot x}$ comes with an action of H/H_x, and there is an isomorphism $\overline{H \cdot x} \cong \operatorname{Spec} \mathbb{K}[S_x]$ of H/H_x-varieties.*

Proof. Write for short $Y := \overline{H \cdot x}$ and $V := H \cdot x$. Then $V \subseteq Y$ is an affine open subset, isomorphic to H/H_x, and we have a commutative diagram

$$\begin{array}{ccc} \Gamma(V, \mathcal{O}) & \xrightarrow{\cong} & \mathbb{K}[K_x] \\ {\scriptstyle f \mapsto f_{|V}}\uparrow & & \uparrow \\ \Gamma(Y, \mathcal{O}) & \xrightarrow[\cong]{} & \mathbb{K}[S_x] \end{array}$$

of graded homomorphisms, where the horizontal arrows send a homogeneous f of degree w to $f(x)\chi^w$. The assertion is part of this. □

Proposition 2.2.10. *Let A be an integral K-graded affine algebra and consider the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $X = \operatorname{Spec} A$. Then there is a nonempty invariant open subset $U \subseteq X$ with*

$$S_x \;=\; S(A), \qquad K_x \;=\; K(A) \qquad \textit{for all } x \in U.$$

Proof. Let $f_1, \ldots, f_r$ be homogeneous generators for A. Then the set $U \subseteq X$ obtained by removing the zero sets $V(X, f_i)$ from X for $i = 1, \ldots, r$ is as wanted. □

Recall that an action of a group G on a set X is said to be *effective* if $g \cdot x = x$ for all $x \in X$ implies $g = e_G$.

Corollary 2.2.11. *Let A be an integral K-graded affine algebra and consider the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $X = \operatorname{Spec} A$. Then the action of H on X is effective if and only if $K = K(A)$ holds.*

2.3. Good quotients. We summarize the basic facts on good quotients. Everything takes place over an algebraically closed field $\mathbb{K}$ of characteristic zero. Besides varieties, we consider more generally possibly non-separated prevarieties. By definition, a *($\mathbb{K}$-)prevariety* is a space X with a sheaf $\mathcal{O}_X$ of $\mathbb{K}$-valued functions covered by open subspaces $X_1, \ldots, X_r$, each of which is an affine ($\mathbb{K}$-)variety.

Let an algebraic group G act on a prevariety X, where, here and later, we always assume that this action is given by a morphism $G \times X \to X$. Recall that a morphism $\varphi\colon X \to Y$ is said to be *G-invariant* if it is constant along the orbits. Moreover, a morphism $\varphi\colon X \to Y$ is called *affine* if for any open affine $V \subseteq Y$ the preimage $\varphi^{-1}(V)$ is an affine variety. When we speak of a *reductive* algebraic group, we mean a not necessarily connected affine algebraic group G such that every rational representation of G splits into irreducible ones.

Definition 2.3.1. Let G be a reductive algebraic group G act on a prevariety X. A morphism $p\colon X \to Y$ of prevarieties is called a *good quotient* for this action if it has the following properties:

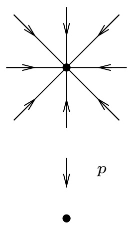
(i) $p\colon X \to Y$ is affine and G-invariant,
(ii) the pullback $p^*\colon \mathcal{O}_Y \to (p_*\mathcal{O}_X)^G$ is an isomorphism.

A morphism $p\colon X \to Y$ is called a *geometric quotient* if it is a good quotient and its fibers are precisely the G-orbits.

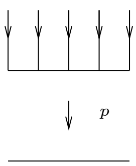
Remark 2.3.2. Let $X = \operatorname{Spec} A$ be an affine G-variety with a reductive algebraic group G. The finiteness theorem of Classical Invariant Theory ensures that the algebra of invariants $A^G \subseteq A$ is finitely generated [**100**, Section II.3.2]. This guarantees existence of a good quotient $p\colon X \to Y$, where $Y := \operatorname{Spec} A^G$. The notion of a good quotient is locally modeled on this concept, because for any good quotient $p'\colon X' \to Y'$ and any affine open $V \subseteq Y'$ the variety V is isomorphic to $\operatorname{Spec} \Gamma(p'^{-1}(V), \mathcal{O})^G$, and the restricted morphism $p'^{-1}(V) \to V$ is the morphism just described.

Example 2.3.3. Consider the $\mathbb{K}^*$-action $t \cdot (z, w) = (t^a z, t^b w)$ on $\mathbb{K}^2$. The following three cases are typical.

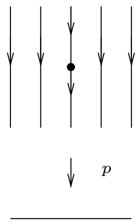
(i) We have $a = b = 1$. Every $\mathbb{K}^*$-invariant function is constant and the constant map $p\colon \mathbb{K}^2 \to \{\mathrm{pt}\}$ is a good quotient.



(ii) We have $a = 0$ and $b = 1$. The algebra of $\mathbb{K}^*$-invariant functions is generated by z and the map $p\colon \mathbb{K}^2 \to \mathbb{K}$, $(z, w) \mapsto z$ is a good quotient.



(iii) We have $a = 1$ and $b = -1$. The algebra of $\mathbb{K}^*$-invariant functions is generated by zw and $p\colon \mathbb{K}^2 \to \mathbb{K}$, $(z, w) \mapsto zw$ is a good quotient.



Note that the general p-fiber is a single $\mathbb{K}^*$-orbit, whereas $p^{-1}(0)$ consists of three orbits and is reducible.

Example 2.3.4. Let A be a K-graded affine algebra. Consider a homomorphism $\psi\colon K \to L$ of abelian groups and the coarsified grading

$$A = \bigoplus_{u \in L} A_u, \qquad A_u = \bigoplus_{w \in \psi^{-1}(u)} A_w.$$

Then the diagonalizable group $H = \operatorname{Spec} \mathbb{K}[L]$ acts on $X = \operatorname{Spec} A$, and for the algebra of invariants we have

$$A^H = \bigoplus_{w \in \ker(\psi)} A_w.$$

Note that in this special case, Proposition 1.2.2 ensures finite generation of the algebra of invariants.

Example 2.3.5 (Veronese subalgebras). Let A be a K-graded affine algebra and $L \subseteq K$ a subgroup. Then we have the corresponding Veronese subalgebra

$$B = \bigoplus_{w \in L} A_w \subseteq \bigoplus_{w \in K} A_w = A.$$

By the preceding example, the morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is a good quotient for the action of $\operatorname{Spec} \mathbb{K}[K/L]$ on $\operatorname{Spec} A$.

We list basic properties of good quotients. The key to most of the statements is the following central observation.

Theorem 2.3.6. *Let a reductive algebraic group G act on a prevariety X. Then any good quotient $p\colon X \to Y$ has the following properties.*

(i) *G-closedness: If $Z \subseteq X$ is G-invariant and closed, then its image $p(Z) \subseteq Y$ is closed.*
(ii) *G-separation: If $Z, Z' \subseteq X$ are G-invariant, closed and disjoint, then $p(Z)$ and $p(Z')$ are disjoint.*

Proof. Since $p\colon X \to Y$ is affine and the statements are local with respect to Y, it suffices to prove them for affine X. This is done in [**100**, Section II.3.2], or [**132**, Theorems 4.6 and 4.7]. □

As an immediate consequence, one obtains basic information on the structure of the fibers of a good quotient.

Corollary 2.3.7. *Let a reductive algebraic group G act on a prevariety X, and let $p\colon X \to Y$ be a good quotient. Then p is surjective and for any $y \in Y$ one has:*

(i) *There is exactly one closed G-orbit $G \cdot x$ in the fiber $p^{-1}(y)$.*
(ii) *Every orbit $G \cdot x' \subseteq p^{-1}(y)$ has $G \cdot x$ in its closure.*

The first statement means that a good quotient $p\colon X \to Y$ parametrizes the closed orbits of the G-prevariety X.

Corollary 2.3.8. *Let a reductive algebraic group G act on a prevariety X, and let $p\colon X \to Y$ be a good quotient.*

(i) *The quotient space Y carries the quotient topology with respect to the map $p\colon X \to Y$.*
(ii) *For every G-invariant morphism of prevarieties $\varphi\colon X \to Z$, there is a unique morphism $\psi\colon Y \to Z$ with $\varphi = \psi \circ p$.*

Proof. The first assertion follows from Theorem 2.3.6 (i). The second one follows from Corollary 2.3.7, Property 2.3.1 (ii) and the first assertion. □

A morphism $p\colon X \to Y$ with the last property is also called a *categorical quotient*. The fact that a good quotient is categorical implies in particular, that the good quotient space is unique up to isomorphy. This justifies the notation $X \to X /\!\!/ G$ for good and $X \to X/G$ for geometric quotients.

Proposition 2.3.9. *Let a reductive algebraic group G act on a prevariety X, and let $p\colon X \to Y$ be a good quotient.*

(i) *Let $V \subseteq Y$ be an open subset. Then the restriction $p\colon p^{-1}(V) \to V$ is a good quotient for the restricted G-action.*
(ii) *Let $Z \subseteq X$ be a closed G-invariant subset. Then the restriction $p\colon Z \to p(Z)$ is a good quotient for the restricted G-action.*

Proof. The first statement is clear and the second one follows immediately from the corresponding statement on the affine case, see [**100**, Section II.3.2]. □

Example 2.3.10 (The Proj construction). Let $A = \oplus A_d$ be a $\mathbb{Z}_{\geq 0}$-graded affine algebra. The *irrelevant ideal* in A is defined as

$$A_{>0} \ := \ \langle f;\ f \in A_d \text{ for some } d > 0 \rangle \ \subseteq \ A.$$

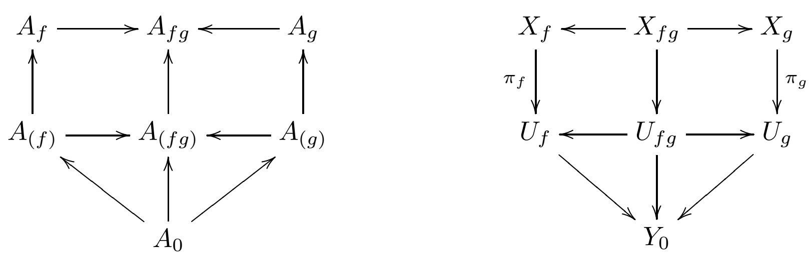
For any homogeneous $f \in A_{>0}$ the localization A_f is a $\mathbb{Z}$-graded affine algebra; concretely, the grading is given by

$$A_f \ = \ \bigoplus_{d \in \mathbb{Z}} (A_f)_d, \qquad (A_f)_d \ := \ \{h/f^l \in A_f;\ \deg(h) - l\deg(f) = d\}.$$

In particular, we have the, again finitely generated, degree zero part of A_f; it is given by

$$A_{(f)} \ := \ (A_f)_0 \ = \ \{h/f^l \in A_f;\ \deg(h) = l\deg(f)\}.$$

Set $X := \operatorname{Spec}(A)$ and $Y_0 := \operatorname{Spec}(A_0)$, and, for a homogeneous $f \in A_{>0}$, set $X_f := \operatorname{Spec} A_f$ and $U_f := \operatorname{Spec} A_{(f)}$. Then, for any two homogeneous $f, g \in A_{>0}$, we have the commutative diagrams



where the second one arises from the first one by applying the Spec-functor. The morphisms $U_{fg} \to U_f$ are open embeddings and gluing the U_f gives the variety $Y = \operatorname{Proj}(A)$. With the zero set $F := V(X, A_{>0})$ of the ideal $A_{>0}$, we have canonical morphisms, where the second one is projective:

$$X \setminus F \xrightarrow{\ \pi\ } Y \longrightarrow Y_0 \ .$$

Geometrically the following happened. The subset $F \subseteq X$ is precisely the fixed point set of the $\mathbb{K}^*$-action on X given by the grading. Thus, $\mathbb{K}^*$ acts with closed orbits on $W := X \setminus F$. The maps $X_f \to U_f$ are geometric quotients, and glue together to a geometric quotient $\pi\colon W \to Y$. Moreover, the $\mathbb{K}^*$-equivariant inclusion $W \subseteq X$ induces the morphism of quotients $Y \to Y_0$.

3. Divisorial algebras

3.1. Sheaves of divisorial algebras. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. We will not only deal with varieties over $\mathbb{K}$ but more generally with prevarieties.

Let X be an irreducible prevariety. The group of *Weil divisors* of X is the free abelian group $\mathrm{WDiv}(X)$ generated by all *prime divisors*, i.e., irreducible subvarieties $D \subseteq X$ of codimension one. To a non-zero rational function $f \in \mathbb{K}(X)^*$ one associates a Weil divisor using its *order* along prime divisors D; recall that, if f belongs to the local ring $\mathcal{O}_{X,D}$, then $\mathrm{ord}_D(f)$ is the length of the $\mathcal{O}_{X,D}$-module $\mathcal{O}_{X,D}/\langle f\rangle$, and otherwise one writes $f = g/h$ with $g, h \in \mathcal{O}_{X,D}$ and defines the order of f to be the difference of the orders of g and h. The divisor of $f \in \mathbb{K}(X)^*$ then is

$$\mathrm{div}(f) \quad := \sum_{D \text{ prime}} \mathrm{ord}_D(f) \cdot D.$$

The assignment $f \mapsto \mathrm{div}(f)$ is a homomorphism $\mathbb{K}(X)^* \to \mathrm{WDiv}(X)$, and its image $\mathrm{PDiv}(X) \subseteq \mathrm{WDiv}(X)$ is called the subgroup of *principal divisors*. The *divisor class group* of X is the factor group

$$\mathrm{Cl}(X) \quad := \quad \mathrm{WDiv}(X) \ / \ \mathrm{PDiv}(X).$$

A Weil divisor $D = a_1D_1 + \ldots + a_sD_s$ with prime divisors D_i is called *effective*, denoted as $D \geq 0$, if $a_i \geq 0$ holds for $i = 1, \ldots, s$. To every divisor $D \in \mathrm{WDiv}(X)$, one associates a sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules by defining its sections over an open $U \subseteq X$ as

$$\Gamma(U, \mathcal{O}_X(D)) \quad := \quad \{f \in \mathbb{K}(X)^*;\ (\mathrm{div}(f) + D)_{|U} \geq 0\} \ \cup \ \{0\},$$

where the restriction map $\mathrm{WDiv}(X) \to \mathrm{WDiv}(U)$ is defined for a prime divisor D as $D_{|U} := D \cap U$ if it intersects U and $D_{|U} := 0$ otherwise. Note that for any two functions $f_1 \in \Gamma(U, \mathcal{O}_X(D_1))$ and $f_2 \in \Gamma(U, \mathcal{O}_X(D_2))$ the product f_1f_2 belongs to $\Gamma(U, \mathcal{O}_X(D_1 + D_2))$.

Definition 3.1.1. The *sheaf of divisorial algebras* associated to a subgroup $K \subseteq \mathrm{WDiv}(X)$ is the sheaf of K-graded $\mathcal{O}_X$-algebras

$$\mathcal{S} \ := \ \bigoplus_{D \in K} \mathcal{S}_D, \qquad\qquad \mathcal{S}_D \ := \ \mathcal{O}_X(D),$$

where the multiplication in $\mathcal{S}$ is defined by multiplying homogeneous sections in the field of functions $\mathbb{K}(X)$.

Example 3.1.2. On the projective line $X = \mathbb{P}_1$, consider $D := \{\infty\}$, the group $K := \mathbb{Z}D$, and the associated K-graded sheaf of algebras $\mathcal{S}$. Then we have isomorphisms

$$\varphi_n\colon \mathbb{K}[T_0, T_1]_n \ \to \ \Gamma(\mathbb{P}_1, \mathcal{S}_{nD}), \qquad f \ \mapsto \ f(1, z),$$

where $\mathbb{K}[T_0, T_1]_n \subseteq \mathbb{K}[T_0, T_1]$ denotes the vector space of all polynomials homogeneous of degree n. Putting them together we obtain a graded isomorphism

$$\mathbb{K}[T_0, T_1] \ \cong \ \Gamma(\mathbb{P}_1, \mathcal{S}).$$

Fix a normal (irreducible) prevariety X, a subgroup $K \subseteq \mathrm{WDiv}(X)$ on the normal prevariety X and let $\mathcal{S}$ be the associated divisorial algebra. We collect first properties.

Remark 3.1.3. If $V \subseteq U \subseteq X$ are open subsets such that $U \setminus V$ is of codimension at least two in U, then we have an isomorphism

$$\Gamma(U, \mathcal{S}) \;\to\; \Gamma(V, \mathcal{S}).$$

In particular, the algebra $\Gamma(U, \mathcal{S})$ equals the algebra $\Gamma(U_{\mathrm{reg}}, \mathcal{S})$, where $U_{\mathrm{reg}} \subseteq U$ denotes the set of smooth points.

Remark 3.1.4. Assume that $D_1, \ldots, D_s$ is a basis for $K \subseteq \mathrm{WDiv}(X)$ and suppose that $U \subseteq X$ is an open subset on which each D_i is principal, say $D_i = \mathrm{div}(f_i)$. Then, with $\deg(T_i) = D_i$ and $f_i^{-1} \in \Gamma(X, \mathcal{S}_{D_i})$, we have a graded isomorphism

$$\Gamma(U, \mathcal{O}) \otimes_{\mathbb{K}} \mathbb{K}[T_1^{\pm 1}, \ldots, T_s^{\pm 1}] \;\to\; \Gamma(U, \mathcal{S}), \qquad g \otimes T_1^{\nu_1} \cdots T_s^{\nu_s} \;\mapsto\; g f_1^{-\nu_1} \cdots f_s^{-\nu_s}.$$

Remark 3.1.5. If K is of finite rank, say s, then the algebra $\Gamma(X, \mathcal{S})$ of global sections can be realized as a graded subalgebra of the Laurent polynomial algebra $\mathbb{K}(X)[T_1^{\pm 1}, \ldots, T_s^{\pm 1}]$. Indeed, let $D_1, \ldots, D_s$ be a basis for K. Then we obtain a monomorphism

$$\Gamma(X, \mathcal{S}) \;\to\; \mathbb{K}(X)[T_1^{\pm 1}, \ldots, T_s^{\pm 1}], \qquad \Gamma(X, \mathcal{S}_{a_1 D_1 + \ldots + a_s D_s}) \ni f \;\mapsto\; f T_1^{a_1} \cdots T_s^{a_s}.$$

In particular, $\Gamma(X, \mathcal{S})$ is an integral ring and we have an embedding of the associated quotient fields

$$\mathrm{Quot}(\Gamma(X, \mathcal{S})) \;\to\; \mathbb{K}(X)(T_1, \ldots, T_s).$$

For quasiaffine X, we have $\mathbb{K}(X) \subseteq \mathrm{Quot}(\Gamma(X, \mathcal{S}))$ and for each variable T_i there is a non-zero function $f_i \in \Gamma(X, \mathcal{S}_{D_i})$. Thus, for quasiaffine X, one obtains

$$\mathrm{Quot}(\Gamma(X, \mathcal{S})) \;\cong\; \mathbb{K}(X)(T_1, \ldots, T_s).$$

The *support* $\mathrm{Supp}(D)$ of a Weil divisor $D = a_1 D_1 + \ldots + a_s D_s$ with prime divisors D_i is the union of those D_i with $a_i \neq 0$. Moreover, for a Weil divisor D on a normal prevariety X and a non-zero section $f \in \Gamma(X, \mathcal{O}_X(D))$, we define the *$D$-divisor* and the *$D$-localization*

$$\mathrm{div}_D(f) \;:=\; \mathrm{div}(f) + D \;\in\; \mathrm{WDiv}(X), \qquad X_{D,f} \;:=\; X \setminus \mathrm{Supp}(\mathrm{div}_D(f)) \;\subseteq\; X.$$

The D-divisor is always effective. Moreover, given sections $f \in \Gamma(X, \mathcal{O}_X(D))$ and $g \in \Gamma(X, \mathcal{O}_X(E))$, we have

$$\mathrm{div}_{D+E}(fg) \;=\; \mathrm{div}_D(f) + \mathrm{div}_E(g), \qquad f^{-1} \;\in\; \Gamma(X_{D,f}, \mathcal{O}_X(-D)).$$

Remark 3.1.6. Let $D \in K$ and consider a non-zero homogeneous section $f \in \Gamma(X, \mathcal{S}_D)$. Then one has a canonical isomorphism of K-graded algebras

$$\Gamma(X_{D,f}, \mathcal{S}) \;\cong\; \Gamma(X, \mathcal{S})_f.$$

Indeed, the canonical monomorphism $\Gamma(X, \mathcal{S})_f \to \Gamma(X_{D,f}, \mathcal{S})$ is surjective, because for any $g \in \Gamma(X_{D,f}, \mathcal{S}_E)$, we have $g f^m \in \Gamma(X, \mathcal{S}_{mD+E})$ with some $m \in \mathbb{Z}_{\geq 0}$.

3.2. The relative spectrum. Again we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let X be a normal prevariety. As any quasicoherent sheaf of $\mathcal{O}_X$-algebras, the sheaf of divisorial algebras $\mathcal{S}$ associated to a group $K \subseteq \mathrm{WDiv}(X)$ of Weil divisors defines in a natural way a geometric object, its relative spectrum $\widetilde{X} := \mathrm{Spec}_X \mathcal{S}$. We briefly recall how to obtain it.

Construction 3.2.1. Let $\mathcal{S}$ be any quasicoherent sheaf of reduced $\mathcal{O}_X$-algebras on a prevariety X, and suppose that $\mathcal{S}$ is locally of finite type, i.e., X is covered by open affine subsets $X_1, \ldots, X_r \subseteq X$ with $\Gamma(X_i, \mathcal{S})$ finitely generated. Cover each intersection $X_{ij} := X_i \cap X_j$ by open subsets $(X_i)_{f_{ijk}}$, where

$f_{ijk} \in \Gamma(X_i, \mathcal{O})$. Set $\widetilde{X}_i := \operatorname{Spec}\Gamma(X_i, \mathcal{S})$ and let $\widetilde{X}_{ij} \subseteq \widetilde{X}_i$ be the union of the open subsets $(\widetilde{X}_i)_{f_{ijk}}$. Then we obtain commutative diagrams

$$\begin{array}{ccccccc} \widetilde{X}_i & \longleftarrow & \widetilde{X}_{ij} & \overset{\cong}{\longleftrightarrow} & \widetilde{X}_{ji} & \longrightarrow & \widetilde{X}_j \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_i & \longleftarrow & X_{ij} & = & X_{ji} & \longrightarrow & X_j \end{array}$$

This allows us to glue together the $\widetilde{X}_i$ along the $\widetilde{X}_{ij}$, and we obtain a prevariety $\widetilde{X} = \operatorname{Spec}_X \mathcal{S}$ coming with a canonical morphism $p\colon \widetilde{X} \to X$. Note that $p_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{S}$ holds. In particular, $\Gamma(\widetilde{X}, \mathcal{O})$ equals $\Gamma(X, \mathcal{S})$. Moreover, the morphism p is affine and $\widetilde{X}$ is separated if X is so. Finally, the whole construction does not depend on the choice of the X_i.

Before specializing this construction to the case of our sheaf of divisorial algebras $\mathcal{S}$ on X, we provide two criteria for $\mathcal{S}$ being locally of finite type. The first one is an immediate consequence of Remark 3.1.6.

Proposition 3.2.2. *Let X be a normal prevariety, $K \subseteq \operatorname{WDiv}(X)$ a finitely generated subgroup, and $\mathcal{S}$ the associated sheaf of divisorial algebras. If $\Gamma(X, \mathcal{S})$ is finitely generated and X is covered by affine open subsets of the form $X_{D,f}$, where $D \in K$ and $f \in \Gamma(X, \mathcal{S}_D)$, then $\mathcal{S}$ is locally of finite type.*

A Weil divisor $D \in \operatorname{WDiv}(X)$ on a prevariety X is called *Cartier* if it is locally a principal divisor, i.e., locally of the form $D = \operatorname{div}(f)$ with a rational function f. The prevariety X is *locally factorial*, i.e., all local rings $\mathcal{O}_{X,x}$ are unique factorization domains if and only if every Weil divisor of X is Cartier. Recall that smooth prevarieties are locally factorial. More generally, a normal prevariety is called $\mathbb{Q}$-*factorial* if for any Weil divisor some positive multiple is Cartier.

Proposition 3.2.3. *Let X be a normal prevariety and $K \subseteq \operatorname{WDiv}(X)$ a finitely generated subgroup. If X is $\mathbb{Q}$-factorial, then the associated sheaf $\mathcal{S}$ of divisorial algebras is locally of finite type.*

Proof. By $\mathbb{Q}$-factoriality, the subgroup $K^0 \subseteq K$ consisting of all Cartier divisors is of finite index in K. Choose a basis $D_1, \ldots, D_s$ for K such that with suitable $a_i > 0$ the multiples $a_i D_i$, where $1 \le i \le s$, form a basis for K^0. Moreover, cover X by open affine subsets $X_1, \ldots, X_r \subseteq X$ such that for any $D \in K^0$ all restrictions $D_{|X_i}$ are principal. Let $\mathcal{S}^0$ be the sheaf of divisorial algebras associated to K^0. Then $\Gamma(X_i, \mathcal{S}^0)$ is the Veronese subalgebra of $\Gamma(X_i, \mathcal{S})$ defined by $K^0 \subseteq K$. By Remark 3.1.4, the algebra $\Gamma(X_i, \mathcal{S}^0)$ is finitely generated. Since $K^0 \subseteq K$ is of finite index, we can apply Proposition 1.2.4 and obtain that $\Gamma(X_i, \mathcal{S})$ is finitely generated. □

Construction 3.2.4. Let X be a normal prevariety, $K \subseteq \operatorname{WDiv}(X)$ a finitely generated subgroup and $\mathcal{S}$ the associated sheaf of divisorial algebras. We assume that $\mathcal{S}$ is locally of finite type. Then, in the notation of Construction 3.2.1, the algebras $\Gamma(X_i, \mathcal{S})$ are K-graded. This means that each affine variety $\widetilde{X}_i$ comes with an action of the torus $H := \operatorname{Spec}\mathbb{K}[K]$, and, because of $\mathcal{S}_0 = \mathcal{O}_X$, the canonical map $\widetilde{X}_i \to X_i$ is a good quotient for this action. Since the whole gluing process is equivariant, we end up with an H-prevariety $\widetilde{X} = \operatorname{Spec}_X\mathcal{S}$ and $p\colon \widetilde{X} \to X$ is a good quotient for the H-action.

Example 3.2.5. Consider once more the projective line $X = \mathbb{P}_1$, the group $K := \mathbb{Z}D$, where $D := \{\infty\}$, and the associated sheaf $\mathcal{S}$ of divisorial algebras. For the affine charts $X_0 = \mathbb{K}$ and $X_1 = \mathbb{K}^* \cup \{\infty\}$ we have the graded isomorphisms

$$\mathbb{K}[T_0^{\pm 1}, T_1] \;\to\; \Gamma(X_0, \mathcal{S}), \qquad \mathbb{K}[T_0^{\pm 1}, T_1]_n \ni f \;\mapsto\; f(1, z) \in \Gamma(X_0, \mathcal{S}_{nD}),$$

$$\mathbb{K}[T_0, T_1^{\pm 1}] \;\to\; \Gamma(X_1, \mathcal{S}), \qquad \mathbb{K}[T_0, T_1^{\pm 1}]_n \ni f \;\mapsto\; f(z, 1) \in \Gamma(X_1, \mathcal{S}_{nD}).$$

Thus, the corresponding spectra are $\mathbb{K}^2_{T_0}$ and $\mathbb{K}^2_{T_1}$. The gluing takes place along $(\mathbb{K}^*)^2$ and gives $\widetilde{X} = \mathbb{K}^2 \setminus \{0\}$. The action of $\mathbb{K}^* = \operatorname{Spec} \mathbb{K}[K]$ on $\widetilde{X}$ is the usual scalar multiplication.

The above example fits into the more general context of sheaves of divisorial algebras associated to groups generated by a very ample divisor, i.e., the pullback of a hyperplane with respect to an embedding into a projective space.

Example 3.2.6. Suppose that X is projective and $K = \mathbb{Z}D$ holds with a very ample divisor D on X. Then $\Gamma(X, \mathcal{S})$ is finitely generated and thus we have the affine cone $\overline{X} := \operatorname{Spec} \Gamma(X, \mathcal{S})$ over X. It comes with a $\mathbb{K}^*$-action and an attractive fixed point $\overline{x}_0 \in \overline{X}$, i.e., $\overline{x}_0$ lies in the closure of any $\mathbb{K}^*$-orbit. The relative spectrum $\widetilde{X} = \operatorname{Spec}_X \mathcal{S}$ equals $\overline{X} \setminus \{\overline{x}_0\}$.

Remark 3.2.7. In the setting of 3.2.4, let $U \subseteq X$ be an open subset such that all divisors $D \in K$ are principal over U. Then there is a commutative diagram of H-equivariant morphisms

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\;\cong\;} & H \times U \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_U} & \\ & U & \end{array}$$

where H acts on $H \times U$ by multiplication on the first factor. In particular, if K consists of Cartier divisors, e.g. if X is locally factorial, then $\widetilde{X} \to X$ is a locally trivial H-principal bundle.

Proposition 3.2.8. *Situation as in Construction 3.2.4. The prevariety $\widetilde{X}$ is normal. Moreover, for any closed $A \subseteq X$ of codimension at least two, $p^{-1}(A) \subseteq \widetilde{X}$ is as well of codimension at least two.*

Proof. For normality, we have to show that for every affine open $U \subseteq X$ the algebra $\Gamma(p^{-1}(U), \mathcal{O})$ is a normal ring. According to Remark 3.1.3, we have

$$\Gamma(p^{-1}(U), \mathcal{O}) \quad = \quad \Gamma(p^{-1}(U_{\mathrm{reg}}), \mathcal{O}).$$

Using Remark 3.1.4, we see that the latter ring is normal. The supplement is then an immediate consequence of Remark 3.1.3. $\square$

3.3. Unique factorization in the global ring. Here we investigate divisibility properties of the ring of global sections of the sheaf of divisorial algebras $\mathcal{S}$ associated to a subgroup $K \subseteq \operatorname{WDiv}(X)$ on a normal prevariety X. The key statement is the following.

Theorem 3.3.1. *Let X be a smooth prevariety, $K \subseteq \operatorname{WDiv}(X)$ a finitely generated subgroup, $\mathcal{S}$ the associated sheaf of divisorial algebras and $\widetilde{X} = \operatorname{Spec}_X \mathcal{S}$. Then the following statements are equivalent.*

(i) *The canonical map $K \to \operatorname{Cl}(X)$ is surjective.*
(ii) *The divisor class group $\operatorname{Cl}(\widetilde{X})$ is trivial.*

We need a preparing observation concerning the pullback of Cartier divisors. Recall that for any dominant morphism $\varphi\colon X \to Y$ of normal prevarieties, there is a pullback of Cartier divisors: if a Cartier divisor E on Y is locally given as $E = \mathrm{div}(g)$, then the pullback divisor $\varphi^*(E)$ is the Cartier divisor locally defined by $\mathrm{div}(\varphi^*(g))$.

Lemma 3.3.2. *Situation as in Construction 3.2.4. Suppose that $D \in K$ is Cartier and consider a non-zero section $f \in \Gamma(X, \mathcal{S}_D)$. Then one has*

$$p^*(D) \quad = \quad \mathrm{div}(f) - p^*(\mathrm{div}(f)),$$

where on the right hand side f is firstly viewed as a homogeneous function on $\widetilde{X}$, and secondly as a rational function on X. In particular, $p^*(D)$ is principal.

Proof. On suitable open sets $U_i \subseteq X$, we find defining equations f_i^{-1} for D and thus may write $f = h_i f_i$, where $h_i \in \Gamma(U_i, \mathcal{S}_0) = \Gamma(U_i, \mathcal{O})$ and $f_i \in \Gamma(U_i, \mathcal{S}_D)$. Then, on $p^{-1}(U_i)$, we have $p^*(h_i) = h_i$ and the function f_i is homogeneous of degree D and invertible. Thus, we obtain

$$\begin{aligned} p^*(D) &= p^*(\mathrm{div}(f) + D) - p^*(\mathrm{div}(f)) \\ &= p^*(\mathrm{div}(h_i)) - p^*(\mathrm{div}(f)) \\ &= \mathrm{div}(h_i) - p^*(\mathrm{div}(f)) \\ &= \mathrm{div}(h_i f_i) - p^*(\mathrm{div}(f)) \\ &= \mathrm{div}(f) - p^*(\mathrm{div}(f)). \end{aligned}$$

□

We are almost ready for proving the Theorem. Recall that, given an action of an algebraic group G on a normal prevariety X, we obtain an induced action of G on the group of Weil divisors by sending a prime divisor $D \subseteq X$ to $g \cdot D \subseteq X$. In particular, we can speak about invariant Weil divisors.

Proof of Theorem 3.3.1. Suppose that (i) holds. It suffices to show that every effective divisor $\widetilde{D}$ on $\widetilde{X}$ is principal. We work with the action of the torus $H = \mathrm{Spec}\,\mathbb{K}[K]$ on $\widetilde{X}$. Choosing an H-linearization of $\widetilde{D}$, see [**95**, Section 2.4], we obtain a representation of H on $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{D}))$ such that for any section $\widetilde{f} \in \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\widetilde{D}))$ one has

$$\mathrm{div}_{\widetilde{D}}(h \cdot \widetilde{f}) \quad = \quad h \cdot \mathrm{div}_{\widetilde{D}}(\widetilde{f}).$$

Taking a non-zero $\widetilde{f}$, which is homogeneous with respect to this representation, we obtain that $\widetilde{D}$ is linearly equivalent to the H-invariant divisor $\mathrm{div}_{\widetilde{D}}(\widetilde{f})$. This reduces the problem to the case of an invariant divisor $\widetilde{D}$; compare also [**9**, Theorem 4.2]. Now, consider any invariant prime divisor $\widetilde{D}$ on $\widetilde{X}$. Let $D := p(\widetilde{D})$ be the image under the good quotient $p\colon \widetilde{X} \to X$. Remark 3.2.7 gives $\widetilde{D} = p^*(D)$. By assumption, D is linearly equivalent to a divisor $D' \in K$. Thus, $\widetilde{D}$ is linearly equivalent to $p^*(D')$, which in turn is principal by Lemma 3.3.2.

Now suppose that (ii) holds. It suffices to show that any effective $D \in \mathrm{WDiv}(X)$ is linearly equivalent to some $D' \in K$. The pullback $p^*(D)$ is the divisor of some function $f \in \Gamma(\widetilde{X}, \mathcal{O})$. We claim that f is K-homogeneous. Indeed

$$F\colon H \times \widetilde{X} \ \to \ \mathbb{K}, \qquad (h, x) \ \mapsto \ f(h \cdot x)/f(x)$$

is an invertible function. By Rosenlicht's Lemma [**96**, Section 1.1], we have $F(h, x) = \chi(h) g(x)$ with $\chi \in \mathbb{X}(H)$ and $g \in \Gamma(\widetilde{X}, \mathcal{O}^*)$. Plugging $(1, x)$ into F yields

$g = 1$ and, consequently, $f(h \cdot x) = \chi(h)f(x)$ holds. Thus, we have $f \in \Gamma(X, \mathcal{S}_{D'})$ for some $D' \in K$. Lemma 3.3.2 gives

$$p^*(D) \;=\; \mathrm{div}(f) \;=\; p^*(D') + p^*(\mathrm{div}(f)),$$

where in the last term, f is regarded as a rational function on X. We conclude $D = D' + \mathrm{div}(f)$ on X. In other words, D is linearly equivalent to $D' \in K$. □

As an immediate consequence, we obtain factoriality of the ring of global sections provided $K \to \mathrm{Cl}(X)$ is surjective, see also [**28**], [**68**] and [**15**].

THEOREM 3.3.3. *Let X be a normal prevariety, $K \subseteq \mathrm{WDiv}(X)$ a finitely generated subgroup and $\mathcal{S}$ the associated sheaf of divisorial algebras. If the canonical map $K \to \mathrm{Cl}(X)$ is surjective, then the algebra $\Gamma(X, \mathcal{S})$ is a unique factorization domain.*

PROOF. According to Remark 3.1.3, the algebra $\Gamma(X, \mathcal{S})$ equals $\Gamma(X_{\mathrm{reg}}, \mathcal{S})$ and thus we may apply Theorem 3.3.1. □

Divisibility and primality in the ring of global sections $\Gamma(X, \mathcal{S})$ can be characterized purely in terms of X.

PROPOSITION 3.3.4. *Let X be a normal prevariety, $K \subseteq \mathrm{WDiv}(X)$ a finitely generated subgroup projecting onto $\mathrm{Cl}(X)$ and let $\mathcal{S}$ be the associated sheaf of divisorial algebras.*

(i) *An element $0 \neq f \in \Gamma(X, \mathcal{S}_D)$ divides an element $0 \neq g \in \Gamma(X, \mathcal{S}_E)$ if and only if $\mathrm{div}_D(f) \leq \mathrm{div}_E(g)$ holds.*
(ii) *An element $0 \neq f \in \Gamma(X, \mathcal{S}_D)$ is prime if and only if the divisor $\mathrm{div}_D(f) \in \mathrm{WDiv}(X)$ is prime.*

PROOF. We may assume that X is smooth. Then $\widetilde{X} = \mathrm{Spec}_X \mathcal{S}$ exists, and Lemma 3.3.2 reduces (i) and (ii) to the corresponding statements on regular functions on $\widetilde{X}$, which in turn are well known. □

3.4. Geometry of the relative spectrum. We collect basic geometric properties of the relative spectrum of a sheaf of divisorial algebras. We will use the following pullback construction for Weil divisors.

REMARK 3.4.1. Consider any morphism $\varphi\colon \widetilde{X} \to X$ of normal prevarieties such that the closure of $X \setminus \varphi(\widetilde{X})$ is of codimension at least two in X. Then we may define a pullback homomorphism for Weil divisors

$$\varphi^*\colon\ \mathrm{WDiv}(X) \;\to\; \mathrm{WDiv}(\widetilde{X})$$

as follows: Given $D \in \mathrm{WDiv}(X)$, consider its restriction D' to X_{reg}, the usual pullback $\varphi^*(D')$ of Cartier divisors on $\varphi^{-1}(X_{\mathrm{reg}})$ and define $\varphi^*(D)$ to be the Weil divisor obtained by closing the support of $\varphi^*(D')$. Note that we always have

$$\mathrm{Supp}(\varphi^*(D)) \quad \subseteq \quad \varphi^{-1}(\mathrm{Supp}(D)).$$

If for any closed $A \subseteq X$ of codimension at least two, $\varphi^{-1}(A) \subseteq \widetilde{X}$ is as well of codimension at least two, then φ^* maps principal divisors to principal divisors, and we obtain a pullback homomorphism

$$\varphi^*\colon\ \mathrm{Cl}(X) \;\to\; \mathrm{Cl}(\widetilde{X}).$$

EXAMPLE 3.4.2. Consider $X = V(\mathbb{K}^4;\ T_1T_2 - T_3T_4)$ and $\widetilde{X} = \mathbb{K}^4$. Then we have a morphism

$$p\colon \widetilde{X} \;\to\; X, \qquad z \;\mapsto\; (z_1z_2, z_3z_4, z_1z_3, z_2z_4).$$

For the prime divisor $D = \mathbb{K} \times 0 \times \mathbb{K} \times 0$ on X, we have

$$\mathrm{Supp}(p^*(D)) \;=\; V(\widetilde{X}; Z_4) \;\subsetneq\; V(\widetilde{X}; Z_4) \cup V(\widetilde{X}; Z_2, Z_3) \;=\; p^{-1}(\mathrm{Supp}(D)).$$

In fact, $p\colon \widetilde{X} \to X$ is the morphism determined by the sheaf of divisorial algebras associated to $K = \mathbb{Z}D$.

We say that a prevariety X is of *affine intersection* if for any two affine open subsets $U, U' \subseteq X$ the intersection $U \cap U'$ is again affine. For example, every variety is of affine intersection. Note that a prevariety X is of affine intersection if it can be covered by open affine subsets $X_1, \ldots, X_s \subseteq X$ such that all intersections $X_i \cap X_j$ are affine. Moreover, if X is of affine intersection, then the complement of any affine open subset $U \subsetneq X$ is of pure codimension one.

PROPOSITION 3.4.3. *In the situation of 3.2.4, consider the pullback homomorphism $p^*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(\widetilde{X})$ defined in 3.4.1. Then, for every $D \in K$ and every non-zero $f \in \Gamma(X, \mathcal{S}_D)$, we have*

$$\mathrm{div}(f) \;=\; p^*(\mathrm{div}_D(f)),$$

where on the left hand side f is a function on $\widetilde{X}$, and on the right hand side a function on X. If X is of affine intersection and $X_{D,f}$ is affine, then we have moreover

$$\mathrm{Supp}(\mathrm{div}(f)) \;=\; p^{-1}(\mathrm{Supp}(\mathrm{div}_D(f))).$$

PROOF. By Lemma 3.3.2, the first equation holds on $p^{-1}(X_{\mathrm{reg}})$. By Proposition 3.2.8, the complement $\widetilde{X} \setminus p^{-1}(X_{\mathrm{reg}})$ is of codimension at least two and thus the first equation holds on the whole $\widetilde{X}$. For the proof of the second one, consider

$$X_{D,f} \;=\; X \setminus \mathrm{Supp}(\mathrm{div}_D(f)), \qquad \widetilde{X}_f \;=\; \widetilde{X} \setminus V(\widetilde{X}, f).$$

Then we have to show that $p^{-1}(X_{D,f})$ equals $\widetilde{X}_f$. Since f is invertible on $p^{-1}(X_{D,f})$, we obtain $p^{-1}(X_{D,f}) \subseteq \widetilde{X}_f$. Moreover, Lemma 3.3.2 yields

$$p^{-1}(X_{D,f}) \;\cap\; p^{-1}(X_{\mathrm{reg}}) \;=\; \widetilde{X}_f \cap p^{-1}(X_{\mathrm{reg}}).$$

Thus the complement $\widetilde{X}_f \setminus p^{-1}(X_{D,f})$ of the affine subset $p^{-1}(X_{D,f}) \subseteq \widetilde{X}_f$ is of codimension at least two. Since $p\colon \widetilde{X} \to X$ is affine, the prevariety $\widetilde{X}$ inherits the property to be of affine intersection from X and hence $\widetilde{X}_f \setminus p^{-1}(X_{D,f})$ must be empty. □

COROLLARY 3.4.4. *Situation as in Construction 3.2.4. Let $\widetilde{x} \in \widetilde{X}$ be a point such that $H \cdot \widetilde{x} \subseteq \widetilde{X}$ is closed, and let $0 \neq f \in \Gamma(X, \mathcal{S}_D)$. Then we have*

$$f(\widetilde{x}) \;=\; 0 \quad\Longleftrightarrow\quad p(\widetilde{x}) \;\in\; \mathrm{Supp}(\mathrm{div}_D(f)).$$

PROOF. Remark 3.4.1 and Proposition 3.4.3 show that $p(\mathrm{Supp}(\mathrm{div}(f)))$ is contained in $\mathrm{Supp}(\mathrm{div}_D(f))$. Moreover, they coincide along the smooth locus of X and Theorem 2.3.6 ensures that $p(\mathrm{Supp}(\mathrm{div}(f)))$ is closed. This gives

$$p(\mathrm{Supp}(\mathrm{div}(f))) \;=\; \mathrm{Supp}(\mathrm{div}_D(f)).$$

Thus, $f(\widetilde{x}) = 0$ implies $p(\widetilde{x}) \in \mathrm{Supp}(\mathrm{div}_D(f))$. If $p(\widetilde{x}) \in \mathrm{Supp}(\mathrm{div}_D(f))$ holds, then some $\widetilde{x}' \in \mathrm{Supp}(\mathrm{div}(f))$ lies in the p-fiber of $\widetilde{x}$. Since $H \cdot \widetilde{x}$ is closed, it is contained in the closure of $H \cdot \widetilde{x}'$, see Corollary 2.3.7. This implies $\widetilde{x} \in \mathrm{Supp}(\mathrm{div}(f))$. □

COROLLARY 3.4.5. *Situation as in Construction 3.2.4. If X is of affine intersection and covered by affine open subsets of the form $X_{D,f}$, where $D \in K$ and $f \in \Gamma(X, \mathcal{S}_D)$, then $\widetilde{X}$ is a quasiaffine variety.*

Proof. According to Proposition 3.4.3, the prevariety $\widetilde{X}$ is covered by open affine subsets of the form $\widetilde{X}_f$ and thus is quasiaffine. □

Corollary 3.4.6. *Situation as in Construction 3.2.4. If X is of affine intersection and $K \to \mathrm{Cl}(X)$ is surjective, then $\widetilde{X}$ is a quasiaffine variety.*

Proof. Cover X by affine open sets $X_1, \ldots, X_r$. Since X is of affine intersection, every complement $X \setminus X_i$ is of pure codimension one. Since $K \to \mathrm{Cl}(X)$ is surjective, we obtain that $X \setminus X_i$ is the support of the D-divisor of some $f \in \Gamma(X, \mathcal{S}_D)$. The assertion thus follows from Corollary 3.4.5. □

Proposition 3.4.7. *Situation as in Construction 3.2.4. For $x \in X$, let $K_x^0 \subseteq K$ be the subgroup of divisors that are principal near x and let $\widetilde{x} \in p^{-1}(x)$ be a point with closed H-orbit. Then the isotropy group $H_{\widetilde{x}} \subseteq H$ is given by $H_{\widetilde{x}} = \mathrm{Spec}\,\mathbb{K}[K/K_x^0]$.*

Proof. Replacing X with a suitable affine neighbourhood of x, we may assume that $\widetilde{X}$ is affine. By Proposition 2.2.8, the isotropy group $H_{\widetilde{x}}$ is $\mathrm{Spec}\,\mathbb{K}[K/K_{\widetilde{x}}]$ with the orbit group

$$K_{\widetilde{x}} \ = \ \langle D \in K;\ f(\widetilde{x}) \neq 0 \text{ for some } f \in \Gamma(X, \mathcal{S}_D) \rangle \ \subseteq \ K.$$

Using Corollary 3.4.4, we obtain that there exists an $f \in \Gamma(X, \mathcal{S}_D)$ with $f(\widetilde{x}) \neq 0$ if and only if $D \in K_x^0$ holds. The assertion follows. □

Corollary 3.4.8. *Situation as in Construction 3.2.4.*

(i) *If X is locally factorial, then H acts freely on $\widetilde{X}$.*
(ii) *If X is $\mathbb{Q}$-factorial, then H acts with at most finite isotropy groups on $\widetilde{X}$.*

4. Cox sheaves and Cox rings

4.1. Free divisor class group. As before, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. We introduce Cox sheaves and Cox rings for a prevariety with a free finitely generated divisor class group. As an example, we compute in 4.1.6 the Cox ring of a non-separated curve, the projective line with multipled points.

Construction 4.1.1. Let X be a normal prevariety with free finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that the canonical map $c\colon K \to \mathrm{Cl}(X)$ sending $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is an isomorphism. We define the *Cox sheaf* associated to K to be

$$\mathcal{R} \ := \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} \ := \ \mathcal{O}_X(D),$$

where $D \in K$ represents $[D] \in \mathrm{Cl}(X)$ and the multiplication in $\mathcal{R}$ is defined by multiplying homogeneous sections in the field of rational functions $\mathbb{K}(X)$. The sheaf $\mathcal{R}$ is a quasicoherent sheaf of normal integral $\mathcal{O}_X$-algebras and, up to isomorphy, it does not depend on the choice of the subgroup $K \subseteq \mathrm{WDiv}(X)$. The *Cox ring* of X is the algebra of global sections

$$\mathcal{R}(X) \ := \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}(X), \qquad \mathcal{R}_{[D]}(X) \ := \ \Gamma(X, \mathcal{O}_X(D)).$$

Proof of Construction 4.1.1. Given two subgroups $K, K' \subseteq \mathrm{WDiv}(X)$ projecting isomorphically onto $\mathrm{Cl}(X)$, we have to show that the corresponding sheaves of divisorial algebras $\mathcal{R}$ and $\mathcal{R}'$ are isomorphic. Choose a basis $D_1, \ldots, D_s$ for K and define a homomorphism

$$\eta\colon K \ \to \ \mathbb{K}(X)^*, \qquad a_1D_1 + \ldots + a_sD_s \ \mapsto \ f_1^{a_1} \cdots f_s^{a_s},$$

where $f_1, \dots, f_s \in \mathbb{K}(X)^*$ are such that the divisors $D_i - \mathrm{div}(f_i)$ form a basis of K'. Then we obtain an isomorphism $(\psi, \widetilde{\psi})$ of the sheaves of divisorial algebras $\mathcal{R}$ and $\mathcal{R}'$ by setting

$$\begin{aligned} \widetilde{\psi}\colon K &\to K', & D &\mapsto -\mathrm{div}(\eta(D)) + D, \\ \psi\colon \mathcal{R} &\to \mathcal{R}', & \Gamma(U, \mathcal{R}_{[D]}) \ni f &\mapsto \eta(D) f \in \Gamma(U, \mathcal{R}_{[\widetilde{\psi}(D)]}). \end{aligned}$$

□

EXAMPLE 4.1.2. Let X be the projective space $\mathbb{P}_n$ and $D \subseteq \mathbb{P}_n$ be a hyperplane. The class of D generates $\mathrm{Cl}(\mathbb{P}_n)$ freely. We take K as the subgroup of $\mathrm{WDiv}(\mathbb{P}_n)$ generated by D, and the Cox ring $\mathcal{R}(\mathbb{P}_n)$ is the polynomial ring $\mathbb{K}[z_0, z_1, \dots, z_n]$ with the standard grading.

REMARK 4.1.3. If $X \subseteq \mathbb{P}_n$ is a closed normal subvariety whose divisor class group is generated by a hyperplane section, then $\mathcal{R}(X)$ coincides with $\Gamma(\overline{X}, \mathcal{O})$, where $\overline{X} \subseteq \mathbb{K}^{n+1}$ is the cone over X if and only if X is projectively normal.

REMARK 4.1.4. Let s denote the rank of $\mathrm{Cl}(X)$. Then Remark 3.1.5 realizes the Cox ring $\mathcal{R}(X)$ as a graded subring of the Laurent polynomial ring:

$$\mathcal{R}(X) \quad \subseteq \quad \mathbb{K}(X)[T_1^{\pm 1}, \dots, T_s^{\pm 1}].$$

Using the fact that there are $f \in \mathcal{R}_{[D]}(X)$ with $X_{D,f}$ affine and Remark 3.1.6, we see that this inclusion gives rises to an isomorphism of the quotient fields

$$\mathrm{Quot}(\mathcal{R}(X)) \quad \cong \quad \mathbb{K}(X)(T_1, \dots, T_s).$$

PROPOSITION 4.1.5. *Let X be a normal prevariety with free finitely generated divisor class group.*

(i) *The Cox ring $\mathcal{R}(X)$ is a unique factorization domain.*
(ii) *The units of the Cox ring are given by $\mathcal{R}(X)^* = \Gamma(X, \mathcal{O}^*)$.*

PROOF. The first assertion is a direct consequence of Theorem 3.3.3. To verify the second one, consider a unit $f \in \mathcal{R}(X)^*$. Then $fg = 1 \in \mathcal{R}_0(X)$ holds with some unit $g \in \mathcal{R}(X)^*$. This can only happen, when f and g are homogeneous, say of degree $[D]$ and $-[D]$, and thus we obtain

$$0 \;=\; \mathrm{div}_0(1) \;=\; \mathrm{div}_D(f) + \mathrm{div}_{-D}(g) \;=\; (\mathrm{div}(f) + D) + (\mathrm{div}(g) - D).$$

Since the divisors $(\mathrm{div}(f) + D)$ and $(\mathrm{div}(g) - D)$ are effective, we conclude that $D = -\mathrm{div}(f)$. This means $[D] = 0$ and we obtain $f \in \Gamma(X, \mathcal{O}^*)$. □

EXAMPLE 4.1.6. Compare [**85**, Section 2]. Take the projective line $\mathbb{P}_1$, a tuple $A = (a_0, \dots, a_r)$ of pairwise different points $a_i \in \mathbb{P}_1$ and a tuple $\mathfrak{n} = (n_0, \dots, n_r)$ of integers $n_i \in \mathbb{Z}_{\ge 1}$. We construct a non-separated smooth curve $\mathbb{P}_1(A, \mathfrak{n})$ mapping birationally onto $\mathbb{P}_1$ such that over each a_i lie precisely n_i points. Set

$$X_{ij} \;:=\; \mathbb{P}_1 \setminus \bigcup_{k \ne i} a_k, \qquad 0 \le i \le r, \qquad 1 \le j \le n_i.$$

Gluing the X_{ij} along the common open subset $\mathbb{P}_1 \setminus \{a_0, \dots, a_r\}$ gives an irreducible smooth prevariety $\mathbb{P}_1(A, \mathfrak{n})$ of dimension one. The inclusion maps $X_{ij} \to \mathbb{P}_1$ define a morphism $\pi\colon \mathbb{P}_1(A, \mathfrak{n}) \to \mathbb{P}_1$, which is locally an isomorphism. Writing a_{ij} for the point in $\mathbb{P}_1(A, \mathfrak{n})$ stemming from $a_i \in X_{ij}$, we obtain the fibre over any $a \in \mathbb{P}_1$ as

$$\pi^{-1}(a) \quad = \quad \begin{cases} \{a_{i1}, \dots, a_{in_i}\} & a = a_i \text{ for some } 0 \le i \le r, \\ \{a\} & a \ne a_i \text{ for all } 0 \le i \le r. \end{cases}$$

We compute the divisor class group of $\mathbb{P}_1(A,\mathfrak{n})$. Let K' denote the group of Weil divisors on $\mathbb{P}_1(A,\mathfrak{n})$ generated by the prime divisors a_{ij}. Clearly K' maps onto the divisor class group. Moreover, the group of principal divisors inside K' is

$$K'_0 \;:=\; K' \cap \mathrm{PDiv}(\mathbb{P}_1(A,\mathfrak{n})) \;=\; \left\{ \sum_{\substack{0\le i\le r,\\ 1\le j\le n_i}} c_i a_{ij};\ c_0+\ldots+c_r=0 \right\}.$$

One directly checks that K' is the direct sum of K'_0 and the subgroup $K \subseteq K'$ generated by $a_{01},\ldots,a_{0n_0}$ and the $a_{i1},\ldots,a_{in_i-1}$. Consequently, the divisor class group of $\mathbb{P}_1(A,\mathfrak{n})$ is given by

$$\mathrm{Cl}(\mathbb{P}_1(A,\mathfrak{n})) \;=\; \bigoplus_{j=1}^{n_0} \mathbb{Z}\cdot[a_{0j}] \;\oplus\; \bigoplus_{i=1}^{r} \left(\bigoplus_{j=1}^{n_i-1} \mathbb{Z}\cdot[a_{ij}] \right).$$

We are ready to determine the Cox ring of the prevariety $\mathbb{P}_1(A,\mathfrak{n})$. For every $0\le i\le r$, define a monomial

$$T_i \;:=\; T_{i1}\cdots T_{in_i} \;\in\; \mathbb{K}[T_{ij};\ 0\le i\le r,\ 1\le j\le n_i].$$

Moreover, for every $a_i\in\mathbb{P}_1$ fix a presentation $a_i=[b_i,c_i]$ with $b_i,c_i\in\mathbb{K}$ and for every $0\le i\le r-2$ set $k=j+1=i+2$ and define a trinomial

$$g_i \;:=\; (b_jc_k-b_kc_j)T_i \;+\; (b_kc_i-b_ic_k)T_j \;+\; (b_ic_j-b_jc_i)T_k.$$

We claim that for $r\le 1$ the Cox ring $\mathcal{R}(\mathbb{P}_1(A,\mathfrak{n}))$ is isomorphic to the polynomial ring $\mathbb{K}[T_{ij}]$, and for $r\ge 2$ it has a presentation

$$\mathcal{R}(\mathbb{P}_1(A,\mathfrak{n})) \;\cong\; \mathbb{K}[T_{ij};\ 0\le i\le r,\ 1\le j\le n_i] \;/\; \langle g_i;\ 0\le i\le r-2\rangle,$$

where, in both cases, the grading is given by $\deg(T_{ij})=[a_{ij}]$. Note that all relations are homogeneous of degree

$$\deg(g_i) \;=\; [a_{i1}+\ldots+a_{in_i}] \;=\; [a_{01}+\ldots+a_{0n_0}].$$

Let us verify this claim. Set for short $X:=\mathbb{P}_1(A,\mathfrak{n})$ and $Y:=\mathbb{P}_1$. Let $K\subseteq \mathrm{WDiv}(X)$ be the subgroup generated by all $a_{ij}\in X$ different from $a_{1n_1},\ldots,a_{rn_r}$, and let $L\subseteq\mathrm{WDiv}(Y)$ be the subgroup generated by $a_0\in Y$. Then we may view the Cox rings $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ as the rings of global sections of the sheaves of divisorial algebras $\mathcal{S}_X$ and $\mathcal{S}_Y$ associated to K and L. The canonical morphism $\pi\colon X\to Y$ gives rise to injective pullback homomorphisms

$$\pi^*\colon L \;\to\; K, \qquad\qquad \pi^*\colon \Gamma(Y,\mathcal{S}_Y) \;\to\; \Gamma(X,\mathcal{S}_X).$$

For any divisor $a_{ij}\in K$, let $T_{ij}\in\Gamma(X,\mathcal{S}_X)$ denote its canonical section, i.e., the rational function $1\in\Gamma(X,\mathcal{S}_{X,a_{ij}})$. Moreover, let $[z,w]$ be the homogeneous coordinates on $\mathbb{P}_1$ and consider the sections

$$S_i \;:=\; \frac{b_iw-c_iz}{b_0w-c_0z} \;\in\; \Gamma(Y,\mathcal{S}_{Y,a_0}), \qquad 0\le i\le r.$$

Finally, set $d_{in_i}:=a_{01}+\ldots+a_{0n_0}-a_{i1}-\ldots-a_{in_i-1}\in K$ and define homogeneous sections

$$T_{in_i} \;:=\; \pi^*S_i(T_{i1}\cdots T_{in_i-1})^{-1} \;\in\; \Gamma(X,\mathcal{S}_{X,d_{in_i}}), \qquad 1\le i\le r.$$

We show that the sections T_{ij}, where $0\le i\le r$ and $1\le j\le n_i$, generate the Cox ring $\mathcal{R}(X)$. Note that we have

$$\mathrm{div}_{a_{ij}}(T_{ij}) \;=\; a_{ij}, \qquad \mathrm{div}_{d_{in_i}}(T_{in_i}) \;=\; a_{in_i}.$$

Consider $D\in K$ and $h\in\Gamma(X,\mathcal{S}_D)$. If there occurs an a_{ij} in $\mathrm{div}_D(h)$, then we may divide h in $\Gamma(X,\mathcal{S})$ by the corresponding T_{ij}, use Proposition 3.3.4 (i). Doing this as long as possible, we arrive at some $h' \in \Gamma(X, \mathcal{S}_{D'})$ such that $\mathrm{div}_{D'}(h')$ has no components a_{ij}. But then D' is a pullback divisor and hence h' is contained in

$$\pi^*(\Gamma(Y, \mathcal{S}_Y)) \;=\; \mathbb{K}[\pi^* S_0, \pi^* S_1] \;=\; \mathbb{K}[T_{01} \cdots T_{0n_0}, T_{11} \cdots T_{1n_1}].$$

Finally, we have to determine the relations among the sections $T_{ij} \in \Gamma(X, \mathcal{S}_X)$. For this, we first note that among the $S_i \in \Gamma(Y, \mathcal{S}_Y)$ we have the relations

$$(b_j c_k - b_k c_j) S_i \;+\; (b_k c_i - b_i c_k) S_j \;+\; (b_i c_j - b_j c_i) S_k \;=\; 0, \quad j \;=\; i+1, \quad k \;=\; i+2.$$

Given any nontrivial homogeneous relation $F = \alpha_1 F_1 + \ldots + \alpha_l F_l = 0$ with $\alpha_i \in \mathbb{K}$ and pairwise different monomials F_i in the T_{ij}, we achieve by subtracting suitable multiples of pullbacks of the above relations a homogeneous relation

$$F' \;=\; \alpha_1' F_1'' \pi^* S_0^{k_1} \pi^* S_1^{l_1} + \ldots + \alpha_m' F_m'' \pi^* S_0^{k_m} \pi^* S_1^{l_m} \;=\; 0$$

with pairwise different monomials F_j'', none of which has any factor $\pi^* S_i$. We show that F' must be trivial. Consider the multiplicative group M of Laurent monomials in the T_{ij} and the degree map

$$M \;\to\; K, \qquad T_{ij} \;\mapsto\; \deg(T_{ij}) \;=\; \begin{cases} a_{ij}, & i = 0 \text{ or } j \le n_i - 1, \\ d_{in_i} & i \ge 1 \text{ and } j = n_i. \end{cases}$$

The kernel of this degree map is generated by the Laurent monomials $\pi^* S_0 / \pi^* S_i$, where $1 \le i \le r$. The monomials of F' are all of the same K-degree and thus any two of them differ by a product of (integral) powers of the $\pi^* S_i$. It follows that all the F_j'' coincide. Thus, we obtain the relation

$$\alpha_1' \pi^* S_0^{k_1} \pi^* S_1^{l_1} \;+\; \ldots \;+\; \alpha_m' \pi^* S_0^{k_m} \pi^* S_1^{l_m} \;\;=\;\; 0.$$

This relation descends to a relation in $\Gamma(Y, \mathcal{S}_Y)$, which is the polynomial ring $\mathbb{K}[S_0, S_1]$. Consequently, we obtain $\alpha_1' = \ldots = \alpha_m' = 0$.

4.2. Torsion in the class group. Again we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. We extend the definition of Cox sheaf and Cox ring to normal prevarieties X having a finitely generated divisor class group $\mathrm{Cl}(X)$ with torsion. The idea is to take a subgroup $K \subseteq \mathrm{WDiv}(X)$ projecting onto $\mathrm{Cl}(X)$, to consider its associated sheaf of divisorial algebras $\mathcal{S}$ and to identify in a systematic manner homogeneous components $\mathcal{S}_D$ and $\mathcal{S}_{D'}$, whenever D and D' are linearly equivalent.

Construction 4.2.1. Let X be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that the map $c\colon K \to \mathrm{Cl}(X)$ sending $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is surjective. Let $K^0 \subseteq K$ be the kernel of c, and let $\chi\colon K^0 \to \mathbb{K}(X)^*$ be a character, i.e. a group homomorphism, with

$$\mathrm{div}(\chi(E)) \;=\; E, \qquad \text{for all } E \in K^0.$$

Let $\mathcal{S}$ be the sheaf of divisorial algebras associated to K and denote by $\mathcal{I}$ the sheaf of ideals of $\mathcal{S}$ locally generated by the sections $1 - \chi(E)$, where 1 is homogeneous of degree zero, E runs through K^0 and $\chi(E)$ is homogeneous of degree $-E$. The *Cox sheaf* associated to K and χ is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} \;=\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad\qquad \mathcal{R}_{[D]} \;:=\; \pi\left(\bigoplus_{D' \in c^{-1}([D])} \mathcal{S}_{D'}\right).$$

where $\pi\colon \mathcal{S} \to \mathcal{R}$ denotes the projection. The Cox sheaf $\mathcal{R}$ is a quasicoherent sheaf of $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras. The *Cox ring* is the ring of global sections

$$\mathcal{R}(X) \;:=\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]}(X), \qquad\qquad \mathcal{R}_{[D]}(X) \;:=\; \Gamma(X, \mathcal{R}_{[D]}).$$

For any open set $U \subseteq X$, the canonical homomorphism $\Gamma(U,\mathcal{S})/\Gamma(U,\mathcal{I}) \to \Gamma(U,\mathcal{R})$ is an isomorphism. In particular, we have

$$\mathcal{R}(X) \quad \cong \quad \Gamma(X,\mathcal{S})/\Gamma(X,\mathcal{I}).$$

All the claims made in this construction will be verified as separate Lemmas in the next subsection. The assumption $\Gamma(X,\mathcal{O}^*) = \mathbb{K}^*$ is crucial for the following uniqueness statement on Cox sheaves and rings.

Proposition 4.2.2. *Let X be a normal prevariety with $\Gamma(X,\mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\mathrm{Cl}(X)$. If K,χ and K',χ' are data as in Construction 4.2.1, then there is a graded isomorphism of the associated Cox sheaves.*

Also this will be proven in the next subsection. The construction of Cox sheaves (and thus also Cox rings) of a prevariety X can be made canonical by fixing a suitable point $x \in X$.

Construction 4.2.3. Let X be a normal prevariety with $\Gamma(X,\mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a point $x \in X$ with factorial local ring $\mathcal{O}_{X,x}$. For the subgroup

$$K^x \quad := \quad \{D \in \mathrm{WDiv}(X);\ x \notin \mathrm{Supp}(D)\}$$

let $\mathcal{S}^x$ be the associated sheaf of divisorial algebras and let $K^{x,0} \subseteq K^x$ denote the subgroup consisting of principal divisors. Then, for each $E \in K^{x,0}$, there is a unique section $f_E \in \Gamma(X,\mathcal{S}_{-E})$, which is defined near x and satisfies

$$\mathrm{div}(f_E) \;=\; E, \qquad\qquad f_E(x) \;=\; 1.$$

The map $\chi^x\colon K^x \to \mathbb{K}(X)^*$ sending E to f_E is a character as in Construction 4.2.1. We call the Cox sheaf $\mathcal{R}^x$ associated to K^x and χ^x the *canonical Cox sheaf of the pointed space* (X,x).

Example 4.2.4 (An affine surface with torsion in the divisor class group). Consider the two-dimensional affine quadric

$$X \;:=\; V(\mathbb{K}^3;\ T_1T_2 - T_3^2) \;\subseteq\; \mathbb{K}^3.$$

We have the functions $f_i := T_{i|X}$ on X and with the prime divisors $D_1 := V(X;f_1)$ and $D_2 := V(X;f_2)$ on X, we have

$$\mathrm{div}(f_1) \;=\; 2D_1, \qquad \mathrm{div}(f_2) \;=\; 2D_2, \qquad \mathrm{div}(f_3) \;=\; D_1 + D_2.$$

The divisor class group $\mathrm{Cl}(X)$ is of order two; it is generated by $[D_1]$. For $K := \mathbb{Z}D_1$, let $\mathcal{S}$ denote the associated sheaf of divisorial algebras. Consider the sections

$$g_1 := 1 \;\in\; \Gamma(X,\mathcal{S}_{D_1}), \qquad g_2 := f_3f_1^{-1} \;\in\; \Gamma(X,\mathcal{S}_{D_1}),$$
$$g_3 := f_1^{-1} \;\in\; \Gamma(X,\mathcal{S}_{2D_1}), \qquad g_4 := f_1 \;\in\; \Gamma(X,\mathcal{S}_{-2D_1}).$$

Then g_1, g_2 generate $\Gamma(X,\mathcal{S}_{D_1})$ as a $\Gamma(X,\mathcal{S}_0)$-module, and g_3, g_4 are inverse to each other. Moreover, we have

$$f_1 \;=\; g_1^2g_4, \qquad f_2 \;=\; g_2^2g_4, \qquad f_3 \;=\; g_1g_2g_4.$$

Thus, g_1, g_2, g_3 and g_4 generate the $\mathbb{K}$-algebra $\Gamma(X,\mathcal{S})$. Setting $\deg(Z_i) := \deg(g_i)$, we obtain a K-graded epimorphism

$$\mathbb{K}[Z_1,Z_2,Z_3^{\pm 1}] \;\to\; \Gamma(X,\mathcal{S}), \qquad Z_1 \mapsto g_1,\ Z_2 \mapsto g_2,\ Z_3 \mapsto g_3,$$

which, by dimension reasons, is even an isomorphism. The kernel of the projection $K \to \mathrm{Cl}(X)$ is $K^0 = 2\mathbb{Z}D_1$ and a character as in Construction 4.2.1 is

$$\chi\colon K^0 \;\to\; \mathbb{K}(X)^*, \qquad 2nD_1 \;\mapsto\; f_1^n.$$

The ideal $\mathcal{I}$ is generated by $1 - f_1$, where $f_1 \in \Gamma(X,\mathcal{S}_{-2D_1})$, see Remark 4.3.2 below. Consequently, the Cox ring of X is given as

$$\mathcal{R}(X) \;\cong\; \Gamma(X,\mathcal{S})/\Gamma(X,\mathcal{I}) \;\cong\; \mathbb{K}[Z_1,Z_2,Z_3^{\pm 1}]/\langle 1 - Z_3^{-1}\rangle \;\cong\; \mathbb{K}[Z_1,Z_2],$$

where the $\mathrm{Cl}(X)$-grading on the polynomial ring $\mathbb{K}[Z_1, Z_2]$ is given by $\deg(Z_1) = \deg(Z_2) = [D_1]$.

4.3. Well-definedness. Here we prove the claims made in Construction 4.2.1 and Proposition 4.2.2. In particular, we show that, up to isomorphy, Cox sheaf and Cox ring do not depend on the choices made in their construction.

LEMMA 4.3.1. *Situation as in Construction 4.2.1. Consider the* $\mathrm{Cl}(X)$*-grading of the sheaf* $\mathcal{S}$ *defined by*

$$\mathcal{S} \;=\; \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{S}_{[D]}, \qquad\qquad \mathcal{S}_{[D]} \;:=\; \bigoplus_{D'\in c^{-1}([D])} \mathcal{S}_{D'}.$$

Given $f \in \Gamma(U, \mathcal{I})$ *and* $D \in K$, *the* $\mathrm{Cl}(X)$*-homogeneous component* $f_{[D]} \in \Gamma(U, \mathcal{S}_{[D]})$ *of* f *has a unique representation*

$$f_{[D]} \;=\; \sum_{E\in K^0} (1-\chi(E))f_E, \qquad \text{where } f_E \in \Gamma(U, \mathcal{S}_D) \text{ and } \chi(E) \in \Gamma(U, \mathcal{S}_{-E}).$$

In particular, the sheaf $\mathcal{I}$ *of ideals is* $\mathrm{Cl}(X)$*-homogeneous. Moreover, if* $f \in \Gamma(U, \mathcal{I})$ *is* K*-homogeneous, then it is the zero section.*

PROOF. To obtain uniqueness of the representation of $f_{[D]}$, observe that for every $0 \neq E \in K^0$, the product $-\chi(E)f_E$ is the K-homogeneous component of degree $D - E$ of $f_{[D]}$. We show existence. By definition of the sheaf of ideals $\mathcal{I}$, every germ $f_x \in \mathcal{I}_x$ can on a suitable neighbourhood U_x be represented by a section

$$g \;=\; \sum_{E\in K^0} (1-\chi(E))g_E, \qquad \text{where } g_E \in \Gamma(U_x, \mathcal{S}).$$

Collecting the $\mathrm{Cl}(X)$-homogeneous parts on the right hand side represents the $\mathrm{Cl}(X)$-homogeneous part $h \in \Gamma(U_x, \mathcal{S}_{[D]})$ of degree $[D]$ of $g \in \Gamma(U_x, \mathcal{S})$ as follows:

$$h \;=\; \sum_{E\in K^0} (1-\chi(E))h_E, \qquad \text{where } h_E \in \Gamma(U_x, \mathcal{S}_{[D]}).$$

Note that we have $h \in \Gamma(U_x, \mathcal{I})$ and h represents $f_{[D],x}$. Now, developping each $h_E \in \Gamma(U_x, \mathcal{S}_{[D]})$ according to the K-grading gives representations

$$h_E \;=\; \sum_{D'\in D+K^0} h_{E,D'}, \qquad \text{where } h_{E,D'} \in \Gamma(U_x, \mathcal{S}_{D'}).$$

The section $h'_{E,D'} := \chi(D'-D)h_{E,D'}$ is K-homogeneous of degree D, and we have the identity

$$(1-\chi(E))h_{E,D'} \;=\; (1-\chi(E+D-D'))h'_{E,D'} - (1-\chi(D-D'))h'_{E,D'}.$$

Plugging this into the representation of h establishes the desired representation of $f_{[D]}$ locally. By uniqueness, we may glue the local representations. □

REMARK 4.3.2. Situation as in Construction 4.2.1. Then, for any two divisors $E, E' \in K^0$, one has the identities

$$\begin{aligned} 1 - \chi(E+E') &= (1 - \chi(E)) + (1 - \chi(E'))\chi(E), \\ 1 - \chi(-E) &= (1 - \chi(E))(-\chi(-E)). \end{aligned}$$

Together with Lemma 4.3.1, this implies that for any basis $E_1, \ldots, E_s$ of K^0 and any open $U \subseteq X$, the ideal $\Gamma(U, \mathcal{I})$ is generated by $1 - \chi(E_i)$, where $1 \leq i \leq s$.

LEMMA 4.3.3. *Situation as in Construction 4.2.1. If* $f \in \Gamma(U, \mathcal{S})$ *is* $\mathrm{Cl}(X)$*-homogeneous of degree* $[D]$ *for some* $D \in K$, *then there is a* K*-homogeneous* $f' \in \Gamma(U, \mathcal{S})$ *of degree* D *with* $f - f' \in \Gamma(U, \mathcal{I})$.

PROOF. Writing the $\mathrm{Cl}(X)$-homogeneous f as a sum of K-homogeneous functions $f_{D'}$, we obtain the assertion by means of the following trick:

$$f \;=\; \sum_{D' \in D+K^0} f_{D'} \;=\; \sum_{D' \in D+K^0} \chi(D'-D) f_{D'} + \sum_{D' \in D+K^0} (1-\chi(D'-D)) f_{D'}.$$

□

LEMMA 4.3.4. *Situation as in Construction 4.2.1. Then, for every $D \in K$, we have an isomorphism of sheaves $\pi_{|\mathcal{S}_D} \colon \mathcal{S}_D \to \mathcal{R}_{[D]}$.*

PROOF. Lemma 4.3.1 shows that the homomorphism $\pi_{|\mathcal{S}_D}$ is stalkwise injective and from Lemma 4.3.3 we infer that it is stalkwise surjective. □

LEMMA 4.3.5. *Situation as in Construction 4.2.1. Then, for every open subset $U \subseteq X$, we have a canonical isomorphism*

$$\Gamma(U, \mathcal{S})/\Gamma(U, \mathcal{I}) \;\cong\; \Gamma(U, \mathcal{S}/\mathcal{I}).$$

PROOF. The canonical map $\psi \colon \Gamma(U, \mathcal{S})/\Gamma(U, \mathcal{I}) \to \Gamma(U, \mathcal{S}/\mathcal{I})$ is injective. In order to see that it is as well surjective, let $h \in \Gamma(U, \mathcal{S}/\mathcal{I})$ be given. Then there are a covering of U by open subsets U_i and sections $g_i \in \Gamma(U_i, \mathcal{S})$ such that $h_{|U_i} = \psi(g_i)$ holds and $g_j - g_i$ belongs to $\Gamma(U_i \cap U_j, \mathcal{I})$. Consider the $\mathrm{Cl}(X)$-homogeneous parts $g_{i,[D]} \in \Gamma(U_i, \mathcal{S}_{[D]})$ of g_i. By Lemma 4.3.1, the ideal sheaf $\mathcal{I}$ is homogeneous and thus also $g_{j,[D]} - g_{i,[D]}$ belongs to $\Gamma(U_i \cap U_j, \mathcal{I})$. Moreover, Lemma 4.3.3 provides K-homogeneous $f_{i,D}$ with $f_{i,D} - g_{i,[D]}$ in $\Gamma(U_i, \mathcal{I})$. The differences $f_{j,D} - f_{i,D}$ lie in $\Gamma(U_i \cap U_j, \mathcal{I})$ and hence, by Lemma 4.3.1, vanish. Thus, the $f_{i,D}$ fit together to K-homogeneous sections $f_D \in \Gamma(U, \mathcal{S})$. By construction, $f = \sum f_D$ satisfies $\psi(f) = h$. □

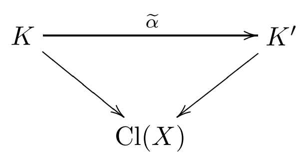
PROOF OF PROPOSITION 4.2.2. In a first step, we reduce to Cox sheaves arising from finitely generated subgroups of $\mathrm{WDiv}(X)$. So, let $K \subseteq \mathrm{WDiv}(X)$ and $\chi \colon K^0 \to \mathbb{K}(X)^*$ be any data as in 4.2.1. Choose a finitely generated subgroup $K_1 \subseteq K$ projecting onto $\mathrm{Cl}(X)$. Restricting χ gives a character $\chi_1 \colon K_1^0 \to \mathbb{K}(X)^*$. The inclusion $K_1 \to K$ defines an injection $\mathcal{S}_1 \to \mathcal{S}$ sending the ideal $\mathcal{I}_1$ defined by χ_1 to the ideal $\mathcal{I}$ defined by χ. This gives a $\mathrm{Cl}(X)$-graded injection $\mathcal{R}_1 \to \mathcal{R}$ of the Cox sheaves associated to K_1, χ_1 and K, χ respectively. Lemma 4.3.3 shows that every $\mathrm{Cl}(X)$-homogeneous section of $\mathcal{R}$ can be represented by a K_1-homogeneous section of $\mathcal{S}$. Thus, $\mathcal{R}_1 \to \mathcal{R}$ is also surjective.

Next we show that for a fixed finitely generated $K \subseteq \mathrm{WDiv}(X)$, any two characters $\chi, \chi' \colon K^0 \to \mathbb{K}(X)^*$ as in 4.2.1 give rise to isomorphic Cox sheaves $\mathcal{R}'$ and $\mathcal{R}$. For this note that the product $\chi^{-1}\chi'$ sends K^0 to $\Gamma(X, \mathcal{O}^*)$. Using $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$, we may extend $\chi^{-1}\chi'$ to a homomorphism $\vartheta \colon K \to \Gamma(X, \mathcal{O}^*)$ and obtain a graded automorphism (α, id) of $\mathcal{S}$ by

$$\alpha_D \colon \mathcal{S}_D \;\to\; \mathcal{S}_D, \qquad f \;\mapsto\; \vartheta(D) f.$$

By construction, this automorphism sends the ideal $\mathcal{I}'$ to the ideal $\mathcal{I}$ and induces a graded isomorphism from $\mathcal{S}/\mathcal{I}'$ onto $\mathcal{S}/\mathcal{I}$.

Now consider two finitely generated subgroups $K, K' \subseteq \mathrm{WDiv}(X)$ both projecting onto $\mathrm{Cl}(X)$. Then we find a homomorphism $\widetilde{\alpha} \colon K \to K'$ such that the following diagram is commutative



This homomorphism $\widetilde{\alpha}\colon K \to K'$ must be of the form $\widetilde{\alpha}(D) = D - \mathrm{div}(\eta(D))$ with a homomorphism $\eta\colon K \to \mathbb{K}(X)^*$. Choose a character $\chi'\colon K'^0 \to \mathbb{K}(X)^*$ as in 4.2.1. Then, for $D \in K^0$, we have

$$D - \mathrm{div}(\eta(D)) \;=\; \widetilde{\alpha}(D) \;=\; \mathrm{div}(\chi'(\widetilde{\alpha}(D))).$$

Thus, D equals the divisor of the function $\chi(D) := \chi'(\widetilde{\alpha}(D))\eta(D)$. This defines a character $\chi\colon K^0 \to \mathbb{K}(X)^*$. Altogether, we obtain a morphism $(\alpha, \widetilde{\alpha})$ of the sheaves of divisorial algebras $\mathcal{S}$ and $\mathcal{S}'$ associated to K and K' by

$$\alpha_D\colon \mathcal{S}_D \;\to\; \mathcal{S}'_{\widetilde{\alpha}(D)}, \qquad f \;\mapsto\; \eta(D)f.$$

By construction, it sends the ideal $\mathcal{I}$ defined by χ to the ideal $\mathcal{I}'$ defined by χ'. Using Lemma 4.3.4, we see that the induced homomorphism $\mathcal{R} \to \mathcal{R}'$ is an isomorphism on the homogeneous components and thus it is an isomorphism. □

4.4. Examples. For a normal prevariety X with a free finitely generated divisor class group, we obtained in Proposition 4.1.5 that the Cox ring is a unique factorization domain having $\Gamma(X, \mathcal{O}^*)$ as its units. Here we provide two examples showing that these statements need not hold any more if there is torsion in the divisor class group. As usual, $\mathbb{K}$ is an algebraically closed field of characteristic zero.

Example 4.4.1 (An affine surface with non-factorial Cox ring). Consider the smooth affine surface

$$Z \;\;:=\;\; V(\mathbb{K}^3; T_1^2 - T_2T_3 - 1).$$

We claim that $\Gamma(Z, \mathcal{O}^*) = \mathbb{K}^*$ and $\mathrm{Cl}(Z) \cong \mathbb{Z}$ hold. To see this, consider $f_i := T_{i|Z}$ and the prime divisors

$$D_+ \;:=\; V(Z; f_1 - 1, f_2) \;=\; \{1\} \times \{0\} \times \mathbb{K},$$
$$D_- \;:=\; V(Z; f_1 + 1, f_2) \;=\; \{-1\} \times \{0\} \times \mathbb{K}.$$

Then we have $\mathrm{div}(f_2) = D_+ + D_-$. In particular, D_+ is linearly equivalent to $-D_-$. Moreover, we have

$$Z \setminus \mathrm{Supp}(\mathrm{div}(f_2)) \;=\; Z_{f_2} \;\cong\; \mathbb{K}^* \times \mathbb{K}.$$

This gives $\Gamma(Z, \mathcal{O})^* = \mathbb{K}^*$, and shows that $\mathrm{Cl}(Z)$ is generated by the class $[D_+]$. Now suppose that $n[D_+] = 0$ holds for some $n > 0$. Then we have $nD_+ = \mathrm{div}(f)$ with $f \in \Gamma(Z, \mathcal{O})$ and $f_2^n = fh$ holds with some $h \in \Gamma(Z, \mathcal{O})$ satisfying $\mathrm{div}(h) = nD_-$. Look at the $\mathbb{Z}$-grading of $\Gamma(Z, \mathcal{O})$ given by

$$\deg(f_1) = 0, \qquad \deg(f_2) = 1, \qquad \deg(f_3) = -1.$$

Any element of positive degree is a multiple of f_2. It follows that in the decomposition $f_2^n = fh$ one of the factors f or h must be a multiple of f_2, a contradiction. This shows that $\mathrm{Cl}(Z)$ is freely generated by $[D_+]$.

Now consider the involution $Z \to Z$ sending z to $-z$ and let $\pi\colon Z \to X$ denote the quotient of the corresponding free $\mathbb{Z}/2\mathbb{Z}$-action. We claim that $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by the class of $D := \pi(D_+)$. Indeed, the subset

$$X \setminus \mathrm{Supp}(D) \;=\; \pi(Z_{f_2}) \;\cong\; \mathbb{K}^* \times \mathbb{K}$$

is factorial and $2D$ equals $\mathrm{div}(f_2^2)$. Moreover, the divisor D is not principal, because $\pi^*(D) = D_+ + D_-$ is not the divisor of a $\mathbb{Z}/2\mathbb{Z}$-invariant function on Z. This verifies our claim. Moreover, we have $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$.

In order to determine the Cox ring of X, take $K = \mathbb{Z}D \subseteq \mathrm{WDiv}(X)$, and let $\mathcal{S}$ denote the associated sheaf of divisorial algebras. Then, as $\Gamma(X, \mathcal{S}_0)$-modules, $\Gamma(X, \mathcal{S}_D)$ and $\Gamma(X, \mathcal{S}_{-D})$ are generated by the sections

$$a_1 := 1,\; a_2 := f_1 f_2^{-1},\; a_3 := f_2^{-1} f_3 \;\in \Gamma(X, \mathcal{S}_D),$$

$$b_1 := f_1f_2,\ b_2 := f_2^2,\ b_3 := f_2f_3\ \in \Gamma(X, \mathcal{S}_{-D}).$$

Thus, using the fact that $f_2^{\pm 2}$ define invertible elements of degree $\mp 2D$, we see that $a_1, a_2, a_3, b_1, b_2, b_3$ generate the algebra $\Gamma(X, \mathcal{S})$. Now, take the character $\chi\colon K^0 \to \mathbb{K}(X)^*$ sending $2nD$ to f_2^{2n}. Then, by Remark 4.3.2, the associated ideal $\Gamma(X, \mathcal{I})$ is generated by $1 - f_2^2$. The generators of the factor algebra $\Gamma(X, \mathcal{S}) \,/\, \Gamma(X, \mathcal{I})$ are

$$Z_1 = a_2 + \mathcal{I} = b_1 + \mathcal{I}, \qquad Z_2 = a_1 + \mathcal{I} = b_2 + \mathcal{I}, \qquad Z_3 = a_3 + \mathcal{I} = b_3 + \mathcal{I}.$$

The defining relation is $Z_1^2 - Z_2Z_3 = 1$. Thus the Cox ring $\mathcal{R}(X)$ is isomorphic to $\Gamma(Z, \mathcal{O})$. In particular, it is not a factorial ring.

Example 4.4.2 (A surface with only constant invertible functions but non-constant invertible elements in the Cox ring). Consider the affine surface

$$X \quad := \quad V(\mathbb{K}^3;\ T_1T_2T_3 - T_1^2 - T_2^2 - T_3^2 + 4).$$

This is the quotient space of the torus $\mathbb{T}^2 := (\mathbb{K}^*)^2$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-action defined by the involution $t \mapsto t^{-1}$; the quotient map is explicitly given as

$$\pi\colon \mathbb{T}^2 \ \to\ X, \qquad t \ \mapsto\ (t_1 + t_1^{-1}, t_2 + t_2^{-1}, t_1t_2 + t_1^{-1}t_2^{-1}).$$

Since every $\mathbb{Z}/2\mathbb{Z}$-invariant invertible function on $\mathbb{T}^2$ is constant, we have $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$. Moreover, using [**96**, Proposition 5.1], one verifies

$$\mathrm{Cl}(X) \ \cong\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{Pic}(X) \ =\ 0.$$

Let us see that the Cox ring $\mathcal{R}(X)$ has non-constant invertible elements. Set $f_i := T_{i|X}$ and consider the divisors

$$D_{\pm} \ :=\ V(X; f_1 \pm 2,\ f_2 \pm f_3), \qquad D \ :=\ D_+ + D_-.$$

Then, using the relations $(f_1 \pm 2)(f_2f_3 - f_1 \pm 2) = (f_2 \pm f_3)^2$, one verifies $\mathrm{div}(f_1 \pm 2) = 2D_{\pm}$. Consequently, we obtain

$$2D \quad = \quad \mathrm{div}(f_1^2 - 4).$$

Moreover, D is not principal, because otherwise $f_1^2 - 4$ must be a square and hence also $\pi^*(f_1^2 - 4)$ is a square, which is impossible due to

$$\pi^*(f_1^2 - 4) \ =\ t_1^2 + t_1^{-2} - 4 \ =\ (t_1 + t_1^{-1} + 2)(t_1 + t_1^{-1} - 2).$$

Now choose Weil divisors D_i on X such that D, D_2, D_3 form a basis for a group $K \subseteq \mathrm{WDiv}(X)$ projecting onto $\mathrm{Cl}(X)$, and let $\mathcal{S}$ be the associated sheaf of divisorial algebras. As usual, let $K^0 \subseteq K$ be the subgroup consisting of principal divisors and fix a character $\chi\colon K^0 \to \mathbb{K}(X)^*$ with $\chi(2D) = f_1^2 - 4$. By Remark 4.3.2, the associated ideal $\Gamma(X, \mathcal{I})$ in $\Gamma(X, \mathcal{S})$ is generated by

$$1 - \chi(2D), \quad 1 - \chi(2D_2), \quad 1 - \chi(2D_3),$$

where $\chi(2D) = f_1^2 - 4$ lives in $\Gamma(X, \mathcal{S}_{-2D})$. Now consider $f_1 \in \Gamma(X, \mathcal{S}_0)$ and the canonical section $1_D \in \Gamma(X, \mathcal{S}_D)$. Then we have

$$(f_1 + 1_D)(f_1 - 1_D) \ =\ f_1^2 - 1_D^2 \ =\ 4 - 1_D^2 \cdot (1 - \chi(2D)) \ \in\ \mathbb{K}^* + \Gamma(X, \mathcal{I}).$$

Consequently, the section $f_1 + 1_D \in \Gamma(X, \mathcal{S})$ defines a unit in $\Gamma(X, \mathcal{R})$. Note that $f_1 + 1_D$ is not $\mathrm{Cl}(X)$-homogeneous.

5. Algebraic properties of the Cox ring

5.1. Integrity and Normality. As before, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. The following statement ensures in particular that the Cox ring is always a normal integral ring.

Theorem 5.1.1. *Let X be a normal prevariety with only constant invertible functions, finitely generated divisor class group, and Cox sheaf $\mathcal{R}$. Then, for every open $U \subseteq X$, the ring $\Gamma(U, \mathcal{R})$ is integral and normal.*

The proof is based on the geometric construction 5.1.4 which is also used later and therefore occurs separately. We begin with two preparing observations.

Lemma 5.1.2. *Situation as in Construction 4.2.1. For any two open subsets $V \subseteq U \subseteq X$ such that $U \setminus V$ is of codimension at least two in U, one has the restriction isomorphism*

$$\Gamma(U, \mathcal{R}) \ \to \ \Gamma(V, \mathcal{R}).$$

In particular, the algebra $\Gamma(U, \mathcal{R})$ equals the algebra $\Gamma(U_{\mathrm{reg}}, \mathcal{R})$, where $U_{\mathrm{reg}} \subseteq U$ denotes the set of smooth points.

Proof. According to Remark 3.1.3, the restriction $\Gamma(U, \mathcal{S}) \to \Gamma(V, \mathcal{S})$ is an isomorphism. Lemma 4.3.1 ensures that $\Gamma(U, \mathcal{I})$ is mapped isomorphically onto $\Gamma(V, \mathcal{I})$ under this isomorphism. By Lemma 4.3.5, we have $\Gamma(U, \mathcal{R}) = \Gamma(U, \mathcal{S})/\Gamma(U, \mathcal{I})$ and $\Gamma(V, \mathcal{R}) = \Gamma(V, \mathcal{S})/\Gamma(V, \mathcal{I})$, which gives the assertion. □

Lemma 5.1.3. *Situation as in Construction 4.2.1. Then for every open $U \subseteq X$, the ideal $\Gamma(U, \mathcal{I}) \subseteq \Gamma(U, \mathcal{S})$ is radical.*

Proof. By Lemma 4.3.5, the ideal $\Gamma(U, \mathcal{I})$ is radical if and only if the algebra $\Gamma(U, \mathcal{R})$ has no nilpotent elements. Proposition 4.2.2 thus allows us to assume that $\mathcal{S}$ arises from a finitely generated group K. Moreover, by Remark 3.1.3, we may assume that X is smooth and it suffices to verify the assertion for affine $U \subseteq X$. We consider $\widetilde{U} = \operatorname{Spec} \Gamma(U, \mathcal{S})$ and the zero set $\widehat{U} \subseteq \widetilde{U}$ of $\Gamma(U, \mathcal{I})$. Note that $\widehat{U}$ is invariant under the action of the quasitorus $H_X = \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(X)]$ on $\widetilde{U}$ given by the $\mathrm{Cl}(X)$-grading.

Now, let $f \in \Gamma(U, \mathcal{S})$ with $f^n \in \Gamma(U, \mathcal{I})$ for some $n > 0$. Then f and thus also every $\mathrm{Cl}(X)$-homogeneous component $f_{[D]}$ of f vanishes along $\widehat{U}$. Consequently, $f_{[D]}^m \in \Gamma(U, \mathcal{I})$ holds for some $m > 0$. By Lemma 4.3.3, we may write $f_{[D]} = f_D + g$ with $f_D \in \Gamma(U, \mathcal{S}_D)$ and $g \in \Gamma(U, \mathcal{I})$. We obtain $f_D^m \in \Gamma(U, \mathcal{I})$. By Lemma 4.3.1, this implies $f_D^m = 0$ and thus $f_D = 0$, which in turn gives $f_{[D]} \in \Gamma(U, \mathcal{I})$ and hence $f \in \Gamma(U, \mathcal{I})$. □

Construction 5.1.4. Situation as in Construction 4.2.1. Assume that $K \subseteq \mathrm{WDiv}(X)$ is finitely generated and X is smooth. Consider $\widetilde{X} := \operatorname{Spec}_X \mathcal{S}$ with the action of the torus $H := \operatorname{Spec} \mathbb{K}[K]$ and the geometric quotient $p\colon \widetilde{X} \to X$ as in Construction 3.2.4. Then, with $\widehat{X} := V(\mathcal{I})$ and $H_X :=$ Spec $\mathbb{K}[\mathrm{Cl}(X)]$, we have a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \widetilde{X} \\ {\scriptstyle q_X}\big\downarrow{\scriptstyle /H_X} & & {\scriptstyle p}\big\downarrow{\scriptstyle /H} \\ X & = & X \end{array}$$

The prevariety $\widehat{X}$ is smooth, and, if X is of affine intersection, then it is quasiaffine. The quasitorus $H_X \subseteq H$ acts freely on $\widehat{X}$ and $q_X\colon \widehat{X} \to X$ is a geometric quotient for this action; in particular, it is an étale H_X-principal bundle. Moreover, we have a canonical isomorphism of sheaves

$$\mathcal{R} \;\cong\; (q_X)_*(\mathcal{O}_{\widehat{X}}).$$

Proof. With the restriction $q_X\colon \widehat{X} \to X$ of $p\colon \widetilde{X} \to X$ we obviously obtain a commutative diagram as above. Moreover, Lemma 5.1.3 gives us $\mathcal{R} \cong q_*(\mathcal{O}_{\widehat{X}})$. Since the ideal $\mathcal{I}$ is $\mathrm{Cl}(X)$-homogeneous, the quasitorus $H_X \subseteq H$ leaves $\widehat{X}$ invariant. Moreover, we see that $q_X\colon \widehat{X} \to X$ is a good quotient for this action, because we have the canonical isomorphisms

$$(q_X)_*(\mathcal{O}_{\widehat{X}})_0 \;\cong\; \mathcal{R}_0 \;\cong\; \mathcal{O}_X \;\cong\; \mathcal{S}_0 \;\cong\; p_*(\mathcal{O}_{\widetilde{X}})_0.$$

Freeness of the H_X-action on $\widehat{X}$ is due to the fact that H_X acts as a subgroup of the freely acting H, see Remark 3.2.7. As a consequence, we see that $q_X\colon \widehat{X} \to X$ is a geometric quotient. Luna's Slice Theorem [**105**] gives commutative diagrams

$$\begin{array}{ccccc}
H_X \times S & \longrightarrow & q_X^{-1}(U) & \subseteq & \widehat{X} \\
\Big\downarrow{\scriptstyle \mathrm{pr}_S} & & \Big\downarrow{\scriptstyle q_X} & & \Big\downarrow{\scriptstyle q_X} \\
S & \longrightarrow & U & \subseteq & X
\end{array}$$

where $U \subseteq X$ are open sets covering X and the horizontal arrows are étale morphisms. By [**111**, Proposition I.3.17], étale morphisms preserve smoothness and thus $\widehat{X}$ inherits smoothness from X. If X is of affine intersection, then $\widetilde{X}$ is quasiaffine, see Corollary 3.4.6, and thus $\widehat{X}$ is quasiaffine. □

Lemma 5.1.5. *Let $\mathbb{L}$ be a field of characteristic zero containing all roots of unity, and assume that $a \in \mathbb{L}$ is not a proper power. Then, for any $n \in \mathbb{Z}_{\geq 1}$, the polynomial $1 - at^n$ is irreducible in $\mathbb{L}[t, t^{-1}]$.*

Proof. Over the algebraic closure of $\mathbb{L}$ we have $1-at^n = (1-a_1t)\cdots(1-a_nt)$, where $a_i^n = a$ and any two a_i differ by a n-th root of unity. If $1 - at^n$ would split over $\mathbb{L}$ non-trivially into $h_1(t)h_2(t)$, then a_1^k must be contained in $\mathbb{L}$ for some $k < n$. But then also a_1^d lies in $\mathbb{L}$ for the greatest common divisor d of n and k. Thus a is a proper power, a contradiction. □

Proof of Theorem 5.1.1. According to Proposition 4.2.2 and Lemma 5.1.2, we may assume that we are in the setting of Construction 5.1.4, where it suffices to prove that $\widehat{X}$ is irreducible. Since $q_X\colon \widehat{X} \to X$ is surjective, some irreducible component $\widehat{X}_1 \subseteq \widehat{X}$ dominates X. We verify that $\widehat{X}_1$ equals $\widehat{X}$ by checking that $q_X^{-1}(U)$ is irreducible for suitable open neighbourhoods $U \subseteq X$ covering X.

Let $D_1, \dots, D_s$ be a basis of K such that $n_1D_1, \dots, n_kD_k$, where $1 \leq k \leq s$, is a basis of K^0. Enlarging K, if necessary, we may assume that the D_i are primitive, i.e., no proper multiples. We take subsets $U \subseteq X$ such that on U every D_i is principal, say $D_i = \mathrm{div}(f_i)$. Then, with $\deg(T_i) := D_i$, Remark 3.1.4 provides a K-graded isomorphism

$$\Gamma(U, \mathcal{O}) \otimes_{\mathbb{K}} \mathbb{K}[T_1^{\pm 1}, \dots, T_s^{\pm 1}] \;\to\; \Gamma(U, \mathcal{S}), \qquad g \otimes T_1^{\nu_1} \cdots T_s^{\nu_s} \;\mapsto\; g f_1^{-\nu_1} \cdots f_s^{-\nu_s}.$$

In particular, this identifies $p^{-1}(U)$ with $U \times \mathbb{T}^s$, where $\mathbb{T}^s := (\mathbb{K}^*)^s$. According to Remark 4.3.2, the ideal $\Gamma(U, \mathcal{I})$ is generated by $1 - \chi(n_iD_i)$, where $1 \leq i \leq k$. Thus $q_X^{-1}(U)$ is given in $U \times \mathbb{T}^s$ by the equations

$$1 - \chi(n_iD_i) f_i^{n_i} T_i^{n_i} \;=\; 0, \qquad 1 \leq i \leq k.$$

To obtain irreducibility of $q_X^{-1}(U)$, it suffices to show that each $1 - \chi(n_iD_i)f_i^{n_i}T_i^{n_i}$ is irreducible in $\mathbb{K}(X)[T_i^{\pm 1}]$. With respect to the variable $S_i := f_iT_i$, this means to verify irreducibility of

$$1 \; - \; \chi(n_iD_i)S_i^{n_i} \quad \in \quad \mathbb{K}(X)[S_i^{\pm 1}].$$

In view of Lemma 5.1.5, we have to show that $\chi(n_iD_i)$ is not a proper power in $\mathbb{K}(X)$. Assume the contrary. Then we obtain $n_iD_i = k_i\mathrm{div}(h_i)$ with some $h_i \in \mathbb{K}(X)$. Since D_i is primitive, k_i divides n_i and thus, n_i/k_iD_i is principal. A contradiction to the choice of n_i.

The fact that each ring $\Gamma(U, \mathcal{R})$ is normal follows directly from the fact that it is the ring of functions of an open subset of the smooth prevariety $\widehat{X}$. □

5.2. Localization and units. We treat localization by homogeneous elements and consider the units of the Cox ring $\mathcal{R}(X)$ of a normal prevariety X defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The main tool is the divisor of a homogeneous element of $\mathcal{R}(X)$, which we first define precisely.

In the setting of Construction 4.2.1, consider a divisor $D \in K$ and a non-zero element $f \in \mathcal{R}_{[D]}(X)$. According to Lemma 4.3.3, there is a (unique) element $\widetilde{f} \in \Gamma(X, \mathcal{S}_D)$ with $\pi(\widetilde{f}) = f$, where $\pi\colon \mathcal{S} \to \mathcal{R}$ denotes the projection. We define the $[D]$-*divisor* of f to be the effective Weil divisor

$$\mathrm{div}_{[D]}(f) \; := \; \mathrm{div}_D(\widetilde{f}) \; = \; \mathrm{div}(\widetilde{f}) + D \; \in \; \mathrm{WDiv}(X).$$

Lemma 5.2.1. *The $[D]$-divisor depends neither on representative $D \in K$ nor on the choices made in 4.2.1. Moreover, the following holds.*

- (i) *For every effective $E \in \mathrm{WDiv}(X)$ there are $[D] \in \mathrm{Cl}(X)$ and $f \in \mathcal{R}_{[D]}(X)$ with $E = \mathrm{div}_{[D]}(f)$.*
- (ii) *Let $[D] \in \mathrm{Cl}(X)$ and $0 \neq f \in \mathcal{R}_{[D]}(X)$. Then $\mathrm{div}_{[D]}(f) = 0$ implies $[D] = 0$ in $\mathrm{Cl}(X)$.*
- (iii) *For any two non-zero homogeneous elements $f \in \mathcal{R}_{[D_1]}(X)$ and $g \in \mathcal{R}_{[D_2]}(X)$, we have*

$$\mathrm{div}_{[D_1]+[D_2]}(fg) \quad = \quad \mathrm{div}_{[D_1]}(f) \; + \; \mathrm{div}_{[D_2]}(g).$$

Proof. Let $f \in \mathcal{R}_{[D]}(X)$, consider any two isomorphisms $\varphi_i\colon \mathcal{O}_X(D_i) \to \mathcal{R}_{[D]}$ and let $\widetilde{f}_i$ be the sections with $\varphi_i(\widetilde{f}_i) = f$. Then $\varphi_2^{-1} \circ \varphi_1$ is multiplication with some $h \in \mathbb{K}(X)^*$ satisfying $\mathrm{div}(h) = D_1 - D_2$. Well-definedness of the $[D]$-divisor thus follows from

$$\mathrm{div}_{D_1}(\widetilde{f}_1) \; = \; \mathrm{div}(h\widetilde{f}_1) + D_2 \; = \; \mathrm{div}_{D_2}(\widetilde{f}_2).$$

If $\mathrm{div}_{[D]}(f) = 0$ holds as in (ii), then, for a representative $\widetilde{f} \in \Gamma(X, \mathcal{O}_X(D))$ of $f \in \mathcal{R}_{[D]}(X)$, we have $\mathrm{div}_D(\widetilde{f}) = 0$ and hence D is principal. Observations (i) and (iii) are obvious. □

For every non-zero homogeneous element $f \in \mathcal{R}_{[D]}(X)$, we define the $[D]$-*localization* of X by f to be the open subset

$$X_{[D],f} \; := \; X \setminus \mathrm{Supp}(\mathrm{div}_{[D]}(f)) \; \subseteq \; X.$$

Proposition 5.2.2. *Let X be a normal prevariety with only constant invertible functions, finitely generated divisor class group and Cox ring $\mathcal{R}(X)$. Then, for every non-zero homogeneous $f \in \mathcal{R}_{[D]}(X)$, we have a canonical isomorphism*

$$\Gamma(X_{[D],f}, \mathcal{R}) \quad \cong \quad \Gamma(X, \mathcal{R})_f.$$

Proof. Let the divisor $D \in K$ represent $[D] \in \mathrm{Cl}(X)$ and consider the section $\widetilde{f} \in \Gamma(X, \mathcal{S}_D)$ with $\pi(\widetilde{f}) = f$. According to Remark 3.1.6, we have

$$\Gamma(X_{D,\widetilde{f}}, \mathcal{S}) \quad \cong \quad \Gamma(X, \mathcal{S})_{\widetilde{f}}.$$

The assertion thus follows from Lemma 4.3.5 and the fact that localization is compatible with passing to the factor ring. □

We turn to the units of the Cox ring $\mathcal{R}(X)$; the following result says in particular, that for a complete normal variety X they are all constant.

Proposition 5.2.3. *Let X be a normal prevariety with only constant invertible functions, finitely generated divisor class group and Cox ring $\mathcal{R}(X)$.*

(i) *Every homogeneous invertible element of $\mathcal{R}(X)$ is constant.*
(ii) *If $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds, then every invertible element of $\mathcal{R}(X)$ is constant.*

Proof. For (i), let $f \in \mathcal{R}(X)^*$ be homogeneous of degree $[D]$. Then its inverse $g \in \mathcal{R}(X)^*$ is homogeneous of degree $-[D]$, and $fg = 1$ lies in $\mathcal{R}(X)^*_0 = \mathbb{K}^*$. By Lemma 5.2.1 (iii), we have

$$0 \; = \; \mathrm{div}_0(fg) \; = \; \mathrm{div}_{[D]}(f) + \mathrm{div}_{[-D]}(g).$$

Since the divisors $\mathrm{div}_{[D]}(f)$ and $\mathrm{div}_{[-D]}(g)$ are effective, they both vanish. Thus, Lemma 5.2.1 (ii) yields $[D] = 0$. This implies $f \in \mathcal{R}(X)^*_0 = \mathbb{K}^*$ as wanted.

For (ii), we have to show that any invertible $f \in \mathcal{R}(X)$ is of degree zero. Choose a decomposition $\mathrm{Cl}(X) = K_0 \oplus K_t$ into a free part and the torsion part, and consider the coarsified grading

$$\mathcal{R}(X) \; = \; \bigoplus_{w \in K_0} R_w, \qquad\qquad R_w \; := \; \bigoplus_{u \in K_t} \mathcal{R}(X)_{w+u}.$$

Then, as any invertible element of the K_0-graded integral ring $\mathcal{R}(X)$, also f is necessarily K_0-homogeneous of some degree $w \in K_0$. Decomposing f and f^{-1} into $\mathrm{Cl}(X)$-homogeneous parts we get representations

$$f \; = \; \sum_{u \in K_t} f_{w+u}, \qquad\qquad f^{-1} \; = \; \sum_{u \in K_t} f^{-1}_{-w+u}.$$

Because of $ff^{-1} = 1$, we have $f_{w+v} f^{-1}_{-w-v} \neq 0$ for at least one $v \in K_t$. Since $\Gamma(X, \mathcal{O}) = \mathbb{K}$ holds, $f_{w+v} f^{-1}_{-w-v}$ must be a non-zero constant. Using Lemma 5.2.1 we conclude $w + v = 0$ as before. In particular, $w = 0$ holds and thus each f_{w+u} has a torsion degree. For a suitable power f^n_{w+u} we have $n\mathrm{div}_{w+u}(f_{w+u}) = 0$, which implies $f_{w+u} = 0$ for any $u \neq 0$. □

Remark 5.2.4. The affine surface X treated in Example 4.4.2 shows that requiring $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ is in general not enough in order to ensure that all units of the Cox ring are constant.

5.3. Divisibility properties. For normal prevarieties X with a free finitely generated divisor class group, we saw that the Cox ring admits unique factorization. If we have torsion in the divisor class group this does not need to hold any more. However, restricting to homogeneous elements leads to a framework for a reasonable divisibility theory; the precise notions are the following.

Definition 5.3.1. Consider an abelian group K and a K-graded integral $\mathbb{K}$-algebra $R = \bigoplus_{w \in K} R_w$.

(i) A non-zero non-unit $f \in R$ is *K-prime* if it is homogeneous and $f|gh$ with homogeneous $g, h \in R$ implies $f|g$ or $f|h$.
(ii) We say that R is *factorially graded* if every homogeneous non-zero non-unit $f \in R$ is a product of K-primes.

(iii) An ideal $\mathfrak{a} \lhd R$ is *K-prime* if it is homogeneous and for any two homogeneous $f, g \in R$ with $fg \in \mathfrak{a}$ one has either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.
(iv) A K-prime ideal $\mathfrak{a} \lhd R$ has *K-height* d if d is maximal admitting a chain $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \ldots \subset \mathfrak{a}_d = \mathfrak{a}$ of K-prime ideals.

Let us look at these concepts also from the geometric point of view. Consider a prevariety Y with an action of an algebraic group H. Then H acts also on the group $\mathrm{WDiv}(Y)$ of Weil divisors via

$$h \cdot \sum a_D D \quad := \quad \sum a_D (h \cdot D).$$

By an *H-prime divisor* we mean a non-zero sum $\sum a_D D$ with prime divisors D such that $a_D \in \{0,1\}$ always holds and the D with $a_D = 1$ are transitively permuted by H. Note that every H-invariant divisor is a unique sum of H-prime divisors. We say that Y is *H-factorial* if every H-invariant Weil divisor on Y is principal.

Proposition 5.3.2. *Let $H = \operatorname{Spec} \mathbb{K}[K]$ be a quasitorus and W an irreducible normal quasiaffine H-variety. Consider the K-graded algebra $R := \Gamma(W, \mathcal{O})$ and assume $R^* = \mathbb{K}^*$. Then the following statements are equivalent.*

(i) *Every K-prime ideal of K-height one in R is principal.*
(ii) *The variety W is H-factorial.*
(iii) *The ring R is factorially graded.*

Moreover, if one of these statements holds, then a homogeneous non-zero non-unit $f \in R$ is K-prime if and only if the divisor $\operatorname{div}(f)$ is H-prime, and every H-prime divisor is of the form $\operatorname{div}(f)$ with a K-prime $f \in R$.

Proof. Assume that (i) holds and let D be an H-invariant Weil divisor on W. Write $D = D_1 + \ldots + D_r$ with H-prime divisors D_i. Then the vanishing ideal $\mathfrak{a}_i$ of D_i is of K-height one, and (i) guarantees that it is principal, say $\mathfrak{a}_i = \langle f_i \rangle$. Thus $D_i = \operatorname{div}(f_i)$ and $D = \operatorname{div}(f_1 \cdots f_r)$ hold, which proves (ii).

Assume that (ii) holds. Given a homogeneous element $0 \neq f \in R \setminus R^*$, write $\operatorname{div}(f) = D_1 + \ldots + D_r$ with H-prime divisors D_i. Then $D_i = \operatorname{div}(f_i)$ holds, where, because of $R^* = \mathbb{K}^*$, the elements f_i are homogeneous. One verifies directly that the f_i are K-prime. Thus we have $f = \alpha f_1 \cdots f_r$ with $\alpha \in \mathbb{K}^*$ as required in (iii).

If (iii) holds and $\mathfrak{a}$ is a K-prime ideal of K-height one, then we take any homogeneous $0 \neq f \in \mathfrak{a}$ and find a K-prime factor f_1 of f with $f_1 \in \mathfrak{a}$. This gives inclusions $0 \subsetneq \langle f_1 \rangle \subseteq \mathfrak{a}$ of K-prime ideals, which implies $\mathfrak{a} = \langle f_1 \rangle$. □

Corollary 5.3.3. *Under the assumptions of Proposition 5.3.2, factoriality of the algebra R implies that it is factorially graded.*

We are ready to study the divisibility theory of the Cox ring. Here comes the main result; it applies in particular to complete varieties, see Corollary 5.3.8.

Theorem 5.3.4. *Let X be an irreducible normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. If the Cox ring $\mathcal{R}(X)$ satisfies $\mathcal{R}(X)^* = \mathbb{K}^*$, then it is factorially graded.*

Lemma 5.3.5. *In the situation of Construction 5.1.4, every non-zero element $f \in \Gamma(X, \mathcal{R}_{[D]})$ satisfies*

$$\operatorname{div}(f) \quad = \quad q_X^*(\operatorname{div}_{[D]}(f)),$$

where on the left hand side f is a regular function on $\widehat{X}$ and on the right hand side f is an element on $\mathcal{R}(X)$.

Proof. In the notation of 5.1.4, let $D \in K$ represent $[D] \in \mathrm{Cl}(X)$, and let $\widetilde{f} \in \Gamma(X, \mathcal{S}_D)$ project to $f \in \Gamma(X, \mathcal{R}_{[D]})$. The commutative diagram of 5.1.4 yields

$$\mathrm{div}(f) \;=\; \imath^*(\mathrm{div}(\widetilde{f})) \;=\; \imath^*(p^*(\mathrm{div}_D(\widetilde{f}))) \;=\; q_X^*(\mathrm{div}_{[D]}(f)),$$

where $\imath\colon \widehat{X} \to \widetilde{X}$ denotes the inclusion and the equality $\mathrm{div}(\widetilde{f}) = p^*(\mathrm{div}_D(\widetilde{f}))$ was established in Lemma 3.3.2. □

Lemma 5.3.6. *In the situation of Construction 5.1.4, the prevariety $\widehat{X}$ is irreducible, smooth and H-factorial.*

Proof. As remarked in Construction 5.1.4, the prevariety $\widehat{X}$ is smooth and due to Proposition 5.1.1, it is irreducible. Let $\widehat{D}$ be an invariant Weil divisor on $\widehat{X}$. Using, for example, the fact that $q_X : \widehat{X} \to X$ is an étale principal bundle, we see that $\widehat{D} = q_X^*(D)$ holds with a Weil divisor D on X. Thus, we have to show that all pullback divisors $q_X^*(D)$ are principal. For this, it suffices to consider effective divisors D on X, and these are treated by Lemmas 5.2.1 and 5.3.5. □

Proof of Theorem 5.3.4. According to Lemma 5.1.2, we may assume that X is smooth. Then $\mathcal{R}(X)$ is the algebra of regular functions of the quasiaffine variety $\widehat{X}$ constructed in 5.1.4. Lemma 5.3.6 tells us that $\widehat{X}$ is irreducible, smooth and H-factorial. Thus, Proposition 5.3.2 gives the assertion. □

Corollary 5.3.7. *Let X be a normal prevariety of affine intersection with $\Gamma(X, \mathcal{O}) = \mathbb{K}$ and finitely generated divisor class group. Then the Cox ring $\mathcal{R}(X)$ is factorially graded.*

Proof. According to Proposition 5.2.3, the assumption $\Gamma(X, \mathcal{O}) = \mathbb{K}$ ensures $\mathcal{R}(X)^* = \mathbb{K}^*$. Thus Theorem 5.3.4 applies. □

Corollary 5.3.8. *Let X be a complete normal variety with finitely generated divisor class group. Then the Cox ring $\mathcal{R}(X)$ is factorially graded.*

As in the torsion free case, see Proposition 3.3.4, divisibility and primality of homogeneous elements in the Cox ring $\mathcal{R}(X)$ can be characterized in terms of X.

Proposition 5.3.9. *Let X be a normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. Suppose that the Cox ring $\mathcal{R}(X)$ satisfies $\mathcal{R}(X)^* = \mathbb{K}^*$.*

(i) *An element $0 \neq f \in \Gamma(X, \mathcal{R}_{[D]})$ divides $0 \neq g \in \Gamma(X, \mathcal{R}_{[E]})$ if and only if $\mathrm{div}_{[D]}(f) \le \mathrm{div}_{[E]}(g)$ holds.*
(ii) *An element $0 \neq f \in \Gamma(X, \mathcal{R}_{[D]})$ is $\mathrm{Cl}(X)$-prime if and only if the divisor $\mathrm{div}_{[D]}(f) \in \mathrm{WDiv}(X)$ is prime.*

Proof. According to Lemma 5.1.2, we may assume that X is smooth. Then Construction 5.1.4 presents X as the geometric quotient of the smooth quasiaffine H_X-variety $\widehat{X}$, which has $\mathcal{R}(X)$ as its algebra of regular functions. The first statement follows immediately from Lemma 5.3.5 and, for the second one, we additionally use Proposition 5.3.2. □

Remark 5.3.10. Let X be a prevariety of affine intersection with only constant invertible functions, finitely generated divisor class group and a Cox ring $\mathcal{R}(X)$ with only constant invertible elements. Then the assignement $f \mapsto \mathrm{div}_{[D]}(f)$ induces an isomorphism from the multiplicative semigroup of homogeneous elements of $\mathcal{R}(X)$ modulo units onto the semigroup $\mathrm{WDiv}^+(X)$ of effective Weil divisors on X. The fact that $\mathcal{R}(X)$ is factorially graded reflects the fact that every effective Weil divisor is a unique non-negative linear combination of prime divisors.

Remark 5.3.11. For the affine surface X considered in Example 4.4.1, the Cox ring $\mathcal{R}(X)$ is factorially $\mathbb{Z}/2\mathbb{Z}$-graded but it is not a factorial ring.

6. Geometric realization of the Cox sheaf

6.1. Characteristic spaces. We study the geometric realization of a Cox sheaf, its relative spectrum, which we call a *characteristic space*. For locally factorial varieties, e.g. smooth ones, this concept coincides with the universal torsor introduced by Colliot-Thélène and Sansuc in [**53**], see also [**56**] and [**144**]. As soon as we have non-factorial singularities, the characteristic space is not any more a torsor, i.e. a principal bundle, as we will see later. As before, we work with normal prevarieties defined over an algebraically closed field of characteristic zero. First we provide two statements on local finite generation of Cox sheaves.

Proposition 6.1.1. *Let X be a normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. If the Cox ring $\mathcal{R}(X)$ is finitely generated, then the Cox sheaf $\mathcal{R}$ is locally of finite type.*

Proof. The assumption that X is of affine intersection guarantees that it is covered by open affine subsets $X_{[D],f}$, where $[D] \in \mathrm{Cl}(X)$ and $f \in \mathcal{R}_{[D]}(X)$. By Proposition 5.2.2, we have $\Gamma(X_{[D],f}, \mathcal{R}) = \mathcal{R}(X)_f$, which gives the assertion. □

Proposition 6.1.2. *Let X be a normal prevariety with only constant invertible functions and finitely generated divisor class group. If X is $\mathbb{Q}$-factorial, then any Cox sheaf $\mathcal{R}$ is locally of finite type.*

Proof. By definition, the Cox sheaf $\mathcal{R}$ is the quotient of a sheaf of divisorial algebras $\mathcal{S}$ by some ideal sheaf $\mathcal{I}$. According to Proposition 4.2.2, we may assume that $\mathcal{S}$ arises from a finitely generated subgroup $K \subseteq \mathrm{WDiv}(X)$. Proposition 3.2.3 then tells us that $\mathcal{S}$ is locally of finite type, and Lemma 4.3.5 ensures that the quotient $\mathcal{R} = \mathcal{S}/\mathcal{I}$ can be taken at the level of sections. □

We turn to the relative spectrum of a Cox sheaf. The following generalizes Construction 5.1.4, where the smooth case is considered.

Construction 6.1.3. Let X be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group, and let $\mathcal{R}$ be a Cox sheaf. Suppose that $\mathcal{R}$ is locally of finite type, e.g., X is $\mathbb{Q}$-factorial or $\mathcal{R}(X)$ is finitely generated. Taking the relative spectrum gives an irreducible normal prevariety

$$\widehat{X} \quad := \quad \mathrm{Spec}_X(\mathcal{R}).$$

The $\mathrm{Cl}(X)$-grading of the sheaf $\mathcal{R}$ defines an action of the diagonalizable group $H_X := \mathrm{Spec}\,\mathbb{K}[\mathrm{Cl}(X)]$ on $\widehat{X}$, the canonical morphism $q_X \colon \widehat{X} \to X$ is a good quotient for this action, and we have an isomorphism of sheaves

$$\mathcal{R} \quad \cong \quad (q_X)_*(\mathcal{O}_{\widehat{X}}).$$

We call $q_X \colon \widehat{X} \to X$ the *characteristic space* associated to $\mathcal{R}$, and H_X the *characteristic quasitorus* of X.

Proof. Everything is standard except irreducibility and normality, which follow from Theorem 5.1.1. □

The Cox sheaf $\mathcal{R}$ was defined as the quotient of a sheaf $\mathcal{S}$ of divisorial algebras by a sheaf $\mathcal{I}$ of ideals. Geometrically this means that the characteristic space comes embedded into the relative spectrum of a sheaf of divisorial algebras; compare 5.1.4 for the case of a smooth X. Before making this precise in the general case, we have to relate local finite generation of the sheaves $\mathcal{R}$ and $\mathcal{S}$ to each other.

PROPOSITION 6.1.4. *Let X be a normal prevariety with only constant invertible functions, finitely generated divisor class group and Cox sheaf $\mathcal{R}$. Moreover, let $\mathcal{S}$ be the sheaf of divisorial algebras associated to a finitely generated subgroup $K \subseteq \mathrm{WDiv}(X)$ projecting onto $\mathrm{Cl}(X)$ and $U \subseteq X$ an open affine subset. Then the algebra $\Gamma(U, \mathcal{R})$ is finitely generated if and only if the algebra $\Gamma(U, \mathcal{S})$ is finitely generated.*

PROOF. Lemma 4.3.5 tells us that $\Gamma(U, \mathcal{R})$ is a factor algebra of $\Gamma(U, \mathcal{S})$. Thus, if $\Gamma(U, \mathcal{S})$ is finitely generated then the same holds for $\Gamma(U, \mathcal{R})$. Moreover, Lemma 4.3.4 says that the projection $\Gamma(U, \mathcal{S}) \to \Gamma(U, \mathcal{R})$ defines isomorphisms along the homogeneous components. Thus, Proposition 1.2.6 shows that finite generation of $\Gamma(U, \mathcal{R})$ implies finite generation of $\Gamma(U, \mathcal{S})$. □

CONSTRUCTION 6.1.5. Let X be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group, and let $K \subseteq \mathrm{WDiv}(X)$ be a finitely generated subgroup projecting onto $\mathrm{Cl}(X)$. Consider the sheaf of divisorial algebras $\mathcal{S}$ associated to K and the Cox sheaf $\mathcal{R} = \mathcal{S}/\mathcal{I}$ as constructed in 4.2.1, and suppose that one of these sheaves is locally of finite type. Then the projection $\mathcal{S} \to \mathcal{R}$ of $\mathcal{O}_X$-algebras defines a commutative diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\imath} & \widetilde{X} \\ & {\scriptstyle q_X} \searrow \quad \swarrow {\scriptstyle p} & \\ & X & \end{array}$$

for the relative spectra $\widehat{X} = \mathrm{Spec}_X \mathcal{R}$ and $\widetilde{X} = \mathrm{Spec}_X \mathcal{S}$. We have the actions of $H_X = \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$ on $\widehat{X}$ and $H = \mathrm{Spec}\, \mathbb{K}[K]$ on $\widetilde{X}$. The map $\imath \colon \widehat{X} \to \widetilde{X}$ is a closed embedding and it is H_X-invariant, where H_X acts on $\widetilde{X}$ via the inclusion $H_X \subseteq H$ defined by the projection $K \to \mathrm{Cl}(X)$. The image $\imath(\widehat{X}) \subseteq \widetilde{X}$ is precisely the zero set of the ideal sheaf $\mathcal{I}$.

PROPOSITION 6.1.6. *Situation as in Construction 6.1.3.*

(i) *The inverse image $q_X^{-1}(X_{\mathrm{reg}}) \subseteq \widehat{X}$ of the set of smooth points $X_{\mathrm{reg}} \subseteq X$ is smooth, the group H_X acts freely on $q_X^{-1}(X_{\mathrm{reg}})$ and the restriction $q_X : q_X^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is an étale H_X-principal bundle.*
(ii) *For any closed $A \subseteq X$ of codimension at least two, $q_X^{-1}(A) \subseteq \widehat{X}$ is as well of codimension at least two.*
(iii) *The prevariety $\widehat{X}$ is H_X-factorial.*
(iv) *If X is of affine intersection, then $\widehat{X}$ is a quasiaffine variety.*

PROOF. For (i), we refer to the proof of Construction 5.1.4. To obtain (ii) consider an affine open set $U \subseteq X$ and $\widehat{U} := q_X^{-1}(U)$. By Lemma 5.1.2, the open set $\widehat{U} \setminus q_X^{-1}(A)$ has the same regular functions as $\widehat{U}$. Since $\widehat{X}$ is normal, we conclude that $\widehat{U} \cap q_X^{-1}(A)$ is of codimension at least two in $\widehat{U}$. Now, cover X by affine $U \subseteq X$ and Assertion (ii) follows. We turn to (iii). According to (ii) we may assume that X is smooth. In this case, the statement was proven in Lemma 5.3.6. We show (iv). We may assume that we are in the setting of Construction 6.1.5. Corollary 3.4.6 then ensures that $\widetilde{X}$ is quasiaffine and Construction 6.1.5 gives that $\widehat{X}$ is a closed subvariety of $\widetilde{X}$. □

The following statement relates the divisor of a $[D]$-homogeneous function on $\widehat{X}$ to its $[D]$-divisor on X; the smooth case was settled in Lemma 5.3.5.

Proposition 6.1.7. *In the situation of 6.1.3, consider the pullback homomorphism* $q_X^* \colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(\widehat{X})$ *defined in 3.4.1. Then, for every* $[D] \in \mathrm{Cl}(X)$ *and every* $f \in \Gamma(X, \mathcal{R}_{[D]})$*, we have*

$$\mathrm{div}(f) \quad = \quad q_X^*(\mathrm{div}_{[D]}(f)),$$

where on the left hand side f *is a function on* $\widehat{X}$*, and on the right hand side a function on* X*. If* X *is of affine intersection and* $X \setminus \mathrm{Supp}(\mathrm{div}_{[D]}(f))$ *is affine, then we have moreover*

$$\mathrm{Supp}(\mathrm{div}(f)) \quad = \quad q_X^{-1}(\mathrm{Supp}(\mathrm{div}_{[D]}(f))).$$

Proof. We may assume that we are in the setting of Construction 6.1.5. Let the divisor $D \in K$ represent the class $[D] \in \mathrm{Cl}(X)$, and let $\widetilde{f} \in \Gamma(X, \mathcal{S}_D)$ project to $f \in \Gamma(X, \mathcal{R}_{[D]})$. The commutative diagram of 6.1.5 yields

$$\mathrm{div}(f) \;=\; \imath^*(\mathrm{div}(\widetilde{f})) \;=\; \imath^*(p^*(\mathrm{div}_D(\widetilde{f}))) \;=\; q_X^*(\mathrm{div}_{[D]}(f)),$$

where $\imath \colon \widehat{X} \to \widetilde{X}$ denotes the inclusion and the equality $\mathrm{div}(\widetilde{f}) = p^*(\mathrm{div}_D(\widetilde{f}))$ was established in Proposition 3.4.3. Similarly, we have

$$\begin{aligned} \mathrm{Supp}(\mathrm{div}(f)) \quad &= \quad \imath^{-1}(\mathrm{Supp}(\mathrm{div}(\widetilde{f}))) \\ &= \quad \imath^{-1}(p^{-1}(\mathrm{Supp}(\mathrm{div}_D(\widetilde{f})))) \\ &= \quad q_X^{-1}(\mathrm{Supp}(\mathrm{div}_{[D]}(f))) \end{aligned}$$

provided that X is of affine intersection and $X \setminus \mathrm{Supp}(\mathrm{div}_{[D]}(f))$ is affine, because Proposition 3.4.3 then ensures $\mathrm{Supp}(\mathrm{div}(\widetilde{f})) = p^{-1}(\mathrm{Supp}(\mathrm{div}_D(\widetilde{f})))$. □

Corollary 6.1.8. *Situation as in Construction 6.1.3. Let* $\widehat{x} \in \widehat{X}$ *be a point such that* $H_X \cdot \widehat{x} \subseteq \widehat{X}$ *is closed, and let* $f \in \Gamma(X, \mathcal{R}_{[D]})$*. Then we have*

$$f(\widehat{x}) \;=\; 0 \quad \Longleftrightarrow \quad q_X(\widehat{x}) \;\in\; \mathrm{Supp}(\mathrm{div}_{[D]}(f)).$$

Proof. The image $q_X(\mathrm{Supp}(\mathrm{div}(f))$ is contained in $\mathrm{Supp}(\mathrm{div}_{[D]}(f))$. By the definition of the pullback and Proposition 6.1.7, the two sets coincide in X_{reg}. Thus, $q_X(\mathrm{Supp}(\mathrm{div}(f))$ is dense in $\mathrm{Supp}(\mathrm{div}_{[D]}(f))$. By Theorem 2.3.6, the image $q_X(\mathrm{Supp}(\mathrm{div}(f))$ is closed and thus we have

$$q_X(\mathrm{Supp}(\mathrm{div}(f))) \quad = \quad \mathrm{Supp}(\mathrm{div}_{[D]}(f)).$$

In particular, if $f(\widehat{x}) = 0$ holds, then $q_X(\widehat{x})$ lies in $\mathrm{Supp}(\mathrm{div}_{[D]}(f))$. Conversely, if $q_X(\widehat{x})$ belongs to $\mathrm{Supp}(\mathrm{div}_{[D]}(f))$, then some $\widehat{x}' \in \mathrm{Supp}(\mathrm{div}(f))$ belongs to the fiber of $\widehat{x}$. Since $H_x \cdot \widehat{x}$ is closed, Corollary 2.3.7 tells us that $\widehat{x}$ is contained in the orbit closure of $\widehat{x}'$ and hence belongs to $\mathrm{Supp}(\mathrm{div}(f))$. □

Corollary 6.1.9. *Situation as in Construction 6.1.5 and suppose that* X *is of affine intersection. For* $x \in X$*, let* $\widehat{x} \in q_X^{-1}(x)$ *such that* $H_X \cdot \widehat{x}$ *is closed in* $\widehat{X}$*. Then* $H \cdot \widehat{x}$ *is closed in* $\widetilde{X}$*.*

Proof. Assume that the orbit $H \cdot \widehat{x}$ is not closed in $\widetilde{X}$. Then there is a point $\widetilde{x} \in p^{-1}(x)$ having a closed H-orbit in $\widetilde{X}$, and $\widetilde{x}$ lies in the closure of $H \cdot \widehat{x}$. Since $\widetilde{X}$ is quasiaffine, we find a function $\widetilde{f} \in \Gamma(X, \mathcal{S}_D)$ with $\widetilde{f}(\widetilde{x}) = 0$ but $\widetilde{f}(\widehat{x}) \neq 0$. Corollary 3.4.4 gives $p(\widetilde{x}) \in \mathrm{Supp}(\mathrm{div}_D(\widetilde{f}))$. Since we have $q_X(\widehat{x}) = p(\widetilde{x})$, this contradicts Corollary 6.1.8. □

6.2. Divisor classes and isotropy groups. The aim of this subsection is to relate local properties of a prevariety to properties of the characteristic quasitorus action on its characteristic space. Again, everything takes places over an algebraically closed field $\mathbb{K}$ of characteristic zero.

For a normal prevariety X and a point $x \in X$, let $\mathrm{PDiv}(X,x) \subseteq \mathrm{WDiv}(X)$ denote the subgroup of all Weil divisors, which are principal on some neighbourhood of x. We define the *local class group* of X in x to be the factor group

$$\mathrm{Cl}(X,x) \quad := \quad \mathrm{WDiv}(X)/\,\mathrm{PDiv}(X,x).$$

Obviously the group $\mathrm{PDiv}(X)$ of principal divisors is contained in $\mathrm{PDiv}(X,x)$. Thus, there is a canonical epimorphism $\pi_x\colon \mathrm{Cl}(X) \to \mathrm{Cl}(X,x)$. The *Picard group* of X is the factor group of the group $\mathrm{CDiv}(X)$ of Cartier divisors by the subgroup of principal divisors:

$$\mathrm{Pic}(X) \;=\; \mathrm{CDiv}(X)/\,\mathrm{PDiv}(X) \;=\; \bigcap_{x \in X} \ker(\pi_x).$$

PROPOSITION 6.2.1. *Situation as in Construction 6.1.3. For $x \in X$, let $\widehat{x} \in q_X^{-1}(x)$ be a point with closed H_X-orbit. Define a submonoid*

$$S_x \;:=\; \{[D] \in \mathrm{Cl}(X);\ f(\widehat{x}) \neq 0 \text{ for some } f \in \Gamma(X, \mathcal{R}_{[D]})\} \;\subseteq\; \mathrm{Cl}(X),$$

and let $\mathrm{Cl}_x(X) \subseteq \mathrm{Cl}(X)$ denote the subgroup generated by S_x. Then the local class groups of X and the Picard group are given by

$$\mathrm{Cl}(X,x) \;=\; \mathrm{Cl}(X)/\,\mathrm{Cl}_x(X), \qquad\qquad \mathrm{Pic}(X) \;=\; \bigcap_{x \in X} \mathrm{Cl}_x(X).$$

PROOF. First observe that Corollary 6.1.8 gives us the following description of the monoid S_x in terms of the $[D]$-divisors:

$$\begin{aligned} S_x \quad &= \quad \{[D] \in \mathrm{Cl}(X);\ x \notin \mathrm{div}_{[D]}(f) \text{ for some } f \in \Gamma(X, \mathcal{R}_{[D]})\} \\ &= \quad \{[D] \in \mathrm{Cl}(X);\ D \geq 0,\ x \notin \mathrm{Supp}(D)\}, \end{aligned}$$

where the latter equation is due to the fact that the $[D]$-divisors are precisely the effective divisors with class $[D]$. The assertions thus follow from

$$\begin{aligned} \mathrm{Cl}_x(X) \quad &= \quad \{[D] \in \mathrm{Cl}(X);\ x \notin \mathrm{Supp}(D)\} \\ &= \quad \{[D] \in \mathrm{Cl}(X);\ D \text{ principal near } x\}. \end{aligned}$$

□

PROPOSITION 6.2.2. *Situation as in Construction 6.1.3. Given $x \in X$, let $\widehat{x} \in q_X^{-1}(x)$ be a point with closed H_X-orbit. Then the inclusion $H_{X,\widehat{x}} \subseteq H_X$ of the isotropy group of $\widehat{x} \in \widehat{X}$ is given by the epimorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(X,x)$ of character groups. In particular, we have*

$$H_{X,\widehat{x}} \;=\; \mathrm{Spec}\,\mathbb{K}[\mathrm{Cl}(X,x)], \qquad\qquad \mathrm{Cl}(X,x) \;=\; \mathbb{X}(H_{X,\widehat{x}}).$$

PROOF. Let $U \subseteq X$ be any affine open neighbourhood of $x \in X$. Then U is of the form $X_{[D],f}$ with some $f \in \Gamma(X, \mathcal{R}_{[D]})$ and $\widetilde{U} := q_X^{-1}(U)$ is affine. According to Proposition 5.2.2, we have

$$\Gamma(\widetilde{U}, \mathcal{O}) \;=\; \Gamma(U, \mathcal{R}) \;=\; \Gamma(X, \mathcal{R})_f \;=\; \Gamma(\widehat{X}, \mathcal{O})_f.$$

Corollary 6.1.8 shows that the group $\mathrm{Cl}_x(X)$ is generated by the classes $[E] \in \mathrm{Cl}(X)$ admitting a section $g \in \Gamma(U, \mathcal{R}_{[E]})$ with $g(\widehat{x}) \neq 0$. In other words, $\mathrm{Cl}_x(X)$ is the orbit group of the point $\widehat{x} \in \widetilde{U}$. Now Proposition 2.2.8 gives the assertion. □

A point x of a normal prevariety X is called *factorial* if near x every divisor is principal. Thus, $x \in X$ is factorial if and only if its local ring $\mathcal{O}_{X,x}$ admits unique factorization. Moreover, a point $x \in X$ is called $\mathbb{Q}$-*factorial* if near x for every divisor some multiple is principal.

COROLLARY 6.2.3. *Situation as in Construction 6.1.3.*

(i) *A point $x \in X$ is factorial if and only if the fiber $q_X^{-1}(x)$ is a single H_X-orbit with trivial isotropy.*
(ii) *A point $x \in X$ is $\mathbb{Q}$-factorial if and only if the fiber $q_X^{-1}(x)$ is a single H_X-orbit.*

PROOF. The point $x \in X$ is factorial if and only if $\mathrm{Cl}(X, x)$ is trivial, and it is $\mathbb{Q}$-factorial if and only if $\mathrm{Cl}(X, x)$ is finite. Thus, the statement follows from Proposition 6.2.2 and Corollary 2.3.7. □

COROLLARY 6.2.4. *Situation as in Construction 6.1.3.*

(i) *The action of H_X on $\widehat{X}$ is free if and only if X is locally factorial.*
(ii) *The good quotient $q_X\colon \widehat{X} \to X$ is geometric if and only if X is $\mathbb{Q}$-factorial.*

COROLLARY 6.2.5. *Situation as in Construction 6.1.3. Let $\widehat{H}_X \subseteq H_X$ be the subgroup generated by all isotropy groups $H_{X,\widehat{x}}$, where $\widehat{x} \in \widehat{X}$. Then we have*

$$\ker\bigl(\mathbb{X}(H_X) \to \mathbb{X}(\widehat{H}_X)\bigr) \quad = \quad \bigcap_{\widehat{x} \in \widehat{X}} \ker\bigl(\mathbb{X}(H_X) \to \mathbb{X}(H_{X,\widehat{x}})\bigr)$$

and the projection $H_X \to H_X/\widehat{H}_X$ corresponds to the inclusion $\mathrm{Pic}(X) \subseteq \mathrm{Cl}(X)$ *of character groups.*

COROLLARY 6.2.6. *Situation as in Construction 6.1.3. If the variety $\widehat{X}$ contains an H_X-fixed point, then the Picard group* $\mathrm{Pic}(X)$ *is trivial.*

6.3. Total coordinate space and irrelevant ideal. Here we consider the situation that the Cox ring is finitely generated. This allows us to introduce the total coordinate space as the spectrum of the Cox ring. As always, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

CONSTRUCTION 6.3.1. Let X be a normal prevariety of affine intersection with only constant invertible functions and finitely generated divisor class group. Let $\mathcal{R}$ be a Cox sheaf and assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. Then we have a diagram

$$\begin{array}{ccccccc} \mathrm{Spec}_X\mathcal{R} & = & \widehat{X} & \xrightarrow{\imath} & \overline{X} & = & \mathrm{Spec}(\mathcal{R}(X)) \\ & & {\scriptstyle q_X}\big\downarrow & & & & \\ & & X & & & & \end{array}$$

where the canonical morphism $\widehat{X} \to \overline{X}$ is an H_X-equivariant open embedding, the complement $\overline{X} \setminus \widehat{X}$ is of codimension at least two and $\overline{X}$ is an H_X-factorial affine variety. We call the H_X-variety $\overline{X}$ the *total coordinate space* associated to $\mathcal{R}$.

PROOF. Cover X by affine open sets $X_{[D],f} = X \setminus \mathrm{Supp}(\mathrm{div}_{[D]}(f))$, where $[D] \in \mathrm{Cl}(X)$ and $f \in \Gamma(X, \mathcal{R}_{[D]})$. Then, according to Proposition 6.1.7, the variety $\widehat{X}$ is covered by the affine sets $\widehat{X}_f = q_X^{-1}(X_{[D],f})$. Note that we have

$$\Gamma(\widehat{X}_f, \mathcal{O}) \;=\; \Gamma(\widehat{X}, \mathcal{O})_f \;=\; \Gamma(\overline{X}, \mathcal{O})_f \;=\; \Gamma(\overline{X}_f, \mathcal{O}).$$

Consequently, the canonical morphisms $\widehat{X}_f \to \overline{X}_f$ are isomorphisms. Gluing them together gives the desired open embedding $\widehat{X} \to \overline{X}$. □

Definition 6.3.2. Situation as in Construction 6.3.1. The *irrelevant ideal* of the prevariety X is the vanishing ideal of the complement $\overline{X} \setminus \widehat{X}$ in the Cox ring:

$$\mathcal{J}_{\mathrm{irr}}(X) \ := \ \{f \in \mathcal{R}(X);\ f_{|\overline{X} \setminus \widehat{X}} = 0\} \ \subseteq \ \mathcal{R}(X).$$

Proposition 6.3.3. *Situation as in Construction 6.3.1.*

(i) *For any section $f \in \Gamma(X, \mathcal{R}_{[D]})$, membership in the irrelevant ideal is characterized as follows:*

$$f \in \mathcal{J}_{\mathrm{irr}}(X) \iff \overline{X}_f = \widehat{X}_f \iff \widehat{X}_f \text{ is affine.}$$

(ii) *Let $0 \neq f \in \Gamma(X, \mathcal{R}_{[D]})$. If the $[D]$-localization $X_{[D],f}$ is affine, then we have $f \in \mathcal{J}_{\mathrm{irr}}(X)$.*

(iii) *Let $0 \neq f_i \in \Gamma(X, \mathcal{R}_{[D_i]})$, where $1 \leq i \leq r$ be such that the sets $X_{[D_i],f_i}$ are affine and cover X. Then we have*

$$\mathcal{J}_{\mathrm{irr}}(X) \quad = \quad \sqrt{\langle f_1, \ldots, f_r \rangle}.$$

Proof. The first equivalence in (i) is obvious and the second one follows from the fact that $\overline{X} \setminus \widehat{X}$ is of codimension at least two in $\overline{X}$. Proposition 6.1.7 tells us that for affine $X_{[D],f}$ also $\widehat{X}_f$ is affine, which gives (ii). We turn to (iii). Proposition 6.1.7 and (ii) ensure that the functions $f_1, \ldots, f_r$ have $\overline{X} \setminus \widehat{X}$ as their common zero locus. Thus Hilbert's Nullstellensatz gives the assertion. □

Corollary 6.3.4. *Situation as in Construction 6.3.1. Then X is affine if and only if $\widehat{X} = \overline{X}$ holds.*

Proof. Take $f = 1$ in the characterization 6.3.3 (i). □

Corollary 6.3.5. *Situation as in Construction 6.3.1 and assume that X is $\mathbb{Q}$-factorial. Then $0 \neq f \in \Gamma(X, \mathcal{R}_{[D]})$ belongs to $\mathcal{J}_{\mathrm{irr}}(X)$ if and only if $X_{[D],f}$ is affine. In particular, we have*

$$\mathcal{J}_{\mathrm{irr}}(X) \quad = \quad \mathrm{lin}_{\mathbb{K}}(f \in \Gamma(X, \mathcal{R}_{[D]});\ [D] \in \mathrm{Cl}(X),\ X_{[D],f} \text{ is affine}).$$

Proof. We have to show that for any $[D]$-homogeneous $f \in \mathcal{J}_{\mathrm{irr}}(X)$, the $[D]$-localization $X_{[D],f}$ is affine. Note that $\widehat{X}_f$ is affine by Proposition 6.3.3 (i). The assumption of $\mathbb{Q}$-factoriality ensures that $q_X \colon \widehat{X} \to X$ is a geometric quotient, see Corollary 6.2.4. In particular, all H_X-orbits in $\widehat{X}$ are closed and thus Corollary 6.1.8 gives us $\widehat{X}_f = q_X^{-1}(X_{[D],f})$. Thus, as the good quotient space of the affine variety $\widehat{X}_f$, the set $X_{[D],f}$ is affine. □

Recall that a divisor D on a prevariety X is called *ample* if it admits sections $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X(D))$ such that the sets X_{D,f_i} are affine and cover X.

Corollary 6.3.6. *Situation as in Construction 6.3.1. If $[D] \in \mathrm{Cl}(X)$ is the class of an ample divisor, then we have*

$$\mathcal{J}_{\mathrm{irr}}(X) \quad = \quad \sqrt{\langle \Gamma(X, \mathcal{R}_{[D]}) \rangle}.$$

6.4. Characteristic spaces via GIT. As we saw, the characteristic space of a prevariety X of affine intersection is a quasiaffine variety $\widehat{X}$ with an action of the characteristic quasitorus H_X having X as a good quotient. Our aim is to characterize this situation in terms of Geometric Invariant Theory. The crucial notion is the following.

Definition 6.4.1. Let G be an affine algebraic group and W a G-prevariety. We say that the G-action on W is *strongly stable* if there is an open invariant subset $W' \subseteq W$ with the following properties:

(i) the complement $W \setminus W'$ is of codimension at least two in W,
(ii) the group G acts freely, i.e. with trivial isotropy groups, on W',
(iii) for every $x \in W'$ the orbit $G{\cdot}x$ is closed in W.

REMARK 6.4.2. Let X be a normal prevariety as in Construction 6.1.3 and consider the characteristic space $q_X: \widehat{X} \to X$ introduced there. Then Proposition 6.1.6 shows that the subset $q_X^{-1}(X_{\mathrm{reg}}) \subseteq \widehat{X}$ satisfies the properties of 6.4.1.

Let X and $q_X: \widehat{X} \to X$ be as in Construction 6.1.3. In the sequel, we mean by a *characteristic space* for X more generally a good quotient $q\colon \mathcal{X} \to X$ for an action of a diagonalizable group H on a prevariety $\mathcal{X}$ such that there is an equivariant isomorphism $(\mu, \widetilde{\mu})$ making the following diagram commutative

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \mu\ } & \widehat{X} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q_X} & \\ & X & \end{array}$$

Recall that here $\mu\colon \mathcal{X} \to \widehat{X}$ is an isomorphism of varieties and $\widetilde{\mu}\colon H \to H_X$ is an isomorphism of algebraic groups such that we always have $\mu(h{\cdot}x) = \widetilde{\mu}(h){\cdot}\mu(x)$. Note that a good quotient $q\colon \mathcal{X} \to X$ of a quasiaffine H-variety is a characteristic space if and only if we have an isomorphism of graded sheaves $\mathcal{R} \to q_*(\mathcal{O}_{\mathcal{X}})$, where $\mathcal{R}$ is a Cox sheaf on X.

THEOREM 6.4.3. *Let a quasitorus H act on a normal quasiaffine variety $\mathcal{X}$ with a good quotient $q\colon \mathcal{X} \to X$. Assume that $\Gamma(\mathcal{X}, \mathcal{O}^*) = \mathbb{K}^*$ holds, $\mathcal{X}$ is H-factorial and the H-action is strongly stable. Then X is a normal prevariety of affine intersection, $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds, $\mathrm{Cl}(X)$ is finitely generated, the Cox sheaf of X is locally of finite type, and $q\colon \mathcal{X} \to X$ is a characteristic space for X.*

The proof will be given later in this section. First we also generalize the concept of the *total coordinate space* of a prevariety X of affine intersection with finitely generated Cox ring $\mathcal{R}(X)$: this is from now on any affine H-variety isomorphic to the affine H_X-variety $\overline{X}$ of Construction 6.3.1.

COROLLARY 6.4.4. *Let Z be a normal affine variety with an action of a quasitorus H. Assume that every invertible function on Z is constant, Z is H-factorial, and there exists an open H-invariant subset $W \subseteq Z$ with $\mathrm{codim}_Z(Z \setminus W) \geq 2$ such that the H-action on W is strongly stable and admits a good quotient $q\colon W \to X$. Then Z is a total coordinate space for X.*

A first step in the proof of Theorem 6.4.3 is to describe the divisor class group of the quotient space. Let us prepare the corresponding statement. Consider an irreducible prevariety $\mathcal{X}$ with an action of a quasitorus $H = \operatorname{Spec} \mathbb{K}[M]$. For any H-invariant morphism $q\colon \mathcal{X} \to X$ to an irreducible prevariety X, we have the push forward homomorphism

$$q_*\colon \ \mathrm{WDiv}(\mathcal{X})^H \ \to \ \mathrm{WDiv}(X)$$

from the invariant Weil divisors of $\mathcal{X}$ to the Weil divisors of X sending an H-prime divisor $D \subseteq \mathcal{X}$ to the closure of its image $q(D)$ if the latter is of codimension one and to zero else. By a *homogeneous rational function* we mean an element $f \in \mathbb{K}(\mathcal{X})$ that is defined on an invariant open subset of $\mathcal{X}$ and is homogeneous there. We denote the multiplicative group of non-zero homogeneous rational functions on $\mathcal{X}$ by $E(\mathcal{X})$ and the subset of non-zero rational functions of weight $w \in M$ by $E(\mathcal{X})_w$.

PROPOSITION 6.4.5. *Let a quasitorus $H = \operatorname{Spec} \mathbb{K}[M]$ act on a normal quasiaffine variety $\mathcal{X}$ with a good quotient $q\colon \mathcal{X} \to X$. Assume that $\Gamma(\mathcal{X}, \mathcal{O}^*) = \mathbb{K}^*$ holds, $\mathcal{X}$ is H-factorial and the H-action is strongly stable. Then X is a normal prevariety of affine intersection and there is an epimorphism*

$$\delta\colon E(\mathcal{X}) \ \to\ \operatorname{WDiv}(X), \qquad f \ \mapsto\ q_*(\operatorname{div}(f)).$$

We have $\operatorname{div}(f) = q^*(q_*(\operatorname{div}(f)))$ for every $f \in E(\mathcal{X})$. Moreover, the epimorphism δ induces a well-defined isomorphism

$$M \ \to\ \operatorname{Cl}(X), \qquad w \ \mapsto\ [\delta(f)], \quad \text{with any } f \in E(\mathcal{X})_w.$$

Finally, for every $f \in E(\mathcal{X})_w$, and every open set $U \subseteq X$, we have an isomorphism of $\Gamma(U, \mathcal{O})$-modules

$$\Gamma(U, \mathcal{O}_X(\delta(f))) \ \to\ \Gamma(q^{-1}(U), \mathcal{O}_{\mathcal{X}})_w, \qquad g \ \mapsto\ f q^*(g).$$

PROOF. First of all note that the good quotient space X inherits normality and the property to be of affine intersection from the normal quasiaffine variety $\mathcal{X}$.

Let $\mathcal{X}' \subseteq \mathcal{X}$ be as in Definition 6.4.1. Then, with $X' := q(\mathcal{X}')$, we have $q^{-1}(X') = \mathcal{X}'$. Consequently, $X' \subseteq X$ is open. Moreover, $X \setminus X'$ is of codimension at least two in X, because $\mathcal{X} \setminus \mathcal{X}'$ is of codimension at least two in $\mathcal{X}$. Thus, we may assume that $X = X'$ holds, which means in particular that H acts freely. Then we have homomorphisms of groups:

$$E(\mathcal{X}) \xrightarrow{f \mapsto \operatorname{div}(f)} \operatorname{WDiv}(\mathcal{X})^H \underset{q^*}{\overset{q_*}{\rightleftarrows}} \operatorname{WDiv}(X).$$

The homomorphism from $E(\mathcal{X})$ to the group of H-invariant Weil divisors $\operatorname{WDiv}(\mathcal{X})^H$ is surjective, because $\mathcal{X}$ is H-factorial. Moreover, q^* and q_* are inverse to each other, which follows from the observation that $q\colon \mathcal{X} \to X$ is an étale H-principal bundle. This establishes the first part of the assertion.

We show that δ induces an isomorphism $M \to \operatorname{Cl}(X)$. First we have to check that $[\delta(f)]$ does not depend on the choice of f. So, let $f, g \in E(\mathcal{X})_w$. Then f/g is H-invariant, and hence defines a rational function on X. We infer well-definedness of $w \mapsto [\delta(f)]$ from

$$q_*(\operatorname{div}(f)) - q_*(\operatorname{div}(g)) \ =\ q_*(\operatorname{div}(f) - \operatorname{div}(g)) \ =\ q_*(\operatorname{div}(f/g)) \ =\ \operatorname{div}(f/g).$$

To verify injectivity, let $\delta(f) = \operatorname{div}(h)$ for some $h \in \mathbb{K}(X)^*$. Then we obtain $\operatorname{div}(f) = \operatorname{div}(q^*(h))$. Thus, $f/q^*(h)$ is an invertible homogeneous function on $\mathcal{X}$ and hence is constant. This implies $w = \deg(f/q^*(h)) = 0$. Surjectivity is clear, because $E(\mathcal{X}) \to \operatorname{WDiv}(X)$ is surjective.

We turn to the last statement. First we note that for every $g \in \Gamma(U, \mathcal{O}_X(\delta(f)))$ the function $fq^*(g)$ is regular on $q^{-1}(U)$, because we have

$$\operatorname{div}(fq^*(g)) \ =\ \operatorname{div}(f) + \operatorname{div}(q^*(g)) \ =\ q^*(\delta(f)) + \operatorname{div}(q^*(g)) \ =\ q^*(\delta(f) + \operatorname{div}(g)) \ \geq\ 0.$$

Thus, the homomorphism $\Gamma(U, \mathcal{O}_X(\delta(f))) \to \Gamma(q^{-1}(U), \mathcal{O}_{\mathcal{X}})_w$ sending g to $fq^*(g)$ is well defined. Note that $h \mapsto h/f$ defines an inverse. $\square$

COROLLARY 6.4.6. *Consider the characteristic space $q\colon \widehat{X} \to X$ obtained from a Cox sheaf $\mathcal{R}$. Then, for any non-zero $f \in \Gamma(X, \mathcal{R}_{[D]})$ the push forward $q_*(\operatorname{div}(f))$, equals the $[D]$-divisor $\operatorname{div}_{[D]}(f)$.*

PROOF. Proposition 6.4.5 shows that $q^*(q_*(\operatorname{div}(f))$ equals $\operatorname{div}(f)$ and Proposition 6.1.7 tells us that $q^*(\operatorname{div}_{[D]}(f))$ equals $\operatorname{div}(f)$ as well. $\square$

Proof of Theorem 6.4.3. Writing $H = \operatorname{Spec} \mathbb{K}[M]$ with the character group M of H, we are in the setting of Proposition 6.4.5. Choose a finitely generated subgroup $K \subseteq \operatorname{WDiv}(X)$ mapping onto $\operatorname{Cl}(X)$, and let $D_1, \ldots, D_s \in \operatorname{WDiv}(X)$ be a basis of K. By Proposition 6.4.5, we have $D_i = \delta(h_i)$ with $h_i \in E(\mathcal{X})_{w_i}$. Moreover, the isomorphism $M \to \operatorname{Cl}(X)$ given there identifies $w_i \in M$ with $[D_i] \in \operatorname{Cl}(X)$. For $D = a_1 D_1 + \ldots + a_s D_s$, we have $D = \delta(h_D)$ with $h_D = h_1^{a_1} \cdots h_s^{a_s}$.

Let $\mathcal{S}$ be the sheaf of divisorial algebras associated to K and for $D \in K$, let $w \in M$ correspond to $[D] \in \operatorname{Cl}(X)$. Then, for any open set $U \subseteq X$ and any $D \in K$, Proposition 6.4.5 provides an isomorphism of $\mathbb{K}$-vector spaces

$$\Phi_{U,D} : \Gamma(U, \mathcal{S}_D) \ \to \ \Gamma(q^{-1}(U), \mathcal{O})_w, \qquad g \ \mapsto \ q^*(g)\, h_D.$$

The $\Phi_{U,D}$ fit together to an epimorphism of graded sheaves $\Phi \colon \mathcal{S} \to q_*(\mathcal{O}_{\mathcal{X}})$. Once we know that Φ has the ideal $\mathcal{I}$ of Construction 4.2.1 as its kernel, we obtain an induced isomorphism $\mathcal{R} \to q_* \mathcal{O}_{\mathcal{X}}$, where $\mathcal{R} = \mathcal{S}/\mathcal{I}$ is the associated Cox sheaf; this shows that $\mathcal{R}$ is locally of finite type and gives an isomorphism $\mu \colon \mathcal{X} \to \widehat{X}$.

Thus we are left with showing that the kernel of Φ equals $\mathcal{I}$. Consider a $\operatorname{Cl}(X)$-homogeneous element $f \in \Gamma(U, \mathcal{S})$ of degree $[D]$, where $D \in K$. Let K^0 be the kernel of the surjection $K \to \operatorname{Cl}(X)$. Then we have

$$f \ = \ \sum_{E \in K^0} f_{D+E}, \qquad\qquad \Phi(f) \ = \ \sum_{E \in K^0} q^*(f_{D+E})\, h_{D+E}.$$

With the character $\chi \colon K^0 \to \mathbb{K}(X)^*$ defined by $q^*\chi(E) = h_E$, we may rewrite the image $\Phi(f)$ as

$$\Phi(f) \ = \ \sum_{E \in K^0} q^*(\chi(E) f_{D+E})\, h_D \ = \ q^* \left(\sum_{E \in K^0} \chi(E) f_{D+E} \right) h_D.$$

So, f lies in the kernel of Φ if and only if $\sum \chi(E) f_{D+E}$ vanishes. Now observe that we have

$$f \ = \ \sum_{E \in K^0} (1 - \chi(E))\, f_{D+E} \ + \ \sum_{E \in K^0} \chi(E)\, f_{D+E}.$$

The second summand is K-homogeneous, and thus we infer from Lemma 4.3.1 that $f \in \mathcal{I}$ holds if and only if $\sum \chi(E)\, f_{D+E} = 0$ holds. □

Remark 6.4.7. Consider the isomorphism $(\mu, \widetilde{\mu})$ identifying the characteristic spaces $q \colon \mathcal{X} \to X$ and $q_X \colon \widehat{X} \to X$ in the above proof. Then the isomorphism $\widetilde{\mu}$ identifying the quasitori H and H_X is given by the isomorphism $M \to \operatorname{Cl}(X)$ of their character groups provided by Proposition 6.4.5.

CHAPTER II

Toric varieties and Gale duality

Toric varieties form an important class of examples in Algebraic Geometry, as they admit a complete description in terms of combinatorial data, so-called lattice fans. In Section 1, we briefly recall this description and also some of the basic facts in toric geometry. Then we present Cox's construction of the characteristic space of a toric variety in terms of a defining fan and discuss the basic geometry around this. Section 2 is pure combinatorics. We introduce the notion of a "bunch of cones" and show that, in an appropriate setting, this is the Gale dual version of a fan. Under this duality, the normal fans of polytopes correspond to bunches of cones arising canonically from the chambers of the so-called Gelfand-Kapranov-Zelevinsky decomposition. In Section 3, we discuss the geometric meaning of bunches of cones: they encode the maximal separated good quotients for subgroups of the acting torus on an affine toric variety. In Section 4, we specialize these considerations to toric characteristic spaces, i.e., to the good quotients arising from Cox's construction. This leads to an alternative combinatorial description of toric varieties in terms of "lattice bunches" which turns out to be particularly suitable for phenomena around divisors.

1. Toric varieties

1.1. Toric varieties and fans. We introduce toric varieties and their morphisms and recall that this category admits a complete description in terms of lattice fans. The details can be found in any text book on toric varieties, for example [**51**], [**70**] or [**122**]. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Definition 1.1.1. A *toric variety* is a normal variety X together with an algebraic torus action $T \times X \to X$ and a base point $x_0 \in X$ such that the orbit map $T \to X$, $t \mapsto t{\cdot}x_0$ is an open embedding.

In the above setting, we refer to T as to the *acting torus* of the toric variety X. If we want to specify notation, we sometimes denote a toric variety X with acting torus T and base point x_0 as a triple (X, T, x_0).

Definition 1.1.2. Let X and X' be toric varieties. A *toric morphism* from X to X' is a pair $(\varphi, \widetilde{\varphi})$, where $\varphi\colon X \to X'$ is a morphism with $\varphi(x_0) = x'_0$ and $\widetilde{\varphi}\colon T \to T'$ is a morphism of the respective acting tori such that $\varphi(t{\cdot}x) = \widetilde{\varphi}(t){\cdot}\varphi(x)$ holds for all $t \in T$ and $x \in X$.

Note that for a toric morphism $(\varphi, \widetilde{\varphi})$, the homomorphism $\widetilde{\varphi}\colon T \to T'$ of the acting tori is uniquely determined by the morphism $\varphi\colon X \to X'$ of varieties; we will therefore often denote a toric morphism just by $\varphi\colon X \to X'$.

A first step in the combinatorial description of the category of toric varieties is to relate affine toric varieties to lattice cones. Recall that a *lattice cone* is a pair (σ, N), where N is a lattice and $\sigma \subseteq N_{\mathbb{Q}}$ a pointed, i.e. not conaining any lines, convex polyhedral cone in the rational vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ associated to N; we also refer to this setting less formally as to a cone σ in a lattice N. By a *morphism* *of lattice cones* (σ, N) and (σ', N') we mean a homomorphism $F\colon N \to N'$ with $F(\sigma) \subseteq \sigma'$. To any lattice cone we associate in a functorial way a toric variety.

Construction 1.1.3. Let N be a lattice and $\sigma \subseteq N_{\mathbb{Q}}$ a pointed cone. Set $M := \operatorname{Hom}(N, \mathbb{Z})$ and let $\sigma^\vee \subseteq M_{\mathbb{Q}}$ be the dual cone. Then we have the M-graded affine $\mathbb{K}$-algebra

$$A_\sigma \ := \ \mathbb{K}[\sigma^\vee \cap M] \ = \ \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{K}\chi^u.$$

The corresponding affine variety $X_\sigma = \operatorname{Spec} A_\sigma$ comes with an action of the torus $T_N := \operatorname{Spec} \mathbb{K}[M]$ and is a toric variety with the base point $x_0 \in X$ defined by the maximal ideal

$$\mathfrak{m}_{x_0} \ = \ \langle \chi^u - 1;\ u \in \sigma^\vee \cap M \rangle \ \subseteq \ A_\sigma.$$

Every morphism $F\colon N \to N'$ of lattice cones (σ, N) and (σ', N') induces a morphism $M' \to M$ of the dual lattice cones, hence a morphism of graded algebras from $A_{\sigma'}$ to A_σ, see Construction I. 1.1.5, and thus, finally, a toric morphism $(\varphi_F, \widetilde{\varphi}_F)$ from X_σ to $X_{\sigma'}$, see Theorem I. 2.2.4.

In order to go the other way round, i.e., from affine toric varieties to lattice cones one works with the one-parameter subgroups of the acting torus. Recall that a *one-parameter subgroup* of a torus T is a homomorphism $\lambda\colon \mathbb{K}^* \to T$. The one-parameter subgroups of a torus T form a lattice $\Lambda(T)$ with respect to pointwise multiplication. Note that we have bilinear pairing

$$\mathbb{X}(T) \times \Lambda(T) \ \to \ \mathbb{Z}, \qquad (\chi, \lambda) \ \mapsto \ \langle \chi, \lambda \rangle,$$

where $\langle \chi, \lambda \rangle \in \mathbb{Z}$ is the unique integer satisfying $\chi \circ \lambda(t) = t^{\langle \chi, \lambda \rangle}$ for all $t \in \mathbb{K}^*$. For every morphism $\varphi\colon T \to T'$ of tori, we have a functorial push forward of one-parameter subgroups:

$$\varphi_*\colon \Lambda(T) \ \to \ \Lambda(T'), \qquad \lambda \ \mapsto \ \varphi \circ \lambda.$$

Construction 1.1.4. Let X be an affine toric variety with acting torus T and base point $x_0 \in X$. We call a one-parameter subgroup $\lambda \in \Lambda(T)$ *convergent in* X if the orbit morphism $\mathbb{K}^* \to X$, $t \mapsto \lambda(t) \cdot x_0$ can be extended to a morphism $\mathbb{K} \to X$. In this case the image of $0 \in \mathbb{K}$ is denoted as

$$\lim_{t \to 0} \lambda(t) \cdot x_0 \ \in \ X.$$

The convergent one-parameter subgroups $\lambda \in \Lambda(T)$ in X generate a pointed convex cone $\sigma_X \subseteq \Lambda_{\mathbb{Q}}(T)$. Moreover, for every toric morphism $(\varphi, \widetilde{\varphi})$ from X to X', the push forward of one-parameter subgroups $\widetilde{\varphi}_*\colon \Lambda(T) \to \Lambda(T')$ is a morphism of the lattice cones $(\sigma_X, \Lambda(T))$ and $(\sigma_{X'}, \Lambda(T'))$.

Proposition 1.1.5. *We have covariant functors being essentially inverse to each other:*

$$\begin{array}{rcl}
\{\textit{lattice cones}\} & \longleftrightarrow & \{\textit{affine toric varieties}\} \\
(\sigma, N) & \mapsto & (X_\sigma, T_N, x_0), \\
F & \mapsto & (\varphi_F, \widetilde{\varphi}_F), \\
(\sigma_X, \Lambda(T)) & \mapsfrom & (X, T, x_0), \\
\widetilde{\varphi}_* & \mapsfrom & (\varphi, \widetilde{\varphi}).
\end{array}$$

We are ready to describe general toric varieties. The idea is to glue the affine descriptions. On the combinatorial side this means to consider lattice fans. We first recall this concept. A *quasifan* in a rational vector space $N_{\mathbb{Q}}$ is a finite collection Σ of convex, polyhedral cones in $N_{\mathbb{Q}}$ such that for any $\sigma \in \Sigma$ all faces $\tau \preceq \sigma$ belong to Σ, and for any two $\sigma, \sigma' \in \Sigma$ the intersection $\sigma \cap \sigma'$ is a face of both, σ and σ'. A quasifan is called a *fan* if it consists of pointed cones. A *lattice fan* is a pair (Σ, N), where N is a lattice and Σ is a fan in $N_{\mathbb{Q}}$. A *morphism of lattice fans* (Σ, N) and (Σ', N') is a homomorphism $F\colon N \to N'$ such that for every $\sigma \in \Sigma$, there is a $\sigma' \in \Sigma'$ with $F(\sigma) \subseteq \sigma'$.

Construction 1.1.6. Let (Σ, N) be a lattice fan. Then, for any two $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_{12} := \sigma_1 \cap \sigma_2$ belongs to Σ, and there is a *separating linear form*, i.e., an element $u \in M$ with

$$u_{|\sigma_1} \ \geq \ 0, \qquad u_{|\sigma_2} \ \leq \ 0, \qquad u^{\perp} \cap \sigma_1 \ = \ \sigma_{12} \ = \ u^{\perp} \cap \sigma_2.$$

The subset $X_{\sigma_{12}} \subseteq X_{\sigma_1}$ is the localization of X_{σ_1} by χ^u and $X_{\sigma_{12}} \subseteq X_{\sigma_2}$ is the localization of X_{σ_2} by χ^{-u}. This allows to glue the affine toric varieties X_σ, where $\sigma \in \Sigma$, together to a variety X_Σ. Since this gluing is equivariant and respects base points, X_Σ is a toric variety with acting torus T_N and well-defined base point x_0.

Moreover, given a morphism $F\colon N \to N'$ from a lattice fan (Σ, N) to a lattice fan (Σ', N'), fix for every $\sigma \in \Sigma$ a $\sigma' \in \Sigma'$ with $F(\sigma) \subseteq \sigma'$. Then the associated toric morphisms from X_σ to $X_{\sigma'}$ glue together to a toric morphism $(\varphi_F, \widetilde{\varphi}_F)$ from X_Σ to $X_{\Sigma'}$.

The key for the way from toric varieties to lattice fans is Sumihiro's Theorem [**150**], see also [**95**, Theorem 1], which tells us that every normal variety with torus action can be covered by invariant open affine subvarieties. Using finiteness of orbits for affine toric varieties, one concludes that any toric variety admits only finitely many invariant open affine subvarieties and is covered by them.

Construction 1.1.7. Let X be a toric variety with acting torus T. Consider the T-invariant affine open subsets $X_1, \ldots, X_r \subseteq X$ and let $\Sigma_X := \{\sigma_{X_1}, \ldots, \sigma_{X_r}\}$ be the collection of the corresponding cones of convergent one-parameter subgroups. Then $(\Sigma_X, \Lambda(T))$ is a lattice fan. Moreover, every toric morphism $(\varphi, \widetilde{\varphi})$ to a toric variety X' with acting torus T' defines a morphism $\widetilde{\varphi}_*\colon \Lambda(T) \to \Lambda(T')$ of the lattice fans $(\Sigma_X, \Lambda(T))$ and $(\Sigma_{X'}, \Lambda(T'))$.

Theorem 1.1.8. *We have covariant functors being essentially inverse to each other:*

$$\begin{array}{rcl} \{\textit{lattice fans}\} & \longleftrightarrow & \{\textit{toric varieties}\}, \\[4pt] (\Sigma, N) & \mapsto & (X_\Sigma, T_N, x_0), \\ F & \mapsto & (\varphi_F, \widetilde{\varphi}_F), \\[4pt] (\Sigma_X, \Lambda(T)), & \leftarrow\!\shortmid & (X, T, x_0) \\ \widetilde{\varphi}_* & \leftarrow\!\shortmid & (\varphi, \widetilde{\varphi}). \end{array}$$

1.2. Some toric geometry. The task of toric geometry is to describe geometric properties of a toric variety in terms of its defining fan. We recall here some of the very basic observations. Again we refer to the textbooks [**51**], [**70**] or [**122**] for details and more. The ground field is algebraically closed and of characteristic zero.

As any space with group action, also each toric variety is the disjoint union of its orbits. For an explicit description of this orbit decomposition, one introduces distinguished points as follows; for a cone σ in a rational vector space, we denote by σ° its relative interior.

Construction 1.2.1. Let X be the toric variety arising from a fan Σ in a lattice N. To every cone $\sigma \in \Sigma$, one associates a (well-defined) *distinguished point*:

$$x_\sigma \ := \ \lim_{t \to 0} \lambda_v(t) \cdot x_0 \ \in \ X, \qquad \text{where } v \in \sigma^\circ.$$

On every affine chart $X_\sigma \subseteq X$, where $\sigma \in \Sigma$, the distinguished point is the unique point with the property

$$\chi^u(x_\sigma) \quad = \quad \begin{cases} 1, & \text{where } u \in \sigma^\perp \cap M, \\ 0, & \text{where } u \in \sigma^\vee \cap M \setminus \sigma^\perp. \end{cases}$$

Note that the distinguished points are precisely the possible limits of the one-parameter subgroups of the acting torus passing through the base point. The following statement shows in particular that the distinguished points represent exactly the orbits of a toric variety.

Proposition 1.2.2 (Orbit decomposition). *Let X be the toric variety arising from a fan Σ and let T denote the acting torus of X. Then there is a bijection*

$$\Sigma \ \to \ \{T\text{-orbits of } X\}, \qquad \sigma \ \mapsto \ T \cdot x_\sigma.$$

Moreover, for any two $\sigma_1, \sigma_2 \in \Sigma$, we have $\sigma_1 \preceq \sigma_2$ if and only if $\overline{T \cdot \sigma_1} \supseteq \overline{T \cdot \sigma_2}$ holds. For the affine chart $X_\sigma \subseteq X$ defined by $\sigma \in \Sigma$, we have

$$X_\sigma \quad = \quad \bigcup_{\tau \preceq \sigma} T \cdot x_\tau.$$

Here comes the description of the structure of the toric orbit correponding to a cone of the defining fan.

Proposition 1.2.3 (Orbit structure). *Let X be the toric variety arising from a fan Σ in an n-dimensional lattice N, and denote by T its acting torus. Then, for every $\sigma \in \Sigma$, the inclusion $T_{x_\sigma} \subseteq T$ of the isotropy group is given by the projection $M \to M/(\sigma^\perp \cap M)$ of character lattices. In particular, T_{x_σ} is a torus and we have*

$$\dim(T_{x_\sigma}) \ = \ \dim(\sigma), \qquad \dim(T \cdot x_\sigma) \ = \ n - \dim(\sigma).$$

In order to describe the fibers of a toric morphism, it suffices to describe the fibers over the distinguished points. This works as follows.

Proposition 1.2.4 (Fiber formula). *Let $(\varphi, \widetilde{\varphi})$ be the toric morphism from (X, T, x_0) to (X', T', x_0') defined by a map $F \colon N \to N'$ of fans Σ and Σ' in lattices N and N' respectively. Then the fiber over a distinguished point x_τ, where $\tau \in \Sigma'$, is given by*

$$\varphi^{-1}(x_\tau) \quad = \quad \bigcup_{\substack{\sigma \in \Sigma \\ F(\sigma)^\circ \subseteq \tau^\circ}} \widetilde{\varphi}^{-1}(T'_{x_\tau}) \cdot x_\sigma.$$

We turn to singularities of toric varieties. Recall that a cone σ in a lattice N is said to be *simplicial*, if it is generated by linearly independent family $v_1, \ldots, v_r \in N$. Moreover a cone σ in a lattice N is called *regular* if it is generated by a family $v_1, \ldots, v_r \in N$ that can be completed to a lattice basis of N.

Proposition 1.2.5. *Let X be the toric variety arising from a fan Σ in a lattice N, and let $\sigma \in \Sigma$.*

(i) *The point $x_\sigma \in X$ is $\mathbb{Q}$-factorial if and only if σ is simplicial.*
(ii) *The point $x_\sigma \in X$ is smooth if and only if σ is regular.*

The next subject is completeness and projectivity. The *support* of a quasifan Σ in a vector space $N_\mathbb{Q}$ is the union $\mathrm{Supp}(\Sigma) \subseteq N_\mathbb{Q}$ of its cones. A quasifan is *complete* if its support coincides with the space $N_\mathbb{Q}$. We say that a quasifan in a vector space $N_\mathbb{Q}$ is *normal* if it is the normal quasifan $\mathcal{N}(\Delta)$ of a polyhedron $\Delta \subseteq M_\mathbb{Q}$ in the dual vector space, i.e., its cones arise from the faces of Δ via the bijection

$$\mathrm{faces}(\Delta) \ \to \ \mathcal{N}(\Delta), \qquad \Delta_0 \ \mapsto \ \{v \in N_\mathbb{Q};\ \langle u - u_0, v\rangle \geq 0 \text{ for all } u \in \Delta,\ u_0 \in \Delta_0\}.$$

Moreover, the support of $\mathcal{N}(\Delta)$ is the dual of the recession cone of Δ, i.e., the unique cone $\sigma \subseteq M_{\mathbb{Q}}$ such that $\Delta = B + \sigma$ holds with a polytope $B \subseteq M_{\mathbb{Q}}$. We say that a quasifan is *polytopal* if it is normal and complete. In other words, a polytopal quasifan in $N_{\mathbb{Q}}$ is the normal quasifan of a polytope $\Delta \subseteq M_{\mathbb{Q}}$.

PROPOSITION 1.2.6. *Let X be the toric variety arising from a fan Σ in a lattice N.*

(i) *X is complete if and only if Σ is complete.*
(ii) *X is projective if and only if Σ is polytopal.*

Now we take a look at the divisor class group $\mathrm{Cl}(X)$ of the toric variety X defined by a fan Σ in the lattice N. We assume that Σ is *non-degenerate*, i.e., the primitive lattice vectors $v_1, \ldots, v_r \in N$ of the rays of Σ generate $N_{\mathbb{Q}}$ as a vector space; this just means that on X every globally invertible function is constant. Set $F := \mathbb{Z}^r$, consider the linear map $P\colon F \to N$ sending the i-th canonical base vector $f_i \in F$ to $v_i \in N$ and the dual map $P^*\colon M \to E$. Then, with $L := \ker(P)$ and $K := E/P^*(M)$, we have the following two exact sequences of abelian groups:

$$0 \longrightarrow L \xrightarrow{\ Q^*\ } F \xrightarrow{\ P\ } N$$

$$0 \longleftarrow K \xleftarrow[\ Q\ ]{} E \xleftarrow[\ P^*\ ]{} M \longleftarrow 0$$

The lattice M represents the characters χ^u of the acting torus T of X, and each such character χ^u is a rational function on X; in fact, χ^u is regular on any affine chart X_σ with $u \in \sigma^\vee$. Moreover, the lattice E is isomorphic to the subgroup $\mathrm{WDiv}^T(X)$ of T-invariant Weil divisors via

$$E \ \to\ \mathrm{WDiv}^T(X), \qquad e \ \mapsto\ D(e) \ := \langle e, f_1\rangle D_1 + \ldots + \langle e, f_r\rangle D_r,$$

where the $D_i := \overline{T \cdot x_{\varrho_i}}$ with $\varrho_i = \mathrm{cone}(v_i)$ are the T-invariant prime divisors. Along the open toric orbit, all Weil divisors are principal and hence every Weil divisor is linearly equivalent to a T-invariant one. Thus, denoting by $\mathrm{PDiv}^T(X) \subseteq \mathrm{WDiv}^T(X)$ the subgroup of invariant principal divisors, we arrive at the following description of the divisor class group.

PROPOSITION 1.2.7. *Let X be the toric variety arising from a non-degenerate fan Σ in a lattice N. Then, in the above notation, there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longleftarrow & K & \xleftarrow{\ Q\ } & E & \xleftarrow{\ P^*\ } & M & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle e \mapsto D(e)\ \cong} & & \downarrow{\scriptstyle u \mapsto \mathrm{div}(\chi^u)\ \cong} & & \\
0 & \longleftarrow & \mathrm{Cl}(X) & \longleftarrow & \mathrm{WDiv}^T(X) & \longleftarrow & \mathrm{PDiv}^T(X) & \longleftarrow & 0
\end{array}$$

To compute intersection numbers, we first recall the basic notions from [**71**]. Let X be any n-dimensional variety. The group $Z_k(X)$ of *k-cycles* on X is the free abelian group over the set of *prime k-cycles* of X, i.e., the irreducible k-dimensional subvarieties of X. The subgroup $B_k(X) \subseteq Z_k(X)$ of *k-boundaries* is generated by the divisors $\mathrm{div}(f)$ of rational functions $f \in \mathbb{K}(Y)^*$ living on $k+1$-dimensional subvarieties $Y \subseteq X$. The *k-th Chow group* of X is the factor group $A_k(X) := Z_k(X)/B_k(X)$. There is a well-defined bilinear intersection map

$$\mathrm{Pic}(X) \times A_{k+1}(X) \ \to\ A_k(X), \qquad ([D],[Y]) \ \mapsto\ [D]\cdot[Y] := [\imath^* D],$$

where $\imath\colon Y \to X$ denotes the inclusion of a $(k+1)$-dimensional irreducible subvariety and D is a representative of $[D]$ such that Y is not contained in its support. Now suppose that X is complete and consider Cartier divisors $D_1, \ldots, D_n$ on X. Then the recursively obtained intersection $[D_1]\cdots[D_n]$ is represented by a divisor on a projective curve and has well defined degree $D_1\cdots D_n$, called the *intersection number* of $D_1,\dots,D_n$. Note that $(D_1,\dots,D_n)\mapsto D_1\cdots D_n$ is linear in every argument. If X is $\mathbb{Q}$-factorial, then one defines the intersection number of any n Weil divisors $D_1,\dots,D_n$ to be the rational number $(a_1D_1)\cdots(a_nD_n)/a_1\cdots a_n$, where $a_i\in\mathbb{Z}_{\geq 0}$ is such that a_iD_i is Cartier.

For computing intersection numbers on a $\mathbb{Q}$-factorial toric variety, one has to know the possible intersection numbers of toric prime divisors. In order to express these numbers in terms of the defining fan, we need the following notion for a cone σ in a lattice N: let $v_1,\dots,v_r$ be the primitive vectors on the rays of σ and set

$$\mu(\sigma) \quad := \quad [N\cap \mathrm{lin}_{\mathbb{Q}}(\sigma) : \mathrm{lin}_{\mathbb{Z}}(v_1,\dots,v_r)].$$

Proposition 1.2.8. *Let X be an n-dimensional complete toric variety arising from a simplicial fan Σ in a lattice N. Let $D_1,\dots,D_n$ be pairwise different invariant prime divisors on X corresponding to rays $\varrho_1,\dots,\varrho_n\in\Sigma$ and set $\sigma:=\varrho_1+\dots+\varrho_n$. Then the intersection number of $D_1,\dots,D_n$ is given as*

$$D_1\cdots D_n \quad = \quad \begin{cases}\mu(\sigma)^{-1}, & \sigma\in\Sigma,\\ 0, & \sigma\notin\Sigma.\end{cases}$$

1.3. The Cox ring of a toric variety. Roughly speaking, Cox's Theorem says that, for a toric variety X with only constant globally invertible functions, the Cox ring is given in terms of its invariant prime divisors $D_1,\dots,D_r\subseteq X$ as

$$\mathcal{R}(X)\ \cong\ \mathbb{K}[T_1,\dots,T_r], \qquad \deg(T_i)\ =\ [D_i]\ \in\ \mathrm{Cl}(X).$$

In fact, approaching this from the combinatorial side makes the statement more concrete and allows to determine the $\mathrm{Cl}(X)$-grading of the Cox ring and the characteristic space explicitly, see [**49**, Theorem 2.1] as well as [**21**, Chapter VI, 2.2] for the simplicial case, [**24**, Section 2] for the regular case and [**117**, Theorem 1] for a similar result. As before, $\mathbb{K}$ is algebraically closed of characteristic zero.

Assume that the toric variety X arises from a fan Σ in a lattice N. The condition $\Gamma(X,\mathcal{O}^*)=\mathbb{K}^*$ means that Σ is *non-degenerate*, i.e., the primitive vectors $v_1,\dots,v_r\in N$ on the rays of Σ generate $N_{\mathbb{Q}}$ as a vector space. Set $F:=\mathbb{Z}^r$ and consider the linear map $P\colon F\to N$ sending the i-th canonical base vector $f_i\in F$ to $v_i\in N$. There is a fan $\widehat{\Sigma}$ in F consisting of certain faces of the positive orthant $\delta\subseteq F_{\mathbb{Q}}$, namely

$$\widehat{\Sigma}\ :=\ \{\widehat{\sigma}\preceq\delta;\ P(\widehat{\sigma})\subseteq\sigma \text{ for some } \sigma\in\Sigma\}.$$

The fan $\widehat{\Sigma}$ defines an open toric subvariety $\widehat{X}$ of $\overline{X}\ =\ \mathrm{Spec}(\mathbb{K}[\delta^\vee\cap E])$, where $E:=\mathrm{Hom}(F,\mathbb{Z})$. Note that all rays $\mathrm{cone}(f_1),\dots,\mathrm{cone}(f_r)$ of the positive orthant $\delta\subseteq F_{\mathbb{Q}}$ belong to $\widehat{\Sigma}$ and thus we have

$$\Gamma(\widehat{X},\mathcal{O})\ =\ \Gamma(\overline{X},\mathcal{O})\ =\ \mathbb{K}[\delta^\vee\cap E].$$

As $P\colon F\to N$ is a map of the fans $\widehat{\Sigma}$ and Σ, i.e., sends cones of $\widehat{\Sigma}$ into cones of Σ, it defines a morphism $p\colon\widehat{X}\to X$ of toric varieties. Now, consider the dual map $P^*\colon M\to E$, where $M:=\mathrm{Hom}(N,\mathbb{Z})$, set $K:=E/P^*(M)$ and denote by $Q\colon E\to K$ the projection. Then, by Proposition 1.2.7, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & \mathbb{X}(H) & \longleftarrow & \mathbb{X}(\mathbb{T}) & \xleftarrow{\;p^*\;} & \mathbb{X}(T) & \longleftarrow & 0 \\
 & & \cong \big\uparrow & & {\scriptstyle e\mapsto\chi^e} \big\uparrow \cong & & {\scriptstyle u\mapsto\chi^u} \big\uparrow \cong & & \\
0 & \longleftarrow & K & \xleftarrow[Q]{} & E & \xleftarrow[P^*]{} & M & \longleftarrow & 0 \\
 & & \cong \big\downarrow & & {\scriptstyle e\mapsto D(e)} \big\downarrow \cong & & {\scriptstyle u\mapsto \mathrm{div}(\chi^u)} \big\downarrow \cong & & \\
0 & \longleftarrow & \mathrm{Cl}(X) & \longleftarrow & \mathrm{WDiv}^T(X) & \longleftarrow & \mathrm{PDiv}^T(X) & \longleftarrow & 0
\end{array}
$$

with the acting tori $T = \mathrm{Spec}(\mathbb{K}[M])$ of X and $\mathbb{T} = \mathrm{Spec}(\mathbb{K}[E])$ of $\widehat{X}$ and the characteristic quasitorus $H = \mathrm{Spec}(\mathbb{K}[K])$ of X. The map $Q\colon E \to K$ turns the polynomial ring $\mathbb{K}[E \cap \delta^\vee]$ into a K-graded algebra:

$$\mathbb{K}[E\cap\delta^\vee] \;=\; \bigoplus_{w\in K} \mathbb{K}[E\cap\delta^\vee]_w, \qquad\qquad \mathbb{K}[E\cap\delta^\vee]_w \;=\; \bigoplus_{e\in Q^{-1}(w)\cap\delta^\vee} \mathbb{K}\cdot\chi^e.$$

THEOREM 1.3.1. *In the above situation, the Cox ring of X is isomorphic to the K-graded polynomial ring $\mathbb{K}[E\cap\delta^\vee]$. Moreover, the toric morphism $p\colon \widehat{X} \to X$ is a characteristic space for X and $\overline{X}$ is a total coordinate space.*

PROOF. We follow the construction of the Cox ring performed in I. 4.2.1. The group $E = \mathrm{WDiv}^T(X)$ of invariant Weil divisors projects onto $K = \mathrm{Cl}(X)$ and the kernel of this projection is

$$E^0 \;=\; P^*(M) \;=\; \mathrm{PDiv}^T(X).$$

Let $\mathcal{S}$ be the divisorial sheaf associated to E. Consider $e \in E$, a cone $\sigma \in \Sigma$ and let $\widehat{\sigma} \in \widehat{\Sigma}$ be the cone with $P(\widehat{\sigma}) = \sigma$. Then we have

$$\begin{aligned}
\Gamma(X_\sigma, \mathcal{S}_e) &= \mathrm{lin}_{\mathbb{K}}\left(\chi^u;\ u\in M,\ \mathrm{div}(\chi^u) + D(e) \ge 0 \text{ on } X_\sigma\right) \\
&= \mathrm{lin}_{\mathbb{K}}\left(\chi^u;\ u\in M,\ P^*(u) + e \in \widehat{\sigma}^\vee\right).
\end{aligned}$$

Using this identity, we define an epimorphism $\pi\colon \Gamma(X_\sigma,\mathcal{S}) \to \Gamma(\widehat{X}_{\widehat{\sigma}},\mathcal{O})$ of K-graded algebras by

$$\Gamma(X_\sigma,\mathcal{S}_e) \;\ni\; \chi^u \;\;\mapsto\;\; \chi^{P^*(u)+e} \;\in\; \Gamma(\widehat{X}_{\widehat{\sigma}},\mathcal{O})_{Q(e)}.$$

For $u \in M$, look at $\chi^u \in \Gamma(X_\sigma, \mathcal{S}_{-P^*(u)})$. Then $\pi(1-\chi^u) = 0$ holds. Moreover, for $\chi^{u'} \in \Gamma(X_\sigma,\mathcal{S}_{e'})$ and $e\in E$ with $e'-e = P^*(u)$ we have

$$\chi^{u'} \;=\; \chi^{u'}\chi^u + \chi^{u'}(1-\chi^u), \qquad\qquad \chi^{u'}\chi^u \in \Gamma(X_\sigma,\mathcal{S}_e).$$

It follows that $\ker(\pi)$ equals the ideal associated to the character $E^0 \to \mathbb{K}(X)^*$, $P^*(u) \mapsto \chi^u$, see I. 4.2.1, and we obtain a K-graded isomorphism

$$\Gamma(X_\sigma,\mathcal{R}) \;\;\to\;\; \Gamma(\widehat{X}_{\widehat{\sigma}},\mathcal{O}).$$

This shows that the K-graded sheaves $\mathcal{R}$ and $p_*\mathcal{O}_{\widehat{X}}$ are isomorphic. Consequently $p\colon \widehat{X} \to X$ is a characteristic space for X and the Cox ring of X is

$$\mathcal{R}(X) \;=\; \Gamma(\widehat{X},\mathcal{O}) \;=\; \mathbb{K}[E\cap\delta^\vee].$$

□

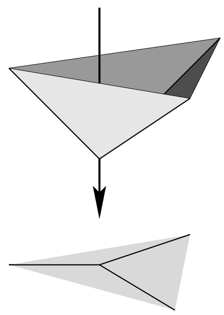
EXAMPLE 1.3.2. Consider the projective plane $X = \mathbb{P}_2$. Its total coordinate space is $\overline{X} = \mathbb{K}^3$, we have $\widehat{X} = \mathbb{K}^3 \setminus \{0\}$, the characteristic torus $H = \mathbb{K}^*$ acts by scalar multiplication, and the quotient map is

$$p\colon \widehat{X} \;\to\; X, \qquad (z_0,z_1,z_2) \;\mapsto\; [z_0,z_1,z_2].$$

In terms of fans, the situation is the following. The fan Σ of X lives in $N = \mathbb{Z}^2$. With $v_0 = (-1,-1)$ and $v_1 = (1,0)$ and $v_2 = (0,1)$ its maximal cones are

$$\operatorname{cone}(v_1,v_2), \qquad \operatorname{cone}(v_2,v_0), \qquad \operatorname{cone}(v_0,v_1).$$

Thus, we have $F = \mathbb{Z}^3$ and the map $P\colon F \to N$ sends the canonical basis vector f_i to v_i, where $i = 0,1,2$. The fan $\widehat{\Sigma}$ of $\widehat{X}$ has the facets of the orthant $\operatorname{cone}(f_0,f_1,f_2)$ as its maximal cones.



In general, the total coordinate space $\overline{X} = \operatorname{Spec} \mathbb{K}[E \cap \delta^\vee]$ is isomorphic to $\mathbb{K}^r$. More precisely, if $e_1,\dots,e_r$ denote the primitive generators of $\delta^\vee$, then a concrete isomorphism $\overline{X} \to \mathbb{K}^r$ is given by the comorphism

$$\mathbb{K}[T_1,\dots,T_r] \ \to \ \mathbb{K}[E \cap \delta^\vee], \qquad\qquad T_i \ \mapsto \ \chi^{e_i}.$$

We want to describe the *irrelevant ideal* of X in the Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_r]$ in terms of the defining fan Σ; recall from Definition I. 6.3.2 that the irrelevant ideal $\mathcal{J}_{\mathrm{irr}}(X)$ is the vanishing ideal of $\overline{X} \setminus \widehat{X}$ in $\Gamma(\overline{X},\mathcal{O}) = \mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_r]$. For any quasifan Σ, we denote by $\Sigma^{\max}$ the set of its maximal cones.

Proposition 1.3.3. *Let X be the toric variety arising from a non-degenerate fan Σ in a lattice N and let $v_1,\dots,v_r$ be the primitive generators of Σ. For every $\sigma \in \Sigma$ define a vector*

$$\nu(\sigma) \ := \ (\varepsilon_1,\dots,\varepsilon_r) \ \in \ \mathbb{Z}^r_{\ge 0} \qquad\qquad \varepsilon_i \ := \ \begin{cases} 1 & v_i \notin \sigma, \\ 0 & v_i \in \sigma. \end{cases}$$

Then the irrelevant ideal $\mathcal{J}_{\mathrm{irr}}(X)$ in the Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1,\dots,T_r]$ is generated by the monomials $T^{\nu(\sigma)} = T_1^{\varepsilon_1} \cdots T_r^{\varepsilon_r}$, where $\sigma \in \Sigma^{\max}$.

Proof. The toric variety $\widehat{X}$ is the union of the affine charts $\widehat{X}_{\widehat{\sigma}}$, where $\sigma \in \Sigma^{\max}$. The complement of $\widehat{X}_{\widehat{\sigma}}$ in $\overline{X}$ is the zero set of $T^{\nu(\sigma)}$. Observing that the monomials $T^{\nu(\sigma)}$, where $\sigma \in \Sigma^{\max}$, generate a radical ideal gives the assertion. □

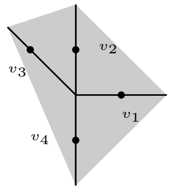
Now we give an explicit description of the complement $\overline{X} \setminus \widehat{X}$ as an arrangement of coordinate subspaces.

Proposition 1.3.4. *Let X be the toric variety arising from a non-degenerate fan Σ in a lattice N, let $v_1,\dots,v_r$ be the primitive generators of Σ and let $\mathcal{G}(\Sigma)$ be the collection of subsets $I \subseteq \{1,\dots,r\}$ which are minimal with the property that the vectors v_i, $i \in I$, are not contained in a common cone of Σ. Then one has*

$$\overline{X} \setminus \widehat{X} \quad = \quad \bigcup_{I \in \mathcal{G}(\Sigma)} V(\mathbb{K}^r\,;\, T_i,\, i \in I).$$

Proof. Take a point $z \in \mathbb{K}^r$ and set $I_z := \{i\,;\, z_i = 0\}$. By Proposition 1.3.3, the point z is not contained in $\widehat{X}$ if and only if the vectors v_i, $i \in I_z$, are not contained in a cone of Σ. This means that $I \subseteq I_z$ for some element I of $\mathcal{G}(\Sigma)$, or that the point z is in $V(\mathbb{K}^r\,;\, T_i, i \in I)$. □

Example 1.3.5. The first Hirzebruch surface is the toric variety X arising from the fan Σ in $N = \mathbb{Z}^2$ given as follows



$$\begin{array}{rclrcl} v_1 & = & (1,0), & v_2 & = & (0,1), \\ v_3 & = & (-1,1), & v_4 & = & (0,-1). \end{array}$$

Thus, we have $F = \mathbb{Z}^4$ and, with respect to the canonical bases, the maps $P\colon F \to N$ and $Q\colon E \to K$ are given by the matrices

$$P \;=\; \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}, \qquad Q \;=\; \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

The total coordinate space of X is $\overline{X} = \mathbb{K}^4$ and, according to Proposition 1.3.4, the open subset $\widehat{X} \subseteq \overline{X}$ is obtained from $\overline{X}$ by removing

$$V(\mathbb{K}^4; T_1, T_3) \;\cup\; V(\mathbb{K}^4; T_2, T_4).$$

This describes the characteristic space $p\colon \widehat{X} \to X$ over X; the action of the characteristic torus $H = \mathbb{K}^* \times \mathbb{K}^*$ on $\widehat{X}$ is given by

$$h \cdot z \;=\; (h_2 z_1, h_1 z_2, h_2 z_3, h_1 h_2 z_4).$$

1.4. Geometry of Cox's construction. We give a self-contained discussion of basic geometric properties of Cox's quotient presentation of toric varieties, i.e., without using general results on characteristic spaces. We begin with an observation on lattices needed also later.

Let F and E be mutually dual lattices, consider an epimorphism $Q\colon E \to K$ onto an abelian group K and a pair of exact sequences

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\;P\;} N$$

$$0 \longleftarrow K \xleftarrow[Q]{} E \xleftarrow[P^*]{} M \longleftarrow 0$$

Lemma 1.4.1. *Let $F^0_{\mathbb{Q}} \subseteq F_{\mathbb{Q}}$ be a vector subspace and let $E^0_{\mathbb{Q}} \subseteq E_{\mathbb{Q}}$ be the annihilating space of $F^0_{\mathbb{Q}}$. Then one has an isomorphism of abelian groups*

$$K/Q(E^0_{\mathbb{Q}} \cap E) \;\;\cong\;\; (L \cap F^0_{\mathbb{Q}}) \;\oplus\; (P(F^0_{\mathbb{Q}}) \cap N)/P(F^0_{\mathbb{Q}} \cap F).$$

Proof. Set $F^0 := F^0_{\mathbb{Q}} \cap F$ and $E^0 := E^0_{\mathbb{Q}} \cap E$. Then E/E^0 is the dual lattice of F_0. Moreover, M/M^0 is the dual lattice of $N^0 := P(F^0_{\mathbb{Q}}) \cap N$, where $M^0 \subseteq M$ is the inverse image of $E^0 \subseteq E$ under P^*. These lattices fit into the exact sequences

$$0 \longrightarrow L^0 \longrightarrow F^0 \xrightarrow{\;P\;} N^0$$

$$0 \longleftarrow K/Q(E^0) \xleftarrow[Q]{} E/E^0 \xleftarrow[P^*]{} M/M^0 \longleftarrow 0$$

where we set $L^0 := L \cap F^0$. In other words, $K/Q(E^0)$ is isomorphic to the cokernel of $M/M^0 \to E/E^0$. The latter is the direct sum of the cokernel and the kernel of the dual map $F^0 \to N^0$. $\square$

We now discuss actions of subgroups of the acting torus of an affine toric variety. Let $\delta \subseteq F_{\mathbb{Q}}$ be any pointed cone, and consider the associated affine toric variety X_δ. Then $Q\colon E \to K$ defines a K-grading of the algebra of regular functions

$$\Gamma(X_\delta, \mathcal{O}) \;=\; \mathbb{K}[E \cap \delta^\vee] \;=\; \bigoplus_{w \in K} \mathbb{K}[E \cap \delta^\vee]_w, \qquad \mathbb{K}[E \cap \delta^\vee]_w \;:=\; \bigoplus_{e \in Q^{-1}(w) \cap \delta^\vee} \mathbb{K} \cdot \chi^e.$$

Thus, the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ acts on X_δ. Note that H acts on X_δ as a subgroup of the acting torus T of X_δ, where the embedding $H \to T$ is given by the map $Q\colon E \to K$ of the character groups.

PROPOSITION 1.4.2. *The inclusion $H_{x_\delta} \subseteq H$ of the isotropy group of $x_\delta \in X_\delta$ is given by the projection $K \to K_\delta$ of character lattices, where*

$$K_\delta \;:=\; K/Q(\delta^\perp \cap E) \;\cong\; (L \cap \operatorname{lin}_{\mathbb{Q}}(\delta)) \oplus (P(\operatorname{lin}_{\mathbb{Q}}(\delta)) \cap N)/P(\operatorname{lin}_{\mathbb{Q}}(\delta) \cap F).$$

PROOF. Use the characterization of the distinguished point $x_\delta \in X_\delta$ given in Construction 1.2.1 to see that $Q(\delta^\perp \cap E)$ is the orbit group of x_δ. Thus, Proposition I. 2.2.8 tells us that $K \to K_\delta$ gives the inclusion $H_{x_\delta} \subseteq H$. The alternative description of K_δ is obtained by applying Lemma 1.4.1 to $F^0_{\mathbb{Q}} := \operatorname{lin}_{\mathbb{Q}}(\delta)$. □

In order to determine a good quotient for the action of H on X_δ, consider the image $P(\delta)$ in $N_{\mathbb{Q}}$. This cone need not be pointed; we denote by $\tau \preceq P(\delta)$ its minimal face. Then, with $N_1 := N/(\tau \cap N)$, we have the projection $P_1\colon F \to N_1$ and $\delta_1 := P_1(\delta)$ is pointed.

PROPOSITION 1.4.3. *In the above notation, the toric morphism $p_1\colon X_\delta \to X_{\delta_1}$ given by $P_1\colon F \to N_1$ is a good quotient for the action of H on X_δ.*

PROOF. The algebra of H-invariant regular functions on X_δ is embedded into the algebra of functions as

$$\Gamma(X_\delta, \mathcal{O})^H \;=\; \mathbb{K}[\delta^\vee \cap M] \;\subseteq\; \mathbb{K}[\delta^\vee \cap E] \;=\; \Gamma(X_\delta, \mathcal{O}).$$

The sublattice M_1 generated by $\delta^\vee \cap M$ is the dual lattice of N_1 and $\delta_1 = P_1(\delta)$ in N_1 is the dual cone of $\delta^\vee \cap (M_1)_{\mathbb{Q}}$ in M_1. □

We turn to Cox's construction. So, in the above setting, we have $F = \mathbb{Z}^r$ and the map $P\colon F \to N$ sends the canonical basis vectors $f_1, \ldots, f_r$ to the primitve generators $v_1, \ldots, v_r \in N$ of the rays of a non-degenerate fan Σ in N. Moreover, we have the othant $\delta = \operatorname{cone}(f_1, \ldots, f_r)$ and the fan $\widehat{\Sigma}$ in F defined by

$$\widehat{\Sigma} \;:=\; \{\widehat{\sigma} \preceq \delta;\; P(\widehat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

As before, we denote by $\widehat{X}$ the open toric subvariety of $\overline{X} = \operatorname{Spec}(\mathbb{K}[\delta^\vee \cap E])$ defined by the fan $\widehat{\Sigma}$. Moreover, $p\colon \widehat{X} \to X$ is the toric morphism defined by $P\colon F \to N$, and we set $H := \operatorname{Spec} \mathbb{K}[K]$. Then H acts on $\overline{X}$ and leaves $\widehat{X}$ invariant. Finally, let $\widehat{\varrho}_i := \operatorname{cone}(f_i)$, where $1 \le i \le r$, denote the rays of $\widehat{\Sigma}$ and set

$$\widehat{W} \;:=\; \overline{X}_{\widehat{\varrho}_1} \cup \ldots \cup \overline{X}_{\widehat{\varrho}_r} \;\subseteq\; \widehat{X}.$$

PROPOSITION 1.4.4. *The toric morphism $p\colon \widehat{X} \to X$ is the good quotient for the action of H on $\widehat{X}$. Moreover, H acts freely on the open subset $\widehat{W} \subseteq \widehat{X}$ and every H-orbit on $\widehat{W}$ is closed in $\widehat{X}$.*

PROOF. We first show that $p\colon \widehat{X} \to X$ is affine. For $\sigma \in \Sigma$, let $\widehat{\sigma} \in \widehat{\Sigma}$ be the (unique) cone with $P(\widehat{\sigma}) = \sigma$. Then we have $p^{-1}(X_\sigma) = \widehat{X}_{\widehat{\sigma}}$ and Proposition 1.4.3 tells us that $p\colon \widehat{X} \to X$ is a good quotient for the H-action. Using Proposition 1.4.2 we see that H acts with trivial isotropy groups on $\widehat{W}$. To see the last assertion, note that the subset $\widehat{W}$ coincides with $p^{-1}(X')$, where $X' = X_{\varrho_1} \cup \ldots \cup X_{\varrho_r} \subseteq X$. □

Note that combining Proposition 1.4.4 with Theorem I. 6.4.3 shows that $p\colon \widehat{X} \to X$ is a characteristic space for X and hence provides another proof of Theorem 1.3.1. We conclude with two observations on the geometry of the H-action and the quotient $p\colon \widehat{X} \to X$; these two statements may as well be obtained as a consequence of Corollary 6.2.4.

PROPOSITION 1.4.5. *For the action of H on X and the quotient $p\colon \widehat{X} \to X$, the following statements are equivalent.*

(i) *The fan Σ is simplicial.*
(ii) *One has $\dim(\widehat{\sigma}) = \dim(P(\widehat{\sigma}))$ for every $\widehat{\sigma} \in \widehat{\Sigma}$.*
(iii) *Any H-orbit in $\widehat{X}$ has at most finite isotropy group.*
(iv) *The quotient $p\colon \widehat{X} \to X$ is geometric.*
(v) *The variety X is $\mathbb{Q}$-factorial.*

PROOF. Statements (i) and (ii) are obviously equivalent. The equivalence of (ii) and (iii) is clear by Proposition 1.4.2. The equivalence of (ii) and (iv) follows from the Fiber Formula 1.2.4. Finally, (i) and (v) are equivalent by standard toric geometry, see Proposition 1.2.5. □

PROPOSITION 1.4.6. *For the action of H on X and the quotient $p\colon \widehat{X} \to X$, the following statements are equivalent.*

(i) *The fan Σ is regular.*
(ii) *$P\colon F \cap \operatorname{lin}_{\mathbb{Q}}(\widehat{\sigma}) \to N \cap \operatorname{lin}_{\mathbb{Q}}(P(\widehat{\sigma}))$ is an isomorphism for every $\widehat{\sigma} \in \widehat{\Sigma}$.*
(iii) *The action of H on $\widehat{X}$ is free.*
(iv) *The variety X is smooth.*

PROOF. Statements (i) and (ii) are obviously equivalent. The equivalence of (ii) and (iii) is clear by Proposition 1.4.2. Finally, (i) and (iv) are equivalent by standard toric geometry, see Proposition 1.2.5. □

2. Linear Gale duality

2.1. Fans and bunches of cones. We introduce the concept of a bunch of cones and state in Theorem 2.1.14 that, in an appropriate setting, this is the Gale dual version of the concept of a fan; the proof is given in Subsection 2.3. Our presentation is in the spirit of [**124**]; in particular, we make no use of the language of oriented matroids. A general reference is [**58**, Chapter 4]. In [**65**, Section 1] some historical aspects are discussed.

DEFINITION 2.1.1. We say that a vector configuration $\mathcal{V} = (v_1, \ldots, v_r)$ in a rational vector space $N_{\mathbb{Q}}$ and a vector configuration $\mathcal{W} = (w_1, \ldots, w_r)$ in a rational vector space $K_{\mathbb{Q}}$ are *Gale dual* to each other if the following holds:

(i) we have $v_1 \otimes w_1 + \ldots + v_r \otimes w_r = 0$ in $N_{\mathbb{Q}} \otimes K_{\mathbb{Q}}$,
(ii) for any rational vector space U and any vectors $u_1, \ldots, u_r \in U$ with $v_1 \otimes u_1 + \ldots + v_r \otimes u_r = 0$ in $N_{\mathbb{Q}} \otimes U$, there is a unique linear map $\psi\colon K_{\mathbb{Q}} \to U$ with $\psi(w_i) = u_i$ for $i = 1, \ldots, r$,
(iii) for any rational vector space U and any vectors $u_1, \ldots, u_r \in U$ with $u_1 \otimes w_1 + \ldots + u_r \otimes w_r = 0$ in $U \otimes K_{\mathbb{Q}}$, there is a unique linear map $\varphi\colon N_{\mathbb{Q}} \to U$ with $\varphi(v_i) = u_i$ for $i = 1, \ldots, r$.

If we fix the first configuration in a Gale dual pair, then the second one is, by the properties of Gale duality, uniquely determined up to isomorphism, and therefore is also called the *Gale transform* of the first one. The following characterization of Gale dual pairs is also used as a definition.

Remark 2.1.2. Consider vector configurations $\mathcal{V} = (v_1, \ldots, v_r)$ and $\mathcal{W} = (w_1, \ldots, w_r)$ in rational vector spaces $N_\mathbb{Q}$ and $K_\mathbb{Q}$ respectively, and let $M_\mathbb{Q}$ be the dual vector space of $N_\mathbb{Q}$. Then Gale duality of $\mathcal{V}$ and $\mathcal{W}$ is characterized by the following property: for any tuple $(a_1, \ldots, a_r) \in \mathbb{Q}^r$ one has

$$a_1 w_1 + \ldots + a_r w_r \ = \ 0 \quad \iff \quad u(v_i) = a_i \text{ for } i = 1, \ldots, r \text{ with some } u \in M_\mathbb{Q}.$$

The following construction shows existence of Gale dual pairs, and, up to isomorphy, produces any pair of Gale dual vector configurations.

Construction 2.1.3. Consider a pair of mutually dual exact sequences of finite dimensional rational vector spaces

$$0 \longrightarrow L_\mathbb{Q} \longrightarrow F_\mathbb{Q} \xrightarrow{P} N_\mathbb{Q} \longrightarrow 0$$

$$0 \longleftarrow K_\mathbb{Q} \xleftarrow[Q]{} E_\mathbb{Q} \longleftarrow M_\mathbb{Q} \longleftarrow 0$$

Let $(f_1, \ldots, f_r)$ be a basis for $F_\mathbb{Q}$, let $(e_1, \ldots, e_r)$ be the dual basis for $E_\mathbb{Q}$ and denote the image vectors by

$$v_i \ := \ P(f_i) \ \in \ N_\mathbb{Q}, \qquad w_i \ := \ Q(e_i) \ \in \ K_\mathbb{Q}, \qquad 1 \le i \le r.$$

Then the vector configurations $\mathcal{V} = (v_1, \ldots, v_r)$ in $N_\mathbb{Q}$ and $\mathcal{W} = (w_1, \ldots, w_r)$ in $K_\mathbb{Q}$ are Gale dual to each other.

Remark 2.1.4. Let $r = n + k$ with integers $n, k \in \mathbb{Z}_{>0}$. Consider matrices $P \in \mathrm{Mat}(n, r; \mathbb{Q})$ and $Q \in \mathrm{Mat}(k, r; \mathbb{Q})$ such that the rows of Q form a basis for the nullspace of P. Then the columns $(v_1, \ldots, v_r)$ of P in $\mathbb{Q}^n$ and the columns $(w_1, \ldots, w_r)$ of Q in $\mathbb{Q}^k$ are Gale dual vector configurations. Note that after fixing one of the matrices, Gale duality determines the other up to multiplicitation by an invertible matrix from the left.

Example 2.1.5. The columns of the following matrices P and Q are Gale dual vector configurations

$$P \ = \ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}, \qquad\qquad Q \ = \ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Now we turn to fans and bunches of cones. We work with convex polyhedral cones generated by elements of a fixed vector configuration. The precise notion is the following.

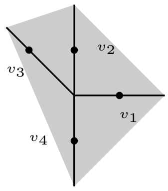
Definition 2.1.6. Let $N_\mathbb{Q}$ be a rational vector space, and $\mathcal{V} = (v_1, \ldots, v_r)$ a family of vectors in $N_\mathbb{Q}$ generating $N_\mathbb{Q}$. A *$\mathcal{V}$-cone* is a convex polyhedral cone generated by some of the $v_1, \ldots, v_r$. The set of all $\mathcal{V}$-cones is denoted by $\Omega(\mathcal{V})$.

Definition 2.1.7. Let $N_\mathbb{Q}$ be a rational vector space, and $\mathcal{V} = (v_1, \ldots, v_r)$ a family of vectors in $N_\mathbb{Q}$ generating $N_\mathbb{Q}$. A *$\mathcal{V}$-quasifan* is a quasifan in $N_\mathbb{Q}$ consisting of $\mathcal{V}$-cones, i.e., a non-empty set $\Sigma \subseteq \Omega(\mathcal{V})$ such that

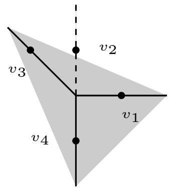
(i) for all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both, σ_1 and σ_2,
(ii) for every $\sigma \in \Sigma$, also all faces $\sigma_0 \preceq \sigma$ belong to Σ.

A *$\mathcal{V}$-fan* is a $\mathcal{V}$-quasifan consisting of pointed cones. We say that a $\mathcal{V}$-(quasi)fan is *maximal* if it cannot be enlarged by adding $\mathcal{V}$-cones. Moreover, we call a $\mathcal{V}$-(quasi)fan *true* if it contains all rays $\mathrm{cone}(v_i)$, where $1 \le i \le r$.

Example 2.1.8. Consider $N_{\mathbb{Q}} = \mathbb{Q}^2$ and let $\mathcal{V} = (v_1, \dots, v_4)$ be the family consisting of the columns of the matrix P given in Example 2.1.5.



A true maximal $\mathcal{V}$-fan Σ_1



A maximal $\mathcal{V}$-fan Σ_2

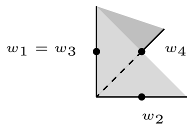
Definition 2.1.9. Let $N_{\mathbb{Q}}$ be a rational vector space, and $\mathcal{V} = (v_1, \dots, v_r)$ a family of vectors in $N_{\mathbb{Q}}$ generating $N_{\mathbb{Q}}$. We say that a $\mathcal{V}$-quasifan $\Sigma_1 \subseteq \Omega(\mathcal{V})$ *refines* a $\mathcal{V}$-quasifan $\Sigma_2 \subseteq \Omega(\mathcal{V})$, written $\Sigma_1 \le \Sigma_2$, if for every $\sigma_1 \in \Sigma_1$ there is a $\sigma_2 \in \Sigma_2$ with $\sigma_1 \subseteq \sigma_2$.

Definition 2.1.10. Let $K_{\mathbb{Q}}$ be a rational vector space, and let $\mathcal{W} = (w_1, \dots, w_r)$ a family of vectors in $K_{\mathbb{Q}}$ generating $K_{\mathbb{Q}}$. A $\mathcal{W}$*-bunch* is a non-empty set $\Theta \subseteq \Omega(\mathcal{W})$ such that

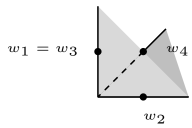
(i) for all $\tau_1, \tau_2 \in \Theta$, one has $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$,
(ii) for every $\tau \in \Theta$, all $\mathcal{W}$-cones τ_0 with $\tau^\circ \subseteq \tau_0^\circ$ belong to Θ.

We say that a $\mathcal{W}$-bunch is *maximal* if it cannot be enlarged by adding $\mathcal{W}$-cones. Moreover, we call a $\mathcal{W}$-bunch Θ *true* if every cone $\vartheta_i = \operatorname{cone}(w_j;\ j \neq i)$, where $1 \le i \le r$, belongs to Θ.

Example 2.1.11. Consider $K_{\mathbb{Q}} = \mathbb{Q}^2$ and let $\mathcal{W} = (w_1, \dots, w_4)$ be the family consisting of the columns of the matrix Q given in Example 2.1.5.



A true maximal $\mathcal{W}$-bunch Θ_1



A maximal $\mathcal{W}$-bunch Θ_2

Definition 2.1.12. Let $K_{\mathbb{Q}}$ be a rational vector space, and $\mathcal{W} = (w_1, \dots, w_r)$ a family of vectors in $K_{\mathbb{Q}}$ generating $K_{\mathbb{Q}}$. We say that a $\mathcal{W}$-bunch $\Theta_1 \subseteq \Omega(\mathcal{W})$ *refines* a $\mathcal{W}$-bunch $\Theta_2 \subseteq \Omega(\mathcal{W})$, written $\Theta_1 \le \Theta_2$, if for every $\tau_2 \in \Theta_2$ there is a $\tau_1 \in \Theta_1$ with $\tau_1 \subseteq \tau_2$.

Definition 2.1.13. Let $\mathcal{V} = (v_1, \dots, v_r)$ in $N_{\mathbb{Q}}$ and $\mathcal{W} = (w_1, \dots, w_r)$ in $K_{\mathbb{Q}}$ be Gale dual vector configurations. Set $\mathfrak{R} := \{1, \dots, r\}$. Then for any collection Σ of $\mathcal{V}$-cones and any collection Θ of $\mathcal{W}$-cones we set

$$\begin{aligned} \Sigma^\sharp &:= \{\operatorname{cone}(w_j;\ j \in \mathfrak{R} \setminus I);\ I \subseteq \mathfrak{R},\ \operatorname{cone}(v_i;\ i \in I) \in \Sigma\}, \\ \Theta^\sharp &:= \{\operatorname{cone}(v_i;\ i \in \mathfrak{R} \setminus J);\ J \subseteq \mathfrak{R},\ \operatorname{cone}(w_j;\ j \in J) \in \Theta\}. \end{aligned}$$

Theorem 2.1.14. *Let $\mathcal{V} = (v_1, \dots, v_r)$ in $N_{\mathbb{Q}}$ and $\mathcal{W} = (w_1, \dots, w_r)$ in $K_{\mathbb{Q}}$ be Gale dual vector configurations. Then we have an order reversing map*

$$\{\mathcal{W}\text{-bunches}\} \ \to \ \{\mathcal{V}\text{-quasifans}\}, \qquad \Theta \ \mapsto \Theta^\sharp.$$

Now assume that $v_1, \dots, v_r$ generate pairwise different one-dimensional cones. Then there are mutually inverse order reversing bijections

$$\begin{aligned} \{\text{true maximal } \mathcal{W}\text{-bunches}\} \quad &\longleftrightarrow \quad \{\text{true maximal } \mathcal{V}\text{-fans}\}, \\ \Theta \quad &\mapsto \quad \Theta^\sharp, \\ \Sigma^\sharp \quad &\leftarrow\!\shortmid \quad \Sigma. \end{aligned}$$

Under these bijections, the simplicial true maximal $\mathcal{V}$-fans correspond to the true maximal $\mathcal{W}$-bunches consisting of full dimensional cones.

EXAMPLE 2.1.15. For the true maximal $\mathcal{W}$-bunch Θ_1 and the true maximal $\mathcal{V}$-fan Σ_1 presented in Examples 2.1.11 and 2.1.8 one has

$$\Sigma_1 \;=\; \Theta_1^\sharp, \qquad\qquad \Theta_1 \;=\; \Sigma_1^\sharp.$$

Moreover, for the maximal $\mathcal{W}$-bunch Θ_2 and the maximal $\mathcal{V}$-fan Σ_2 presented there, we have $\Sigma_2 = \Theta_2^\sharp$ but $\Sigma_2^\sharp$ is not even a $\mathcal{W}$-bunch as it contains $\operatorname{cone}(w_4)$ and $\operatorname{cone}(w_2, w_4)$ contradicting 2.1.10 (i).

2.2. The GKZ-decomposition. Given a vector configuration $\mathcal{V}$, the normal ones among the possible $\mathcal{V}$-fans are obtained from the chambers of the so-called Gelfand-Kapranov-Zelevinsky decomposition of the dual vector configuration $\mathcal{W}$. We make this precise and study it in detail; the proofs are given in Subsection 2.4. A different treatment is given in [**58**, Chapter 5].

CONSTRUCTION 2.2.1. Let $K_{\mathbb{Q}}$ be a rational vector space and $\mathcal{W} = (w_1, \ldots, w_r)$ a family of vectors in $K_{\mathbb{Q}}$ generating $K_{\mathbb{Q}}$. For every $w \in \operatorname{cone}(\mathcal{W})$, we define its *chamber* to be

$$\lambda(w) \;:=\; \bigcap_{\substack{\tau \in \Omega(\mathcal{W}) \\ w \in \tau}} \tau \;=\; \bigcap_{\substack{\tau \in \Omega(\mathcal{W}) \\ w \in \tau^\circ}} \tau.$$

The *Gelfand-Kapranov-Zelevinsky decomposition (GKZ-decomposition)* associated to $\mathcal{W}$ is the collection of all these chambers:

$$\Lambda(\mathcal{W}) \;\;:=\;\; \{\lambda(w);\; w \in \operatorname{cone}(\mathcal{W})\}.$$

Note that for every $w \in \operatorname{cone}(\mathcal{W})$ one has $w \in \lambda(w)^\circ$. Moreover, for any $\lambda \in \Lambda(\mathcal{W})$ and $w \in \lambda^\circ$ one has $\lambda = \lambda(w)$. To every chamber $\lambda = \lambda(w)$, we associate a $\mathcal{W}$-bunch

$$\Theta(\lambda) \;\;:=\;\; \{\tau \in \Omega(\mathcal{W});\; w \in \tau^\circ\}.$$

THEOREM 2.2.2. *Let $\mathcal{V} = (v_1, \ldots, v_r)$ in $N_{\mathbb{Q}}$ and $\mathcal{W} = (w_1, \ldots, w_r)$ in $K_{\mathbb{Q}}$ be Gale dual vector configurations. Then $\Lambda(\mathcal{W})$ is a fan in $K_{\mathbb{Q}}$ with support* $\operatorname{cone}(\mathcal{W})$. *Moreover, the following statements hold.*

(i) *For every chamber $\lambda \in \Lambda(\mathcal{W})$, the associated $\mathcal{W}$-bunch $\Theta(\lambda)$ is maximal and $\Sigma(\lambda) := \Theta(\lambda)^\sharp$ is a normal maximal $\mathcal{V}$-quasifan.*
(ii) *Situation as in Construction 2.1.3 and set $\gamma := \operatorname{cone}(e_1, \ldots, e_r)$. Then the $\mathcal{V}$-quasifan $\Sigma(\lambda)$ associated to $\lambda \in \Lambda(\mathcal{W})$ is the normal quasifan of any polyhedron $B_w \subseteq M_{\mathbb{Q}}$ obtained as follows:*

$$B_w \;:=\; (Q^{-1}(w) \cap \gamma) - e, \qquad w \in \lambda^\circ,\; e \in Q^{-1}(w).$$

(iii) *Let $\lambda_1, \lambda_2 \in \Lambda(\mathcal{W})$. Then $\lambda_1 \preceq \lambda_2$ is equivalent to $\Theta(\lambda_1) \le \Theta(\lambda_2)$. In particular, if λ_1 is a face of λ_2 then $\Sigma(\lambda_2)$ refines $\Sigma(\lambda_1)$.*
(iv) *If Σ is a normal maximal $\mathcal{V}$-quasifan, then $\Sigma = \Sigma(\lambda)$ holds with some chamber $\lambda \in \Lambda(\mathcal{W})$.*

Recall that, given quasifans $\Sigma_1, \ldots, \Sigma_n$ in a rational vector space $N_{\mathbb{Q}}$ such that all Σ_i have the same support, the *coarsest common refinement* of $\Sigma_1, \ldots, \Sigma_n$ is the quasifan

$$\Sigma \;\;:=\;\; \{\sigma_1 \cap \ldots \cap \sigma_n;\; \sigma_1 \in \Sigma_1, \ldots, \sigma_n \in \Sigma_n\}.$$

If each Σ_i is the normal quasifan of a polyhedron $B_i \subseteq M_{\mathbb{Q}}$ in the dual vector space, then the coarsest common refinement of $\Sigma_1, \ldots, \Sigma_n$ is the normal fan of the Minkowski sum $B_1 + \ldots + B_n$.

THEOREM 2.2.3. *Let $\mathcal{V} = (v_1, \ldots, v_r)$ in $N_{\mathbb{Q}}$ and $\mathcal{W} = (w_1, \ldots, w_r)$ in $K_{\mathbb{Q}}$ be Gale dual vector configurations.*

(i) *The GKZ-decomposition $\Lambda(\mathcal{V})$ is the coarsest common refinement of all quasifans $\Sigma(\lambda)$, where $\lambda \in \Lambda(\mathcal{W})$.*

(ii) *The GKZ-decomposition* $\Lambda(\mathcal{V})$ *is the coarsest common refinement of all* $\mathcal{V}$*-quasifans having* $\operatorname{cone}(\mathcal{V})$ *as their support.*
(iii) *In the setting of Theorem 2.2.2 (iii), fix for each chamber* $\lambda \in \Lambda(\mathcal{W})$ *an element* $w(\lambda) \in \lambda^\circ$*. Then the Minkowski sum over the* $B_{w(\lambda)}$ *has* $\Lambda(\mathcal{V})$ *as its normal fan.*

Corollary 2.2.4. *Every complete quasifan in a rational vector space admits a refinement by a polytopal fan.*

Definition 2.2.5. Let $K_\mathbb{Q}$ be a rational vector space and $\mathcal{W} = (w_1, \dots, w_r)$ a family of vectors in $K_\mathbb{Q}$ generating $K_\mathbb{Q}$. The *moving cone* of $\mathcal{W}$ is

$$\operatorname{Mov}(\mathcal{W}) \ := \ \bigcap_{i=1}^{r} \operatorname{cone}(w_j;\ j \neq i) \ \subseteq \ K_\mathbb{Q}.$$

Theorem 2.2.6. *Let* $\mathcal{V} = (v_1, \dots, v_r)$ *in* $N_\mathbb{Q}$ *and* $\mathcal{W} = (w_1, \dots, w_r)$ *in* $K_\mathbb{Q}$ *be Gale dual vector configurations.*

(i) *The cone* $\operatorname{Mov}(\mathcal{W})$ *is of full dimension in* $K_\mathbb{Q}$ *if and only if* $v_1, \dots, v_r$ *generate pairwise different one-dimensional cones in* $N_\mathbb{Q}$*.*
(ii) *Assume that* $\operatorname{Mov}(\mathcal{W})$ *is of full dimension in* $K_\mathbb{Q}$*. Then the quasifan* $\Sigma(\lambda)$ *associated to* $\lambda \in \Lambda(\mathcal{W})$ *is a fan if and only if* $\lambda^\circ \subseteq \operatorname{cone}(\mathcal{W})^\circ$ *holds.*
(iii) *Assume that* $\operatorname{Mov}(\mathcal{W})$ *is of full dimension in* $K_\mathbb{Q}$*. Then we have mutually inverse order preserving bijections*

$$\begin{array}{rcl} \{\lambda \in \Lambda(\mathcal{W});\ \lambda^\circ \subseteq \operatorname{Mov}(\mathcal{W})^\circ\} & \longleftrightarrow & \{\textit{true normal } \mathcal{V}\textit{-fans}\}, \\ \lambda & \mapsto & \Sigma(\lambda), \\ \bigcap_{\tau \in \Sigma^\sharp} \tau & \mapsfrom & \Sigma. \end{array}$$

Under these bijections, the simplicial true normal fans correspond to the full dimensional chambers.

As an immediate consequence, one obtains the following, see [**124**, Corollary 3.8], [**127**, Theorem 4.1] and also [**147**, Theorem 8.3].

Corollary 2.2.7. *Assume that* $v_1, \dots, v_r \in N_\mathbb{Q}$ *generate pairwise different one-dimensional cones. There exist normal true* $\mathcal{V}$*-fans and every such fan admits a refinement by a simplicial normal true* $\mathcal{V}$*-fan.*

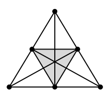
Example 2.2.8. Consider the Gale dual vector configurations $\mathcal{V} = (v_1, \dots, v_6)$ in $N_\mathbb{Q} = \mathbb{Q}^3$ and $\mathcal{W} = (w_1, \dots, w_6)$ in $K_\mathbb{Q} = \mathbb{Q}^3$ given by

$$v_1 = (-1,0,0), \quad v_2 = (0,-1,0), \quad v_3 = (0,0,-1),$$
$$v_4 = (0,1,1), \quad v_5 = (1,0,1), \quad v_6 = (1,1,0)$$

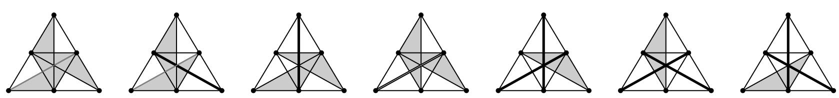
and

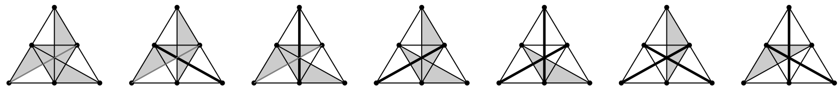
$$w_1 = (1,0,0), \quad w_2 = (0,1,0), \quad w_3 = (0,0,1),$$
$$w_4 = (1,1,0), \quad w_5 = (1,0,1), \quad w_6 = (0,1,1).$$

Then there are 13 chambers $\lambda \in \Lambda(\mathcal{W})$ with $\lambda^\circ \subseteq \operatorname{Mov}(\mathcal{W})^\circ$. These give rise to 13 polytopal true maximal $\mathcal{V}$-fans.



Moreover, one finds 14 true maximal $\mathcal{W}$-bunches not arising from a chamber. Thus we have in total 27 maximal $\mathcal{V}$-fans 14 of them being non-polytopal.





2.3. Proof of Theorem 2.1.14. We work in the setting of Construction 2.1.3. That means that we have a pair of mutually dual exact sequences of finite dimensional rational vector spaces

$$0 \longrightarrow L_{\mathbb{Q}} \longrightarrow F_{\mathbb{Q}} \xrightarrow{\ P\ } N_{\mathbb{Q}} \longrightarrow 0$$

$$0 \longleftarrow K_{\mathbb{Q}} \xleftarrow[\ Q\ ]{} E_{\mathbb{Q}} \longleftarrow M_{\mathbb{Q}} \longleftarrow 0$$

The idea is to decompose the $\sharp$-operation between collections in $K_{\mathbb{Q}}$ and collections in $N_{\mathbb{Q}}$ according to the following scheme of further operations on collections of cones:

$$\begin{array}{ccc} \{\text{collections in } E_{\mathbb{Q}}\} & \xleftrightarrow{\ *\ } & \{\text{collections in } F_{\mathbb{Q}}\} \\ {\scriptstyle Q^{\uparrow}}\big\uparrow\big\downarrow{\scriptstyle Q^{\downarrow}} & & {\scriptstyle P^{\downarrow}}\big\downarrow\big\uparrow{\scriptstyle P^{\uparrow}} \\ \{\text{collections in } K_{\mathbb{Q}}\} & \xleftrightarrow[\ \sharp\ ]{} & \{\text{collections in } N_{\mathbb{Q}}\} \end{array}$$

We enter the detailed discussion. Let $(f_1, \ldots, f_r)$ be a basis for $F_{\mathbb{Q}}$, and let $(e_1, \ldots, e_r)$ be the dual basis for $E_{\mathbb{Q}}$. Then we have the image vectors

$$v_i \ := \ P(f_i) \ \in \ N_{\mathbb{Q}}, \qquad w_i \ := \ Q(e_i) \ \in \ K_{\mathbb{Q}}, \qquad 1 \le i \le r.$$

The vector configurations $\mathcal{V} = (v_1, \ldots, v_r)$ in $N_{\mathbb{Q}}$ and $\mathcal{W} = (w_1, \ldots, w_r)$ in $K_{\mathbb{Q}}$ are Gale dual to each other. Moreover, we set

$$\delta \ := \ \operatorname{cone}(f_1, \ldots, f_r), \qquad \gamma \ := \ \operatorname{cone}(e_1, \ldots, e_r).$$

Then these cones are dual to each other, and we have the face correspondence, i.e., mutually inverse bijections

$$\begin{array}{rcl} \operatorname{faces}(\delta) & \longleftrightarrow & \operatorname{faces}(\gamma), \\ \delta_0 & \mapsto & \delta_0^* := \delta_0^{\perp} \cap \gamma, \\ \gamma_0^{\perp} \cap \delta =: \gamma_0^* & \mapsfrom & \gamma_0. \end{array}$$

Definition 2.3.1. By an *$L_{\mathbb{Q}}$-invariant separating linear form* for two faces $\delta_1, \delta_2 \preceq \delta$, we mean an element $e \in E_{\mathbb{Q}}$ such that

$$e_{|L_{\mathbb{Q}}} \ = \ 0, \qquad e_{|\delta_1} \ \ge \ 0, \qquad e_{|\delta_2} \ \le \ 0, \qquad \delta_1 \cap e^{\perp} \ = \ \delta_1 \cap \delta_2 \ = \ e^{\perp} \cap \delta_2.$$

Lemma 2.3.2 (Invariant Separation Lemma). *Consider two faces $\delta_1, \delta_2 \preceq \delta$ and the correponding faces $\gamma_i := \delta_i^* \preceq \gamma$. Then the following statements are equivalent.*

(i) *There is an $L_{\mathbb{Q}}$-invariant separating linear form for δ_1 and δ_2.*
(ii) *The relative interiors $Q(\gamma_i)^{\circ}$ satisfy $Q(\gamma_1)^{\circ} \cap Q(\gamma_2)^{\circ} \neq \emptyset$.*

Proof. Let $\delta_1 = \operatorname{cone}(f_i;\ i \in I)$ and $\delta_2 = \operatorname{cone}(f_j;\ j \in J)$ with subsets I and J of $\mathfrak{R} := \{1, \ldots, r\}$. Condition (i) is equivalent to existence of a linear form $u \in M_{\mathbb{Q}}$ with

$$u(v_i) > 0 \text{ for } i \in I \setminus J, \qquad u(v_k) = 0 \text{ for } k \in I \cap J, \qquad u(v_j) < 0 \text{ for } j \in J \setminus I.$$

According to Remark 2.1.2 such a linear form $u \in M_{\mathbb{Q}}$ exists if and only if there are $a_1, \ldots, a_r \in \mathbb{Q}$ with $a_1 w_1 + \ldots + a_r w_r = 0$ and

$$a_i > 0 \text{ for } i \in I \setminus J, \qquad a_k = 0 \text{ for } k \in I \cap J, \qquad a_j < 0 \text{ for } j \in J \setminus I.$$

This in turn is possible if and only if we find coefficients $b_m > 0$ for $m \in \mathfrak{R} \setminus I$ and $c_n > 0$ for $n \in \mathfrak{R} \setminus J$ such that $\sum_{m \in \mathfrak{R} \setminus I} b_m w_m$ equals $\sum_{n \in \mathfrak{R} \setminus J} c_n w_n$. The latter is Condition (ii). $\square$

DEFINITION 2.3.3. By a *δ-collection* we mean a collection of faces of δ. We say that a δ-collection $\mathfrak{A}$ is

(i) *separated* if any two $\delta_1, \delta_2 \in \mathfrak{A}$ admit an $L_{\mathbb{Q}}$-invariant separating linear form,
(ii) *saturated* if for any $\delta_1 \in \mathfrak{A}$ and any $\delta_2 \preceq \delta_1$, which is $L_{\mathbb{Q}}$-invariantly separable from δ_1, one has $\delta_2 \in \mathfrak{A}$,
(iii) *true* if all rays of δ belong to $\mathfrak{A}$,
(iv) *maximal* if it is maximal among the separated δ-collections.

DEFINITION 2.3.4. By a *γ-collection* we mean a collection of faces of γ. We say that a γ-collection $\mathfrak{B}$ is

(i) *connected* if for any two $\gamma_1, \gamma_2 \in \mathfrak{B}$ the intersection $Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ$ is not empty,
(ii) *saturated* if for any $\gamma_1 \in \mathfrak{B}$ and any $\gamma \succeq \gamma_2 \succeq \gamma_1$ with $Q(\gamma_1)^\circ \subseteq Q(\gamma_2)^\circ$, one has $\gamma_2 \in \mathfrak{B}$,
(iii) *true* if all facets of γ belong to $\mathfrak{B}$,
(iv) *maximal* if it is maximal among the connected γ-collections.

PROPOSITION 2.3.5. *We have mutually inverse bijections sending separated (saturated, true, maximal) collections to connected (saturated, true, maximal) collections:*

$$\begin{array}{rcl} \{\textit{separated } \delta\textit{-collections}\} & \longleftrightarrow & \{\textit{connected } \gamma\textit{-collections}\}, \\ \mathfrak{A} & \mapsto & \mathfrak{A}^* := \{\delta_0^*;\ \delta_0 \in \mathfrak{A}\}, \\ \mathfrak{B}^* := \{\gamma_0^*;\ \gamma_0 \in \mathfrak{B}\} & \mapsfrom & \mathfrak{B}. \end{array}$$

PROOF. The assertion is an immediate consequence of the Invariant Separation Lemma. □

DEFINITION 2.3.6. By a *$\mathcal{W}$-collection* we mean a set of $\mathcal{W}$-cones. We say that a $\mathcal{W}$-collection Θ is

(i) *connected* if for any two $\tau_1, \tau_2 \in \Theta$ we have $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$,
(ii) *saturated* if for any $\tau \in \Theta$ also all $\mathcal{W}$-cones σ with $\tau^\circ \subseteq \sigma^\circ$ belong to Θ,
(iii) *true* if every $\vartheta_i = \operatorname{cone}(w_j;\ j \neq i)$, where $1 \le i \le r$, belongs to Θ,
(iv) *maximal* if it is maximal among all connected $\mathcal{W}$-collections.

DEFINITION 2.3.7. Consider the set of all $\mathcal{W}$-collections and the set of all γ-collections. We define the *Q-lift* and the *Q-drop* to be the maps

$$\begin{array}{rcl} Q^\uparrow \colon \{\mathcal{W}\text{-collections}\} & \to & \{\gamma\text{-collections}\}, \\ \Theta & \mapsto & Q^\uparrow\Theta := \{\gamma_0 \preceq \gamma;\ Q(\gamma_0) \in \Theta\}, \\[1ex] Q_\downarrow \colon \{\gamma\text{-collections}\} & \to & \{\mathcal{W}\text{-collections}\}, \\ \mathfrak{B} & \mapsto & Q_\downarrow\mathfrak{B} := \{Q(\gamma_0);\ \gamma_0 \in \mathfrak{B}\}. \end{array}$$

PROPOSITION 2.3.8. *The Q-lift is injective and sends connected (saturated, true, maximal) $\mathcal{W}$-collections to Q-connected (saturated, true, maximal) γ-collections. Moreover, we have mutually inverse bijections sending true collections to true collections:*

$$\begin{array}{rcl} \{\textit{maximal } \mathcal{W}\textit{-collections}\} & \longleftrightarrow & \{\textit{maximal } \gamma\textit{-collections}\}, \\ \Theta & \mapsto & Q^\uparrow\Theta, \\ Q_\downarrow\mathfrak{B} & \mapsfrom & \mathfrak{B}. \end{array}$$

PROOF. By definition of the Q-lift and the Q-drop, we have $Q_\downarrow Q^\uparrow\Theta = \Theta$ for every $\mathcal{W}$-collection Θ. In particular, $Q^\uparrow$ is injective. Moreover, $Q^\uparrow$ clearly preserves the properties connected, saturated, true and maximal. If $\mathfrak{B}$ is a maximal γ-collection, then $Q_{\downarrow}\mathfrak{B}$ is a maximal $\mathcal{W}$-collection and we have $Q^{\uparrow}Q_{\downarrow}\mathfrak{B} = \mathfrak{B}$. Thus, restricted to maximal collections, $Q^{\uparrow}$ and $Q_{\downarrow}$ are mutually inverse bijections. Obviously, $Q_{\downarrow}$ sends true collections to true collections. □

Definition 2.3.9. By a *$\mathcal{V}$-collection* we mean a set of $\mathcal{V}$-cones. We say that a $\mathcal{V}$-collection Σ is

(i) *separated* if any two $\sigma_1, \sigma_2 \in \Sigma$ intersect in a common face,
(ii) *saturated* if for any $\sigma \in \Sigma$ also all faces $\sigma_0 \preceq \sigma$ belong to Σ,
(iii) *true* if every ray $\varrho_i = \mathrm{cone}(v_i)$, where $1 \le i \le r$, belongs to Σ,
(iv) *maximal* if it is maximal among all separated $\mathcal{V}$-collections.

Definition 2.3.10. Consider the set of all $\mathcal{V}$-collections and the set of all δ-collections. We define the *P-lift* and the *P-drop* to be the maps

$$\begin{array}{rcl} P^{\uparrow}\colon \{\mathcal{V}\text{-collections}\} & \to & \{\delta\text{-collections}\}, \\ \Sigma & \mapsto & P^{\uparrow}\Sigma := \{\delta_0 \preceq \delta;\ P(\delta_0) \in \Sigma\}, \end{array}$$

$$\begin{array}{rcl} P_{\downarrow}\colon \{\delta\text{-collections}\} & \to & \{\mathcal{V}\text{-collections}\}, \\ \mathfrak{A} & \mapsto & P_{\downarrow}\mathfrak{A} := \{P(\delta_0);\ \delta_0 \in \mathfrak{A}\}. \end{array}$$

Proposition 2.3.11. *The P-drop is surjective and sends separated (saturated, true, maximal) collections to separated (saturated, true, maximal) collections. If $v_1, \ldots, v_r$ generate pairwise different rays, then we have mutually inverse bijections sending saturated (maximal) collections to saturated (maximal) collections:*

$$\begin{array}{rcl} \{\textit{true separated } \delta\textit{-collections}\} & \longleftrightarrow & \{\textit{true separated } \mathcal{V}\textit{-collections}\}, \\ \mathfrak{A} & \mapsto & P_{\downarrow}\mathfrak{A}, \\ P^{\uparrow}\Sigma & \leftarrow\!\shortmid & \Sigma. \end{array}$$

Proof. For every $\mathcal{V}$-collection Σ we have $\Sigma = P_{\downarrow}P^{\uparrow}\Sigma$. In particular, $P_{\downarrow}$ is surjective. The fact that $P_{\downarrow}$ preserves separatedness and saturatedness follows from the observation that an $L_{\mathbb{Q}}$-invariant separating linear form for two faces $\delta_1, \delta_2 \preceq \delta$ induces a separating linear form for the images $P(\delta_1)$ and $P(\delta_2)$. The fact that $P_{\downarrow}$ preserves the properties true and maximal is obvious.

Now assume that $v_1, \ldots, v_r$ generate pairwise different rays. Consider a true separated $\mathcal{V}$-collection Σ. Then, for every $\sigma \in \Sigma$ and every $1 \le i \le r$, we have $v_i \in \sigma$ if and only if $\mathbb{Q}_{\ge 0} \cdot v_i$ is an extremal ray of σ. Consequently, for every $\sigma \in \Sigma$ there is a unique $\delta_0 \preceq \delta$ with $P(\delta_0) = \sigma$. It follows that $P^{\uparrow}(\Sigma)$ is true and separated, and, if Σ is saturated (maximal), then also $P^{\uparrow}(\Sigma)$ saturated (maximal). Moreover, we conclude that $P_{\downarrow}$ restricted to the true separated collections is injective. □

Proof of Theorem 2.1.14. First observe that the (true, maximal) $\mathcal{W}$-bunches are precisely the (true, maximal) connected saturated $\mathcal{W}$-collections and the (true, maximal) $\mathcal{V}$-quasifans are precisely the (true, maximal) separated saturated $\mathcal{V}$-collections. Next observe that we have

$$\Theta^{\sharp} \ = \ P_{\downarrow}((Q^{\uparrow}\Theta)^{*}), \qquad\qquad \Sigma^{\sharp} \ = \ Q_{\downarrow}((P^{\uparrow}\Sigma)^{*}).$$

Then Propositions 2.3.8, 2.3.5 and 2.3.11 provide the statements made on $\Theta \mapsto \Theta^{\sharp}$ and $\Sigma \mapsto \Sigma^{\sharp}$; the fact that these assigments are order-reversing is obvious.

We still have to show that simplicial fans correspond to bunches of full-dimensional cones. A $\mathcal{V}$-fan Σ is simplicial exactly when for every $\mathrm{cone}(v_i\,;\, i \in I)$ in Σ and every subset $I_1 \subseteq I$ the $\mathrm{cone}(v_i\,;\, i \in I_1)$ is in Σ. For true Σ this means that for every $\tau = \mathrm{cone}(w_j\,;\, j \in J)$ in the corresponding bunch Θ and every J_1, $J \subseteq J_1 \subseteq \mathfrak{R}$, one has $\tau^{\circ} \subseteq \mathrm{cone}(w_j\,;\, j \in J_1)^{\circ}$. Since the vectors w_j generate $K_{\mathbb{Q}}$, this is exactly the case when all cones of Θ are of full dimension. □

2.4. Proof of Theorems 2.2.2, 2.2.3 and 2.2.6. The setup is as in the preceding subsection, that means that we have the pair of mutually dual exact sequences

$$0 \longrightarrow L_{\mathbb{Q}} \longrightarrow F_{\mathbb{Q}} \xrightarrow{P} N_{\mathbb{Q}} \longrightarrow 0$$

$$0 \longleftarrow K_{\mathbb{Q}} \xleftarrow{Q} E_{\mathbb{Q}} \longleftarrow M_{\mathbb{Q}} \longleftarrow 0$$

fix a basis $(f_1, \ldots, f_r)$ for $F_{\mathbb{Q}}$ and denote by $(e_1, \ldots, e_r)$ the dual basis for $E_{\mathbb{Q}}$. Then the image vectors

$$v_i := P(f_i) \in N_{\mathbb{Q}}, \qquad w_i := Q(e_i) \in K_{\mathbb{Q}}, \qquad 1 \le i \le r,$$

give Gale dual vector configurations $\mathcal{V} = (v_1, \ldots, v_r)$ in $N_{\mathbb{Q}}$ and $\mathcal{W} = (w_1, \ldots, w_r)$ in $K_{\mathbb{Q}}$. Again, the positive orthants in $F_{\mathbb{Q}}$ and $E_{\mathbb{Q}}$ are denoted by

$$\delta := \mathrm{cone}(f_1, \ldots, f_r), \qquad \gamma := \mathrm{cone}(e_1, \ldots, e_r).$$

We split the proofs of the Theorems into several Propositions. The first two settle Theorem 2.2.2 (i) and (ii).

Proposition 2.4.1. *The collection $\Sigma(\lambda(w))$ equals the normal quasifan $\mathcal{N}(B_w)$. In particular, $\Sigma(\lambda(w))$ is a normal maximal $\mathcal{V}$-quasifan with support* $\mathrm{cone}(\mathcal{V})$.

Proof. The cones of the normal quasifan $\mathcal{N}(B_w)$ correspond to the faces of the polyhedron of $B_w = (Q^{-1}(w) \cap \gamma) - e$, where $e \in E_{\mathbb{Q}}$ is any element with $Q(e) = w$ as follows: given $B_{w,0} \preceq B_w$, the corresponding cone of $\mathcal{N}(B_w)$ is

$$\sigma_0 := \{v \in N_{\mathbb{Q}};\ \langle u - u_0, v\rangle \ge 0,\ u \in B_w,\ u_0 \in B_{w,0}\} \subseteq N_{\mathbb{Q}}.$$

Now, let $\gamma_0 \preceq \gamma$ denote the minimal face with $B_{w,0} + e \subseteq \gamma_0$. Then we have $\gamma_0^\circ \cap Q^{-1}(w) \neq \emptyset$. Thus, γ_0 belongs to the Q-lift of $\Theta(\lambda(w))$. For the corresponding face $\delta_0 = \gamma_0^\perp \cap \delta$ of $\delta = \gamma^\vee$, one directly verifies

$$P(\delta_0) = \sigma_0.$$

Thus, the first assertion follows from the observation that the assignment $B_{w,0} \mapsto \gamma_0$ defines a bijection from the faces of B_w to the Q-lift of $\Theta(\lambda(w))$, and the fact that $\Sigma(\lambda)$ equals $P_\downarrow((Q^\uparrow\Theta(\lambda(w)))^*)$. The rest follows from

$$\mathrm{Supp}(\mathcal{N}(B_w)) = (P^*(M_{\mathbb{Q}}) \cap \gamma)^\vee = \mathrm{cone}(\mathcal{V}).$$

□

Proposition 2.4.2. *For every chamber $\lambda \in \Lambda(\mathcal{W})$, the associated $\mathcal{W}$-bunch $\Theta(\lambda)$ is maximal.*

Proof. Suppose that $\Theta(\lambda)$ is a proper subset of a maximal $\mathcal{W}$-bunch Θ. For the associated saturated separated δ-collections this means that $(Q^\uparrow\Theta(\lambda))^*$ is a proper subset of $(Q^\uparrow\Theta)^*$. Since P induces a bijection $(Q^\uparrow\Theta)^* \to P_\downarrow((Q^\uparrow\Theta)^*)$ we obtain that $\Theta(\lambda)^\sharp$ is a proper subset of $\Theta^\sharp$. This contradicts the fact that, due to Proposition 2.4.1, the quasifan $\Theta(\lambda)^\sharp$ has $\mathrm{cone}(\mathcal{V})$ as its support. □

Now we will see that the GKZ-decomposition $\Lambda(\mathcal{W})$ is in fact a fan. We prove this for the GKZ-decomposition $\Lambda(\mathcal{V})$ of the Gale dual vector configuration and moreover verify the assertions made in Theorem 2.2.3.

Proposition 2.4.3. *The GKZ-decomposition $\Lambda(\mathcal{V})$ is a fan. Moreover, the following statements hold.*

(i) *The GKZ-decomposition $\Lambda(\mathcal{V})$ is the coarsest common refinement of all quasifans $\Sigma(\lambda)$, where $\lambda \in \Lambda(\mathcal{W})$.*

(ii) *The GKZ-decomposition* $\Lambda(\mathcal{V})$ *is the coarsest common refinement of all* $\mathcal{V}$*-quasifans having* $\mathrm{cone}(\mathcal{V})$ *as its support.*
(iii) *Fix for each chamber* $\lambda \in \Lambda(\mathcal{W})$ *an element* $w(\lambda) \in \lambda^\circ$. *Then the Minkowski sum over the* $B_{w(\lambda)}$ *has* $\Lambda(\mathcal{V})$ *as its normal fan.*

Lemma 2.4.4. *For every* $\mathcal{V}$*-cone* $\sigma \subseteq N_{\mathbb{Q}}$*, there exists a chamber* $\lambda \in \Lambda(\mathcal{W})$ *with* $\sigma \in \Sigma(\lambda)$.

Proof. Write $\sigma = \mathrm{cone}(v_i\,;\, i \in I)$, and choose $w \in \mathrm{cone}(w_j\,;\, j \notin I)^\circ$. Then the associated quasifan $\Sigma(\lambda(w)) = \Theta(\lambda(w))^\sharp$ contains σ. □

Proof of Proposition 2.4.3. Lemma 2.4.4 tells us that every $\mathcal{V}$-cone is contained in a normal $\mathcal{V}$-quasifan $\Sigma(\lambda) = \mathcal{N}(B_{w(\lambda)})$, where $\lambda \in \Lambda(\mathcal{W})$ and $w \in \lambda^\circ$ is fixed. It follows that $\Lambda(\mathcal{V})$ is the coarsest common refinement of the $\Sigma(\lambda)$. Since every ray $\mathrm{cone}(v_j) \subseteq N_{\mathbb{Q}}$ belongs to some $\Sigma(\lambda)$, we obtain that $\Lambda(\mathcal{V})$ is even a fan. Moreover, also the second assertion follows from the fact that every $\mathcal{V}$-cone is contained in some $\Sigma(\lambda)$. Finally, we obtained that $\Sigma(\lambda)$ is the normal fan of the Minkowski sum over the $B_{w(\lambda)}$. □

Proposition 2.4.5. *For any two* $\lambda_1, \lambda_2 \in \Lambda(\mathcal{W})$ *we have* $\lambda_1 \preceq \lambda_2$ *if and only if* $\Theta(\lambda_1) \le \Theta(\lambda_2)$ *holds. In particular,* $\lambda_1 \preceq \lambda_2$ *implies* $\Sigma(\lambda_2) \le \Sigma(\lambda_1)$.

Proof. Suppose that $\lambda_1 \preceq \lambda_2$ holds. Then, given any $\tau_2 \in \Theta(\lambda_2)$, we have $\lambda_1 \subseteq \tau_2$ and thus find a face $\tau_1 \preceq \tau_2$ with $\lambda_1^\circ \subseteq \tau_1^\circ$. Consequently $\tau_1 \in \Theta(\lambda_1)$ and $\tau_1 \subseteq \tau_2$ holds. Next suppose that $\Theta(\lambda_1) \le \Theta(\lambda_2)$ holds. For every $\tau_2 \in \Theta(\lambda_2)$, fix a $\tau_1 \in \Theta(\lambda_1)$ with $\tau_1 \subseteq \tau_2$. This yields $\lambda_1 \subseteq \tau_2$ for all $\tau_2 \in \Theta(\lambda_2)$. We conclude $\lambda_1 \subseteq \lambda_2$ and, since $\Lambda(\mathcal{W})$ is a fan, $\lambda_1 \preceq \lambda_2$. The supplement is then clear because we have $\Sigma(\lambda_i) = \Theta(\lambda_i)^\sharp$ and $\Theta \to \Theta^\sharp$ is order reversing. □

Proposition 2.4.6. *Every normal* $\mathcal{V}$*-quasifan is of the form* $\Sigma(\lambda)$ *with a chamber* $\lambda \in \Lambda(\mathcal{W})$.

Proof. Suppose that a $\mathcal{V}$-quasifan Σ is the normal quasifan of a polyhedron $B \subseteq M_{\mathbb{Q}}$. Then the polyhedron B is given by inequalities $\langle u, v_i\rangle \ge c_i$, where $v_i \in \mathcal{V}$ and $c_i \in \mathbb{Q}$. Note that each v_i is the restriction of a coordinate function on $E_{\mathbb{Q}}$ to the subspace $M_{\mathbb{Q}}$. This shows that B may be obtained as the intersection of γ with the parallel translate $e + M_{\mathbb{Q}}$ of the subspace $M_{\mathbb{Q}}$, where $e = -c_1e_1 - \ldots - c_re_r \in E_{\mathbb{Q}}$. Thus $B = B_w$, where $w = Q(e)$, and Proposition 2.4.1 completes the proof. □

We obtained all the assertions of Theorems 2.2.2 and 2.2.3 and now turn to the proof of Theorem 2.2.6. Recall that we set $\vartheta_i = \mathrm{cone}(w_j;\ j \neq i)$ for $1 \le i \le r$.

Proposition 2.4.7. *Assume that* $v_1, \ldots, v_r$ *generate pairwise different one-dimensional cones. Then, for* $\lambda \in \Lambda(\mathcal{W})$*, the quasifan* $\Sigma(\lambda)$ *is a fan if and only if* $\lambda^\circ \subseteq \mathrm{cone}(\mathcal{W})^\circ$ *holds.*

Proof. Take $w \in \lambda^\circ$. Then the normal quasifan $\mathcal{N}(B_w)$ is a fan if and only if the polyhedron $B_w \subseteq M_{\mathbb{Q}}$ is of full dimension. The latter holds if and only if $Q^{-1}(w)$ intersects γ_0° for a face $\gamma_0 \preceq \gamma$ with $M_{\mathbb{Q}} \subseteq \mathrm{lin}(\gamma_0)$. Since all v_i are nonzero, the latter is equivalent to $Q^{-1}(w) \cap \gamma^\circ \neq \emptyset$. This in turn means precisely $w \in \mathrm{cone}(\mathcal{W})^\circ$. □

Proposition 2.4.8. *Assume that* $v_1, \ldots, v_r$ *generate pairwise different one-dimensional cones. Then there exists a true normal* $\mathcal{V}$*-fan* Σ.

Proof. Rescale each v_i to $v_i' = c_iv_i$, where $c_i \in \mathbb{Q}_{>0}$, such that every v_i' is a vertex of the convex hull over $0, v_1', \ldots, v_r'$. Then $\Sigma := \mathcal{N}(B)$ is as wanted for

$$B \ := \ \{u \in M_{\mathbb{Q}};\ \langle u, v_i\rangle \ge -1\} \ \subseteq \ M_{\mathbb{Q}}.$$

□

Proposition 2.4.9. *The cone* $\mathrm{Mov}(\mathcal{W})$ *is of full dimension in* $K_{\mathbb{Q}}$ *if and only if* $v_1, \ldots, v_r$ *generate pairwise different one-dimensional cones. If one of these statements holds, then we have*

$$\mathrm{Mov}(\mathcal{W})^\circ \quad = \quad \vartheta_1^\circ \cap \ldots \cap \vartheta_r^\circ.$$

Proof. First suppose that $v_1, \ldots, v_r$ generate pairwise different one-dimensional cones. Then Proposition 2.4.8 provides us with a true normal $\mathcal{V}$-fan Σ and Proposition 2.4.6 guarantees $\Sigma = \Sigma(\lambda)$ with a chamber $\lambda \in \Lambda(\mathcal{W})$. Take any $w \in \lambda^\circ$. Since each $\mathrm{cone}(v_i)$ belongs to $\Sigma = \Theta(\lambda)^\sharp$, we obtain that each ϑ_i° contains w. Since every v_i is non-zero, the ϑ_i are of full dimension in $K_{\mathbb{Q}}$ and thus $\mathrm{Mov}(\mathcal{W})$ is so.

Conversely, if $\mathrm{Mov}(\mathcal{W})$ is of full dimension in $K_{\mathbb{Q}}$, then $\vartheta_1, \ldots, \vartheta_r$ together with $Q(\gamma)^\circ$ form a true $\mathcal{W}$-bunch. Consequently, $\mathrm{cone}(f_1), \ldots, \mathrm{cone}(f_r)$ together with the zero cone form a true separated δ-collection $\mathfrak{A}$. The P-drop $P_\downarrow\mathfrak{A}$ is the $\mathcal{V}$-fan Σ consisting of the zero cone and $\mathrm{cone}(v_1), \ldots, \mathrm{cone}(v_r)$. Since P induces an injection $\mathfrak{A} \to \Sigma$, we conclude that the $\mathrm{cone}(v_i)$ are pairwise different and one-dimensional. □

Proposition 2.4.10. *Assume that the moving cone* $\mathrm{Mov}(\mathcal{W})$ *is of full dimension in* $K_{\mathbb{Q}}$. *Then we have mutually inverse bijections*

$$\begin{array}{rcl} \{\lambda \in \Lambda;\ \lambda^\circ \subseteq \mathrm{Mov}(\mathcal{W})^\circ\} & \longleftrightarrow & \{\textit{true normal } \mathcal{V}\textit{-fans}\} \\ \lambda & \mapsto & \Sigma(\lambda), \\ \displaystyle\bigcap_{\tau \in \Sigma^\sharp} \tau & \mapsfrom & \Sigma. \end{array}$$

Proof. First we remark that, by Proposition 2.4.9, the vectors $v_1, \ldots, v_r$ generate pairwise different one-dimensional cones. Thus, given $\lambda \in \Lambda(\mathcal{W})$, the associated $\mathcal{V}$-fan $\Sigma(\lambda) = \Theta(\lambda)^\sharp$ is true if and only if $\Theta(\lambda)$ comprises $\vartheta_1, \ldots, \vartheta_r$, where the latter is equivalent to $\lambda^\circ \subseteq \mathrm{Mov}(\mathcal{W})^\circ$. Now the assertion follows directly from Propositions 2.4.1, 2.4.2, 2.4.6 and Theorem 2.1.14. □

3. Good toric quotients

3.1. Characterization of good toric quotients. Consider a toric variety X with acting torus T and a closed subgroup $H \subseteq T$. We present the combinatorial criterion [**152**, Theorem 4.1] for existence of a good quotient for the action of H on X in the following sense.

Definition 3.1.1. Let a reductive group G act on a variety X. We say that a good quotient $p\colon X \to Y$ for the G-action is *separated* if Y is separated.

A first general observation enables us to treat the problem of existence of a separated good quotient entirely in terms of toric geometry.

Proposition 3.1.2. *Let* X *be a toric variety with acting torus* T *and* $p\colon X \to Y$ *a separated good quotient for the action of a closed subgroup* $H \subseteq T$. *Then* Y *admits a unique structure of a toric variety turning* p *into a toric morphism.*

Proof. First observe that Y inherits normality from X. Next consider the product $T \times X$. Then H acts on the second factor as a subgroup of T and we have a commutative diagram

$$\begin{array}{ccc} T \times X & \xrightarrow{\ \mu_X\ } & X \\ {\scriptstyle \mathrm{id}\times p}\downarrow & & \downarrow{\scriptstyle p} \\ T \times Y & \xrightarrow[\ \mu_Y\ ]{} & Y \end{array}$$

where μ_X describes the T-action and the downwards arrows are good quotients for the respective H-actions. One verifies directly that the induced morphism μ_Y is a T-action with an open dense orbit. The assertion follows. □

We fix the setup for the rest of this section. Let Δ be a fan in a lattice F and denote by $X := X_\Delta$ the associated toric variety. Moreover, let $Q\colon E \to K$ be an epimorphism from the dual lattice $E = \mathrm{Hom}(F, \mathbb{Z})$ onto an abelian group K; this specifies an embedding of $H := \mathrm{Spec}\, \mathbb{K}[K]$ into the acting torus $T = \mathrm{Spec}\, \mathbb{K}[E]$ and thus an action of H on X. Denoting by $M \subseteq E$ the kernel of $Q\colon E \to K$, we obtain mutually dual exact sequences

$$0 \longrightarrow L \xrightarrow{Q^*} F \xrightarrow{P} N$$

$$0 \longleftarrow K \xleftarrow[Q]{} E \xleftarrow[P^*]{} M \longleftarrow 0$$

Definition 3.1.3. We say that Δ is *$L_{\mathbb{Q}}$-projectable* if any two $\delta_1, \delta_2 \in \Delta^{\max}$ admit an $L_{\mathbb{Q}}$-invariant separating linear form, i.e., an element $e \in E_{\mathbb{Q}}$ with

$$e_{|L_{\mathbb{Q}}} = 0, \qquad e_{|\delta_1} \geq 0, \qquad e_{|\delta_2} \leq 0, \qquad \delta_1 \cap e^{\perp} = \delta_1 \cap \delta_2 = e^{\perp} \cap \delta_2.$$

Remark 3.1.4. In the above setting, the fan Δ is $L_{\mathbb{Q}}$-projectable if and only if every cone $\delta \in \Delta^{\max}$ satisfies

$$P^{-1}(P(\delta)) \cap \mathrm{Supp}(\Delta) = \delta.$$

Construction 3.1.5. Suppose that the fan Δ is $L_{\mathbb{Q}}$-projectable. Then we have the collection of those cones of Δ which can be separated by an $L_{\mathbb{Q}}$-invariant linear form from a maximal cone:

$$\mathfrak{A}(\Delta) := \{\delta \in \Delta;\ \delta = e^{\perp} \cap \delta_0 \text{ for some } \delta_0 \in \Delta^{\max} \text{ and } e \in \delta_0^{\vee} \cap L_{\mathbb{Q}}^{\perp}\}.$$

The images $P(\delta)$, where $\delta \in \mathfrak{A}(\Delta)$, form a quasifan Σ in $N_{\mathbb{Q}}$ and all share the same minimal face $\tau \subseteq N_{\mathbb{Q}}$. The map $P\colon F \to N$ is a map of the quasifans Δ and Σ, and we have a bijection

$$\mathfrak{A}(\Delta) \to \Sigma, \qquad \delta \mapsto P(\delta).$$

Moreover, with $N_1 := N/(\tau \cap N)$ and the canonical map $P_1\colon F \to N_1$, the cones $P_1(\delta)$, where $\delta \in \mathfrak{A}(\Delta)$, form a fan Σ_1 in N_1. The map $P_1\colon F \to N_1$ is a map of the fans Δ and Σ_1, and we have a bijection

$$\mathfrak{A}(\Delta) \to \Sigma_1, \qquad \delta \mapsto P_1(\delta).$$

Proof. We claim that any two cones $\delta_i \in \mathfrak{A}(\Delta)$ admit an $L_{\mathbb{Q}}$-invariant separating linear form. Indeed, consider maximal cones $\delta_i' \in \Delta$ with $\delta_i \preceq \delta_i'$. Then δ_1' and δ_2' admit an $L_{\mathbb{Q}}$-invariant separating linear form e, the faces δ_i are separated from δ_i' by $L_{\mathbb{Q}}$-invariant linear forms e_i and any linear combination $e_2 - e_1 + ae$ with a big enough provides the wanted $L_{\mathbb{Q}}$-invariant separating linear form for δ_1 and δ_2. This claim directly implies that the cones $P(\delta)$, where $\delta \in \mathfrak{A}(\Delta)$, form a quasifan Σ and the canonical map $\mathfrak{A}(\Delta) \to \Sigma$ is bijective. By construction, Σ_1 is a fan and the canonical map $\Sigma \to \Sigma_1$ is a bijection. □

Proposition 3.1.6. *As above, let X be the toric variety arising from a fan Δ in a lattice F, and consider the action of a subgroup $H \subseteq T$ given by an epimorphism $Q\colon E \to K$. Then the following statements are equivalent.*

(i) *The action of H on X admits a separated good quotient.*
(ii) *The fan Δ is $L_{\mathbb{Q}}$-projectable.*
(iii) *Every $\delta \in \Delta^{\max}$ satisfies $P^{-1}(P(\delta)) \cap \mathrm{Supp}(\Delta) = \delta$.*

Moreover, if one of these statements holds, then the toric morphism $p_1\colon X \to Y_1$ *arising from the map* $P_1\colon F \to N_1$ *of the fans* Δ *and* Σ_1 *as in 3.1.5 is a separated good quotient for the action of* H *on* X.

Specializing this characterization to the case of geometric quotients with separated quotient space gives the following characterization.

Corollary 3.1.7. *As above, let* X *be the toric variety arising from a fan* Δ *in a lattice* F*, and consider the action of a subgroup* $H \subseteq T$ *given by an epimorphism* $Q\colon E \to K$*. Then the following statements are equivalent.*

(i) *The action of* H *on* X *admits a separated geometric quotient.*
(ii) *The restriction* $P\colon \operatorname{Supp}(\Delta) \to N_{\mathbb{Q}}$ *is injective.*

If one of these statements holds, then $\Sigma := \{P(\delta);\ \delta \in \Delta\}$ *is a fan in* $N_{\mathbb{Q}}$ *and the toric morphism* $p\colon X \to Y$ *associated to the map* $P\colon F \to N$ *of the fans* Δ *and* Σ *is a separated geometric quotient for the action of* H *on* X.

We come to the proof of Proposition 3.1.6. The following elementary observation will also be used later.

Lemma 3.1.8. *Situation as in Construction 3.1.5. Then for every* $\delta \in \mathfrak{A}(\Delta)$*, we have* $P_1^{-1}(P_1(\delta)) \ \cap\ \operatorname{Supp}(\Delta) = \delta$.

Proof. Consider $\delta_0 \in \Delta$ with $P_1(\delta_0) \subseteq P_1(\delta)$. Then $P(\delta_0) \subseteq P(\delta)$ holds and thus any $L_{\mathbb{Q}}$-invariant linear form on F that is nonnegative on δ is necessarily nonnegative on δ_0. It follows that δ_0 is a face of δ. □

Proof of Proposition 3.1.6. The equivalence of (ii) and (iii) is elementary. We only show that (i) and (ii) are equivalent. First suppose that the action of H has a good quotient $\pi\colon X \to Y$. By Proposition 3.1.2, the quotient variety Y is toric and π is a toric morphism. So we may assume that π arises from a map $\Pi\colon F \to N'$ from Δ to a fan Σ' in a lattice N'. Note that the sublattice $L \subseteq F$ is contained in the kernel of Π. We claim that there are bijections of the sets $\Delta^{\max}$ and $(\Sigma')^{\max}$ of maximal cones:

$$\Delta^{\max} \ \to\ (\Sigma')^{\max}, \qquad \delta \ \mapsto\ \Pi(\delta), \tag{3.1}$$

$$(\Sigma')^{\max} \ \to\ \Delta^{\max}, \qquad \sigma' \ \mapsto\ \Pi^{-1}(\sigma') \cap \operatorname{Supp}(\Delta) \tag{3.2}$$

To check that the first map is well-defined, let $\delta \in \Delta^{\max}$. Then the image $\Pi(\delta)$ is contained in some maximal cone $\sigma' \in \Sigma'$. In particular, $\pi(X_\delta) \subseteq X'_{\sigma'}$ holds. Since π is affine, the inverse image $\pi^{-1}(X'_{\sigma'})$ is an invariant affine chart of X, and hence equals X_δ. Since π is surjective, we have $\pi(X_\delta) = X'_{\sigma'}$. This means $\Pi(\delta) = \sigma'$. To see that the second map is well defined, let $\sigma' \in (\Sigma')^{\max}$. The inverse image of the associated affine chart $X'_{\sigma'} \subseteq X'$ is given by

$$\pi^{-1}(X'_{\sigma'}) \ = \ \bigcup_{\substack{\tau \in \Delta \\ \Pi(\tau) \subseteq \sigma'}} X_\tau.$$

Since π is affine, this inverse image is an affine invariant chart X_δ given by some cone $\delta \in \Delta$. In follows that

$$\delta \ = \ \operatorname{cone}(\tau \in \Delta;\ \Pi(\tau) \subseteq \sigma') \ = \ \Pi^{-1}(\sigma') \cap \operatorname{Supp}(\Delta).$$

By surjectivity of π, we have $\Pi(\delta) = \sigma'$. Assume that $\delta \subseteq \vartheta$ for some $\vartheta \in \Delta^{\max}$. As seen above, $\Pi(\vartheta)$ is a maximal cone of Σ'. Since $\Pi(\vartheta)$ contains the maximal cone σ', we get $\Pi(\vartheta) = \sigma'$. This implies $\delta = \vartheta$, thus $\delta \in \Delta^{\max}$, and the map (3.2) is well defined. Let $\delta_1, \delta_2 \in \Delta^{\max}$ be two different cones. The maximal cones $\sigma'_i := \Pi(\delta_i)$ can be separated by a linear form u' on N'. Then $u := u' \circ \Pi$ is an $L_{\mathbb{Q}}$-invariant separating linear form for the cones δ_1 and δ_2.

Now suppose that the fan Δ is $L_{\mathbb{Q}}$-projectable. We show that the toric morphism $p_1\colon X \to Y_1$ is affine. Consider an affine chart $Y_1' \subseteq Y_1$ corresponding to a maximal cone $\sigma_1 \subseteq \Sigma_1$. Then there are unique maximal cones $\sigma \in \Sigma$ and $\delta \in \Delta$ projecting onto σ_1. Lemma 3.1.8 and the Fiber Formula tell us that $p_1^{-1}(Y_1')$ equals the affine toric chart of X corresponding to δ and thus $p_1\colon X \to Y_1$ is affine. Proposition 1.4.3 then implies that it is a good quotient. $\square$

Example 3.1.9. Let Σ be a fan in a lattice F and consider a cone $\sigma \in \Sigma$. Denote by $\mathrm{star}(\sigma)$ the set of all cones $\tau \in \Sigma$ that contain σ as a face. Then the closure of the toric orbit corresponding to σ is given by

$$\overline{T \cdot x_\sigma} \quad = \bigcup_{\tau \in \mathrm{star}(\sigma)} T \cdot x_\tau.$$

The union $U(\sigma)$ of the affine charts X_τ, where $\tau \in \mathrm{star}(\sigma)$, is an open T-invariant neighbourhood of the orbit closure $\overline{T \cdot x_\sigma}$. The set of maximal cones of the fan $\Sigma(\sigma)$ corresponding to $U(\sigma)$ coincides with $\Sigma^{\max} \cap \mathrm{star}(\sigma)$.

Let L be the intersection of the linear span $\mathrm{lin}(\sigma)$ of σ in $F_{\mathbb{Q}}$ with the lattice F, and let $P\colon F \to N := F/L$ denote the projection. Then the cones $P(\tau)$, where τ runs through $\Sigma(\sigma)^{\max}$, are the maximal cones of the quotient fan $\widetilde{\Sigma}(\sigma)$ of $\Sigma(\sigma)$ by L. Moreover, $\widetilde{\Sigma}(\sigma)$ is the fan of $\overline{T \cdot x_\sigma}$ viewed as a toric variety with acting torus T/T_{x_σ}. In other words, the projection defines a good quotient $p\colon U(\sigma) \to \overline{T \cdot x_\sigma}$ by the isotropy group T_{x_σ}.

3.2. Combinatorics of good toric quotients. We consider a $\mathbb{Q}$-factorial affine toric variety X with acting torus T and the action of a closed subgroup $H \subseteq T$ on X. Our aim is a combinatorial description of the "maximal" open subsets $U \subseteq X$ admitting a separated good quotient by the action of H; the main result reformulates [**35**] in terms of bunches of cones. First we fix the notion of maximality.

Definition 3.2.1. Let a reductive affine algebraic group G act on a variety X.

(i) By a *good G-set* in X, we mean an open subset $U \subseteq X$ with a separated good quotient $U \to U /\!\!/ G$.
(ii) We say that a subset $U' \subseteq U$ of a good G-set $U \subseteq X$ is *G-saturated* if it satisfies $U' = p^{-1}(p(U'))$, where $p\colon U \to U /\!\!/ G$ is the quotient.
(iii) We say that a subset $U \subseteq X$ is *G-maximal* if it is a good G-set and maximal w.r.t. G-saturated inclusion.

The key to a combinatorial description of H-maximal subsets for subgroup actions on toric varieties is the following, see [**152**, Corollary 2.5].

Theorem 3.2.2. *Let X be a toric variety with the acting torus T and H be a closed subgroup of T. Then every H-maximal subset of X is T-invariant.*

We fix the setup for the rest of the section. Let F be a lattice, $\delta \subseteq F_{\mathbb{Q}}$ a simplicial cone of full dimension with primitive generators $f_1, \ldots, f_r \in F$ and let $Q\colon E \to K$ be an epimorphism from the dual lattice $E = \mathrm{Hom}(F, \mathbb{Z})$ onto an abelian group K. Denoting by $M \subseteq E$ the kernel of $Q\colon E \to K$, we obtain mutually dual exact sequences of rational vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{\mathbb{Q}} & \longrightarrow & F_{\mathbb{Q}} & \xrightarrow{\ P\ } & N_{\mathbb{Q}} & \longrightarrow & 0 \\ \\ 0 & \longleftarrow & K_{\mathbb{Q}} & \xleftarrow[\ Q\ ]{} & E_{\mathbb{Q}} & \longleftarrow & M_{\mathbb{Q}} & \longleftarrow & 0 \end{array}$$

Let $(e_1, \ldots, e_r)$ be the basis for $E_{\mathbb{Q}}$ dual to the basis $(f_1, \ldots, f_r)$ for $F_{\mathbb{Q}}$. With $v_i := P(f_i)$ and $w_i := Q(e_i)$, we obtain Gale dual vector configurations $\mathcal{V} := (v_1, \ldots, v_r)$ and $\mathcal{W} := (w_1, \ldots, w_r)$. Moreover, let $\gamma \subseteq E_{\mathbb{Q}}$ be the dual cone of $\delta \subseteq F_{\mathbb{Q}}$. We denote by $X := X_\delta$ the affine toric variety associated to the cone δ in the lattice F and consider the action of the subgroup $H := \operatorname{Spec} \mathbb{K}[K]$ of the acting torus $T = \operatorname{Spec} \mathbb{K}[E]$ on X.

CONSTRUCTION 3.2.3. To every saturated connected γ-collection $\mathfrak{B}$ and also to every $\mathcal{W}$-bunch Θ, we associate an $L_{\mathbb{Q}}$-projectable subfan of the fan of faces of δ and the corresponding open toric subsets of X:

$$\Delta(\mathfrak{B}) := \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in \mathfrak{B}\}, \qquad U(\mathfrak{B}) := X_{\Delta(\mathfrak{B})},$$
$$\Delta(\Theta) := \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in Q^{\uparrow}\Theta\}, \qquad U(\Theta) := X_{\Delta(\Theta)}.$$

Conversely, any toric good H-set $U \subseteq X$ arises from an $L_{\mathbb{Q}}$-projectable subfan $\Delta(U)$ of the fan of faces of δ and we define an associated saturated connected γ-collection and an associated $\mathcal{W}$-bunch

$$\mathfrak{B}(U) := \mathfrak{A}(\Delta(U))^*, \qquad \Theta(\Delta) := Q_{\downarrow}(\mathfrak{A}(\Delta(U))^*).$$

THEOREM 3.2.4. *As above, let X be the affine toric variety arising from a simplicial cone δ of full dimension in a lattice F, and consider the action of a subgroup $H \subseteq T$ given by an epimorphism $Q\colon E \to K$. Then one has order reversing mutually inverse bijections*

$$\begin{array}{ccc} \{\textit{maximal } \mathcal{W}\textit{-bunches}\} & \longleftrightarrow & \{H\textit{-maximal subsets of } X\}, \\ \Theta & \mapsto & U(\Theta), \\ \Theta(U) & \leftarrow\!\shortmid & U. \end{array}$$

Under these bijections, the bunches arising from GKZ-chambers correspond to the subsets with a quasiprojective quotient space.

LEMMA 3.2.5. *Situation as in Construction 3.1.5. Let $\Delta' \preceq \Delta$ be an $L_{\mathbb{Q}}$-projectable subfan and let X', X denote the toric varieties associated to Δ', Δ respectively. Then X' is H-saturated in X if and only if $\mathfrak{A}(\Delta') \subseteq \mathfrak{A}(\Delta)$ holds.*

PROOF. We work with the good quotient $p_1\colon X \to Y_1$ arising from the map $P_1\colon F \to N_1$ of the fans Δ and Σ_1 provided by Construction 3.1.5. First assume that X' is H-saturated in X. Given $\delta' \in \mathfrak{A}(\Delta')$, consider the associate affine toric chart $U' \subseteq X'$. Since Y_1 carries the quotient topology, $p_1(U) \subseteq Y_1$ is open and we obtain $P_1(\delta') \in \Sigma_1$. Lemma 3.1.8 yields $\delta' \in \mathfrak{A}(\Delta)$. Conversely, if $\mathfrak{A}(\Delta') \subseteq \mathfrak{A}(\Delta)$ holds, combine Lemma 3.1.8 and the Fiber Formula to see that $p^{-1}(p(U'))$ equals U' for every toric affine chart $U' \subseteq X'$. □

LEMMA 3.2.6. *Situation as before Construction 3.2.3. Then we have mutually inverse order reversing bijections*

$$\begin{array}{ccc} \{\textit{saturated connected } \gamma\textit{-collections}\} & \longleftrightarrow & \{L_{\mathbb{Q}}\textit{-projectable fans}\}, \\ \mathfrak{B} & \mapsto & \Delta(\mathfrak{B}), \\ \mathfrak{A}(\Delta)^* & \leftarrow\!\shortmid & \Delta. \\ \{\textit{maximal } \mathcal{W}\textit{-bunches}\} & \longleftrightarrow & \{\textit{maximal } L_{\mathbb{Q}}\textit{-projectable fans}\}, \\ \Theta & \mapsto & \Delta(Q^{\uparrow}\Theta), \\ Q_{\downarrow}(\mathfrak{A}(\Delta)^*) & \leftarrow\!\shortmid & \Delta. \end{array}$$

PROOF. Obviously, the assignment $\Delta \mapsto \mathfrak{A}(\Delta)$ defines a bijection from $L_{\mathbb{Q}}$-projectable fans to saturated separated δ-collections; its inverse is given by

$$\mathfrak{A} \mapsto \{\delta_0 \preceq \delta;\ \delta_0 \preceq \delta_1 \text{ for some } \delta_1 \in \mathfrak{A}\}.$$

Thus, Proposition 2.3.5 establishes the first pair of bijections. Involving also Proposition 2.3.8 gives the second one. □

Proof of Construction 3.2.3 and Theorem 3.2.4. Construction 3.2.3 is clear by Lemma 3.2.6. Theorem 3.2.2 ensures that the H-maximal subsets $U \subseteq X$ are toric. According to Proposition 3.1.6 and Lemma 3.2.5, they correspond to maximal $L_{\mathbb{Q}}$-projectable subfans Δ of the fan of faces of $\delta \subseteq F_{\mathbb{Q}}$. Thus, Lemma 3.2.6 provides the desired bijections. The supplement follows from the characterization of normal fans given in Theorem 2.2.2. □

Proposition 3.2.7. *Situation as in Theorem 3.2.4. Let Θ be a maximal $\mathcal{W}$-bunch and $p\colon U(\Theta) \to Y$ the associated good quotient. Then the following statements are equivalent.*

(i) *The $\mathcal{W}$-bunch Θ consists of full-dimensional cones.*
(ii) *We have $(Q^{\uparrow}\Theta)^* = \Delta(\Theta)$.*
(iii) *The good quotient $p\colon U(\Theta) \to Y$ is geometric.*

Proof. The equivalence of (i) and (ii) is elementary and the equivalence of (ii) and (iii) is a direct consequence of the Fiber Formula and the fact that P induces a bijection $(Q^{\uparrow}\Theta)^* \to P_{\downarrow}((Q^{\uparrow}\Theta)^*)$. □

4. Toric varieties and bunches of cones

4.1. Toric varieties and lattice bunches. We use the concept of a bunch of cones to describe toric varieties. As with fans, this means to enhance bunches with a lattice structure. We obtain a functor from maximal lattice bunches to "maximal" toric varieties which induces a bijection on isomorphy classes. In fact, for later use, we formulate the assignment first in a more general context.

As earlier, given an abelian group K, we write $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ for the associated rational vector space and for any $w \in K$, we denote the associated element $w \otimes 1 \in K_{\mathbb{Q}}$ again by w. Moreover, for a homomorphism $Q\colon E \to K$ of abelian groups we denote the associated linear map $E_{\mathbb{Q}} \to K_{\mathbb{Q}}$ again by Q.

Definition 4.1.1. A *(true) lattice collection* is a triple $(E \xrightarrow{Q} K, \gamma, \mathfrak{B})$, where $Q\colon E \to K$ is an epimorphism from a lattice E with basis $e_1, \ldots, e_r$ onto an abelian group K generated by any $r-1$ of the $w_i := Q(e_i)$, the cone $\gamma \subseteq E_{\mathbb{Q}}$ is generated by $e_1, \ldots, e_r$ and $\mathfrak{B}$ is a (true) saturated connected γ-collection.

Construction 4.1.2. Let $(E \xrightarrow{Q} K, \gamma, \mathfrak{B})$ be a true lattice collection. In particular, E is a lattice with basis $e_1, \ldots, e_r$, we have $\gamma := \mathrm{cone}(e_1, \ldots, e_r)$ and $Q\colon E \to K$ an epimorphism onto an abelian group such that any $r-1$ of the $w_i := Q(e_i)$ generate K as an abelian group. With $M := \ker(Q)$, we have the mutually dual exact sequences

$$0 \longrightarrow L \longrightarrow F \xrightarrow{P} N$$

$$0 \longleftarrow K \xleftarrow[Q]{} E \longleftarrow M \longleftarrow 0$$

Let $f_1, \ldots, f_r$ be the dual basis of $e_1, \ldots, e_r$. Then each $v_i := P(f_i)$ is a primitive lattice vector in N. Let $\delta = \mathrm{cone}(f_1, \ldots, f_r)$ denote the dual cone of $\gamma \subseteq E_{\mathbb{Q}}$ and for $\gamma_0 \preceq \gamma$ let $\gamma_0^* = \gamma_0^{\perp} \cap \delta$ be the corresponding face. Then one has fans in the lattices F and N:

$$\widehat{\Sigma} := \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in \mathfrak{B}\}, \qquad \Sigma := \{P(\gamma_0^*);\ \gamma_0 \in \mathfrak{B}\}.$$

The canonical map $\mathfrak{B} \to \Sigma$, $\gamma_0 \mapsto P(\gamma_0^*)$ is an order reversing bijection. In particular, the fan Σ has exactly r rays, namely $\mathrm{cone}(v_1), \ldots, \mathrm{cone}(v_r)$. The *toric variety associated to* $\mathfrak{B}$ is the toric variety X defined by the fan Σ. The projection $P \colon F \to N$ is a map of the fans $\widehat{\Sigma}$ and Σ and hence defines a toric morphism $\widehat{X} \to X$.

Proof. Since any $r-1$ of the weights w_i generate K as an abelian group, Lemma 1.4.1 shows that each $v_i = P(f_i)$ is a primitive lattice vector. Moreover, Propositions 2.3.5 and 2.3.11 show that Σ is a fan and the canonical map $\mathfrak{B} \to \Sigma$ is an order reversing bijection. □

Note that every non-degenerate lattice fan can be obtained via this construction. Specializing to bunches of cones, we loose a bit of this generality, but obtain a more concise presentation.

Definition 4.1.3. A *(true, maximal) lattice bunch* a triple $(K, \mathcal{W}, \Theta)$, where K is a finitely generated abelian group, $\mathcal{W} = (w_1, \ldots, w_r)$ is a family in K such that any $r-1$ of the w_i generate K as an abelian group and Θ is a (true, maximal) $\mathcal{W}$-bunch.

Construction 4.1.4. Let $(K, \mathcal{W}, \Theta)$ be a true lattice bunch. The associated *projected cone* is $(E \xrightarrow{Q} K, \gamma)$, where $E = \mathbb{Z}^r$, the homomorphism $Q \colon E \to K$ sends the i-th canonical basis vector $e_i \in E$ to $w_i \in K$ and $\gamma \subseteq E_{\mathbb{Q}}$ is the cone generated by $e_1, \ldots, e_r$. We have the following collections of cones:

$$\begin{aligned} Q^{\uparrow}\Theta &= \{\gamma_0 \preceq \gamma;\ Q(\gamma_0) \in \Theta\}, \\ \mathrm{cov}(\Theta) &:= \{\gamma_0 \in Q^{\uparrow}\Theta;\ \gamma_0 \text{ minimal}\}. \end{aligned}$$

The first one is the Q-lift of Θ and we call the second one the *covering collection* of Θ. In the notation of Theorem 2.1.14 and Construction 4.1.2, it defines fans in F and N:

$$\widehat{\Sigma} := \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in Q^{\uparrow}\Theta\}, \qquad \Sigma = \Theta^{\sharp} = \{P(\gamma_0^*);\ \gamma_0 \in Q^{\uparrow}\Theta\}.$$

The *toric variety associated to* $(K, \mathcal{W}, \Theta)$ is the toric variety $X = X_{\Theta}$ defined by the maximal $\mathcal{V}$-fan $\Sigma = \Theta^{\sharp}$ corresponding to Θ. The projection $P \colon F \to N$ defines a map of the fans $\widehat{\Sigma}$ and Σ and thus a toric morphism $\widehat{X} \to X$.

Definition 4.1.5. A *morphism of lattice bunches* $(K_i, \mathcal{W}_i, \Theta_i)$ with associated projected cones $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$ is a homomorphism $\varphi \colon E_1 \to E_2$ such that there is a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \varphi\ } & E_2 \\ {\scriptstyle Q_1}\downarrow & & \downarrow{\scriptstyle Q_2} \\ K_1 & \xrightarrow[\ \bar{\varphi}\ ]{} & K_2, \end{array}$$

where $\varphi(\gamma_1) \subseteq \gamma_2$ holds and for every $\alpha_2 \in \mathrm{cov}(\Theta_2)$ there is $\alpha_1 \in \mathrm{cov}(\Theta_1)$ with $\varphi(\alpha_1) \subseteq \alpha_2$.

Remark 4.1.6. Let $(K_i, \mathcal{W}_i, \Theta_i)$ be two true lattice bunches and φ a morphism between them. Then φ induces a morphism of the corresponding lattice fans $\Sigma_2 = \Theta_2^{\sharp}$ and $\Sigma_1 = \Theta_1^{\sharp}$, and thus a morphism $\psi(\varphi) \colon X_{\Theta_2} \to X_{\Theta_1}$ of the associated toric varieties.

We say that a toric variety X is *maximal* if admits no open toric embedding $X \subsetneq X'$ with $X' \setminus X$ of codimension at least two in X'. Note that in terms of a defining fan Σ of X, maximality means that Σ does not occur as a proper subfan of some fan Σ' having the same rays as Σ. For example, complete as well as affine toric varieties are maximal.

Theorem 4.1.7. *We have a contravariant faithful essentially surjective functor inducing a bijection on the sets of isomorphism classes:*

$$\begin{array}{rcl} \{\textit{true maximal lattice bunches}\} & \longrightarrow & \{\textit{maximal toric varieties}\}, \\ (K, \mathcal{W}, \Theta) & \mapsto & X_\Theta, \\ \varphi & \mapsto & \psi(\varphi). \end{array}$$

Proof. The assertion follows directly from Construction 4.1.4, Remark 4.1.6 and Theorem 2.1.14. □

Example 4.1.8. Consider the true maximal lattice bunch $(K, \mathcal{W}, \Theta)$, where $K := \mathbb{Z}^3$, the family $\mathcal{W} = (w_1, \ldots, w_6)$ is given by

$$w_1 = (1,0,0), \quad w_2 = (0,1,0), \quad w_3 = (0,0,1),$$

$$w_4 = (1,1,0), \quad w_5 = (1,0,1), \quad w_6 = (0,1,1),$$

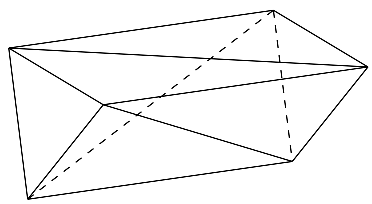
and, finally, the maximal $\mathcal{W}$-bunch Θ in $K_\mathbb{Q}$ has the following four three-dimensional cones as its minimal cones:

$$\mathrm{cone}(w_3, w_4, w_5), \quad \mathrm{cone}(w_1, w_4, w_6), \quad \mathrm{cone}(w_2, w_5, w_6), \quad \mathrm{cone}(w_4, w_5, w_6).$$

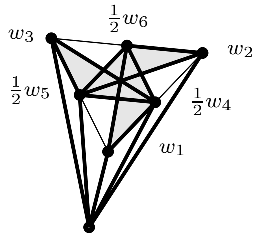
The fan $\Sigma = \Theta^\sharp$ is one of the simplest non-polytopal complete simplicial fans in $N = \mathbb{Z}^3$. It looks as follows. Consider the polytope $B \subset K_\mathbb{Q}$ with the vertices

$$(-1,0,0), \quad (0,-1,0), \quad (0,0,-1), \quad (0,1,1), \quad (1,0,1), \quad (1,1,0)$$

and subdivide the facets of B according to the picture below. Then Σ is the fan generated by the cones over the simplices of this subdivision.



Defining polytope subdivision of Σ



Corresponding bunch Θ

4.2. Toric geometry via bunches. We describe basic geometric properties of a toric variety in terms of a defining lattice collection or lattice bunch. We consider orbit decomposition, divisor class group, Cox ring, local class groups, Picard group, smoothness, $\mathbb{Q}$-factoriality, cones of movable, semiample and ample divisors and intersection numbers.

Proposition 4.2.1 (Orbit decomposition II). *Situation as in Construction 4.1.2. Then we have a bijection*

$$\mathfrak{B} \ \to \ \{T\text{-}\textit{orbits of } X\}, \qquad \gamma_0 \ \mapsto \ T \cdot x_{\gamma_0}, \quad \textit{where} \quad x_{\gamma_0} := x_{P(\gamma_0^*)}.$$

Moreover, for any two $\gamma_0, \gamma_1 \in \mathfrak{B}$, one has $\gamma_0 \preceq \gamma_1$ if and only if $\overline{T \cdot x_{\gamma_0}} \subseteq \overline{T \cdot x_{\gamma_1}}$ holds.

Proof. The collection $\mathfrak{B}$ is in order reversing bijection with the defining fan Σ of X via $\gamma_0 \mapsto P(\gamma_0^*)$. Thus, the usual description 1.2.2 of the orbit decomposition in terms of Σ gives the assertion. □

Proposition 4.2.2 (Divisor class group and Cox ring). *Situation as in Construction 4.1.2. Then there is a commutative diagram of abelian groups*

$$
\begin{array}{ccccccccc}
 & & \mathbb{X}(H_X) & \longleftarrow & \mathbb{X}(\mathbb{T}) & \xleftarrow{\;p_X^*\;} & \mathbb{X}(T) & & \\
 & \swarrow & \uparrow \cong & & e\mapsto\chi^e \uparrow \cong & & u\mapsto\chi^u \uparrow \cong & \nwarrow & \\
0 & \longleftarrow & K & \xleftarrow{\;Q\;} & E & \xleftarrow{\;P^*\;} & M & \longleftarrow & 0 \\
 & \nwarrow & \cong \downarrow & & e_i\mapsto E_i \downarrow \cong & & u\mapsto \mathrm{div}(\chi^u) \downarrow \cong & \swarrow & \\
 & & \mathrm{Cl}(X) & \longleftarrow & \mathrm{WDiv}^T(X) & \longleftarrow & \mathrm{PDiv}^T(X) & &
\end{array}
$$

Moreover, $\widehat{X} \to X$ is a characteristic space and for the Cox ring $\mathcal{R}(X)$, we have the following isomorphism of graded algebras

$$\mathbb{K}[E \cap \gamma] \;=\; \bigoplus_{w \in K} \mathbb{K}[E \cap \gamma]_w \;\cong\; \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}(X)_{[D]} \;=\; \mathcal{R}(X).$$

Proof. The assertion follows directly from Theorem 1.3.1 and the discussion given before this Theorem. $\square$

Proposition 4.2.3. *Situation as in Construction 4.1.2. For $x \in X$, let $\gamma_0 \in \mathfrak{B}$ be the face with $x \in T \cdot x_{\gamma_0}$. Then the local divisor class group of X at x is given by*

$$
\begin{array}{ccc}
\mathrm{Cl}(X) & \longrightarrow & \mathrm{Cl}(X, x) \\
\cong \updownarrow & & \updownarrow \cong \\
K & \longrightarrow & K/Q(\mathrm{lin}(\gamma_0) \cap E)
\end{array}
$$

Proof. We may assume that $x = x_\sigma$ with $\sigma = P(\gamma_0^*)$ holds. Set $\widehat{\sigma} := \gamma_0^*$. Using the Fiber Formula 1.2.4, we see that the distinguished point $x_{\widehat{\sigma}} \in \widehat{\Sigma}$ has a closed H-orbit in the fiber $p^{-1}(x_\sigma)$, where $p\colon \widehat{X} \to X$ denotes the characteristic space. By Proposition 1.4.2, the isotropy group $H_{x_{\widehat{\sigma}}} \subseteq H$ is given by $K \to K/Q(\mathrm{lin}(\gamma_0) \cap E)$. Thus, we may apply Proposition I. 6.2.2 to obtain the assertion. $\square$

Corollary 4.2.4. *Situation as in Construction 4.1.2. Inside the divisor class group $\mathrm{Cl}(X) = K$, the Picard group of X is given by*

$$\mathrm{Pic}(X) \quad = \quad \bigcap_{\gamma_0 \in \mathfrak{B}} Q(\mathrm{lin}(\gamma_0) \cap E).$$

Corollary 4.2.5. *Situation as in Construction 4.1.2. For a point $x \in X$, let $\gamma_0 \in \mathfrak{B}$ be the face with $x \in T \cdot x_{\gamma_0}$.*

(i) *The point x is $\mathbb{Q}$-factorial if and only if* $\dim Q(\gamma_0)$ *equals* $\dim K_{\mathbb{Q}}$.
(ii) *The point x is smooth if and only if* $Q(\mathrm{lin}(\gamma_0) \cap E)$ *equals K.*

We describe cones of divisors in the rational divisor class group. Recall that a divisor on a variety is called *movable* if it has a positive multiple with base locus of codimension at least two and it is called *semiample* if it has a base point free multiple.

Proposition 4.2.6. *Situation as in Construction 4.1.2. The cones of effective, movable, semiample and ample divisor classes of X in $\mathrm{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}}$ are given as*

$$\mathrm{Eff}(X) \;=\; Q(\gamma), \qquad \mathrm{Mov}(X) \;=\; \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0),$$

$$\mathrm{SAmple}(X) \;=\; \bigcap_{\gamma_0 \in \mathfrak{B}} Q(\gamma_0), \qquad \mathrm{Ample}(X) \;=\; \bigcap_{\gamma_0 \in \mathfrak{B}} Q(\gamma_0)^\circ.$$

Moreover, if X arises from a lattice bunch $(K, \mathcal{W}, \Theta)$ as in Construction 4.1.4, then we have

$$\mathrm{SAmple}(X) \ = \ \bigcap_{\tau \in \Theta} \tau, \qquad \mathrm{Ample}(X) \ = \ \bigcap_{\tau \in \Theta} \tau^{\circ}.$$

PROOF. For the descriptions of SAmple(X) and Ample(X), let D be an invariant $\mathbb{Q}$-Cartier divisor on X and let $\widehat{w} \in E_{\mathbb{Q}}$ be the corresponding element. Recall that D is semiample (ample) if and only if it is described by a support function (u_σ), which is convex (strictly convex) in the sense that $u_\sigma - u_{\sigma'}$ is nonnegative (positive) on $\sigma \setminus \sigma'$ for any two $\sigma, \sigma' \in \Sigma$. For $\sigma \in \Sigma$, we denote by $\widehat{\sigma} \in \widehat{\Sigma}$ the cone with $P(\widehat{\sigma}) = \widehat{\sigma}$.

Suppose that D is semiample (ample) with convex (strictly convex) support function (u_σ). In terms of $\ell_\sigma := \widehat{w} - P^*(u_\sigma)$ this means that each $\ell_{\sigma'} - \ell_\sigma$ is nonnegative (positive) on $\widehat{\sigma} \setminus \widehat{\sigma}'$. Since $\ell_\sigma \in \widehat{\sigma}^\perp$ holds, this is equivalent to nonnegativity (positivity) of $\ell_{\sigma'}$ on every $\widehat{\sigma} \setminus \widehat{\sigma}'$.

Since all rays of the cone δ occur in the fan $\widehat{\Sigma}$, the latter is valid if and only if $\ell_\sigma \in \widehat{\sigma}^*$ (resp. $\ell_\sigma \in (\widehat{\sigma}^*)^\circ$) holds for all σ. This in turn implies that for every $\sigma \in \Sigma$ we have

$$w = Q(\widehat{w}) = Q(\ell_\sigma) \in Q(\widehat{\sigma}^*) \quad (\text{resp. } w \in Q((\widehat{\sigma}^*)^\circ)). \tag{4.1}$$

Now, the $\widehat{\sigma}^*$, where $\sigma \in \Sigma$, are precisely the cones of $\mathfrak{B}$. Since any interior $Q(\widehat{\sigma}^*)^\circ$ contains the interior of a cone of Θ, we can conclude that w lies in the respective intersections of the assertion.

Conversely, if w belongs to one of the right hand side intersections, then we surely arrive at (4.1). Thus, for every $\sigma \in \Sigma$, we find an $\ell_\sigma \in \widehat{\sigma}^*$ (an $\ell_\sigma \in (\widehat{\sigma}^*)^\circ$) mapping to w. Reversing the above arguments, we see that $u_\sigma := \widehat{w} - \ell_\sigma$ is a convex (strictly convex) support function describing D. □

COROLLARY 4.2.7. *Situation as in Construction 4.1.4. Assume that Θ arises from a chamber $\lambda \subseteq K_{\mathbb{Q}}$ of the GKZ-decomposition. Then the toric variety X associated to the true maximal bunch Θ has λ as its semiample cone.*

PROPOSITION 4.2.8. *Situation as in Construction 4.1.2. Let $e_1, \ldots, e_r$ be the primitive generators of γ. Then the canonical divisor class of X is given as*

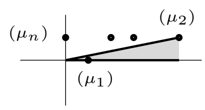
$$\mathcal{K}_X \ = \ -Q(e_1 + \ldots + e_r) \ \in \ K.$$

PROOF. The assertion follows from [**70**, Sec. 4.3] and Proposition 4.2.2. □

EXAMPLE 4.2.9. The toric variety $X := X_\Theta$ given in Example 4.1.8 is $\mathbb{Q}$-factorial and nonprojective. Moreover, the cone SAmple(X) of semiample divisors is spanned by the class of the anticanonical divisor.

EXAMPLE 4.2.10 (Kleinschmidt's classification [**93**]). The smooth maximal toric varieties X with $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ correspond to bunches $(\mathbb{Z}^2, \mathcal{W}, \Theta)$, where

- $\mathcal{W} = (w_{ij};\ 1 \le i \le n,\ 1 \le j \le \mu_i)$ satisfies
  - $w_{1j} := (1, 0)$, and $w_{ij} := (b_i, 1)$ with $0 = b_n < b_{n-1} < \cdots < b_2$,
  - $\mu_1 > 1$, $\mu_n > 0$ and $\mu_2 + \cdots + \mu_n > 1$,
- Θ is the $\mathcal{W}$-bunch arising from the chamber $\lambda = \mathrm{cone}(w_{11}, w_{21})$.



Moreover, the toric variety X defined by such a bunch Θ is always projective, and it is Fano if and only if we have

$$b_2(\mu_2 + \cdots + \mu_n) < \mu_1 + b_2\mu_2 + \cdots + b_{n-1}\mu_{n-1}.$$

Finally, we investigate intersection numbers of a complete $\mathbb{Q}$-factorial toric variety X arising from a lattice bunch $(K, \mathcal{W}, \Theta)$ with $\mathcal{W} = (w_1, \ldots, w_r)$. For $w_{i_1}, \ldots, w_{i_n}$, where $1 < i_1 < \ldots < i_n < r$, denote by $w_{j_1}, \ldots, w_{j_{r-n}}$, where $1 < j_1 < \ldots < j_{r-n} < r$, the complementary weights and set

$$\begin{aligned} \tau(w_{i_1}, \ldots, w_{i_n}) &:= \operatorname{cone}(w_{j_1}, \ldots, w_{j_{r-n}}), \\ \mu(w_{i_1}, \ldots, w_{i_n}) &= [K : \langle w_{j_1}, \ldots, w_{j_{r-n}} \rangle]. \end{aligned}$$

Proposition 4.2.11. *Situation as in Construction 4.1.4 and assume that X is $\mathbb{Q}$-factorial and complete. The intersection number of classes $w_{i_1}, \ldots, w_{i_n}$, where $n = \dim(X)$ and $1 < i_1 < \ldots < i_n < r$, is given by*

$$w_{i_1} \cdots w_{i_n} = \begin{cases} \mu(w_{i_1}, \ldots, w_{i_n})^{-1}, & \tau(w_{i_1}, \ldots, w_{i_n}) \in \Theta, \\ 0, & \tau(w_{i_1}, \ldots, w_{i_n}) \notin \Theta. \end{cases}$$

Proof. Combining Proposition 1.2.8 with Lemma 1.4.1 gives the assertion. □

CHAPTER III

Cox rings and Combinatorics

We present a combinatorial approach to the geometry of varieties with a finitely generated Cox ring. It relies on the observation that basically all varieties X sharing a given K-graded algebra R as their Cox ring arise as good quotients of open sets of $\operatorname{Spec} R$ by the action of the quasitorus $\operatorname{Spec} \mathbb{K}[K]$. The door to combinatorics is opened by Geometric Invariant Theory (GIT), which provides a description of the possible quotients in terms of combinatorial data, certain collections of polyhedral cones living in the rational vector space $K_{\mathbb{Q}}$. In Section 1, we develope the Geometric Invariant Theory of quasitorus actions on affine varieties. To every point we associate a convex polyhedral "orbit cone" and based on these data we build up the combinatorial structures describing the variation of quotients. The variation of (semi-)projective quotients is described in terms of the "GIT-fan" and for the more general case of torically embeddable quotients the description is in terms of "bunches of orbit cones". In the case of a subtorus action on an affine toric variety, these descriptions coincide with the ones obtained via Gale duality in the preceding Chapter. In Section 2, the concept of a bunched ring is presented; this is basically a factorially graded algebra R together with a bunch of cones living in the grading group K. To any such data we associate a normal variety X: we consider the action of $\operatorname{Spec} K$ on $\operatorname{Spec} R$ and take the quotient determined by the bunch of cones. The basic feature of this construction is that the resulting variety X has K as its divisor class group and R as its Cox ring. Moreover, it turns out that X comes with a canonical closed embedding into a toric variety. As first examples we discuss flag varieties and quotients of quadrics. The main task then is to descibe the geometry of the variety X in terms of its defining data, the bunched ring. Section 3 provides basic results on local divisor class groups, the Picard group and singularities. Moreover, we determine base loci of divisors as well as the cones of movable, semiample and ample divisor classes. In the case of a complete intersection, there is a simple formula for the canonical divisor and intersection numbers can easily be computed. Finally, we give a proof of Hu and Keel's characterization of finite generation of the Cox ring.

1. GIT for affine quasitorus actions

1.1. Orbit cones. Here we discuss local properties of quasitorus actions and also touch computational aspects. We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. By K we denote a finitely generated abelian group and we consider an affine K-graded $\mathbb{K}$-algebra

$$A \quad = \quad \bigoplus_{w \in K} A_w.$$

Then the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ acts on the affine variety $X := \operatorname{Spec} A$. For a point $x \in X$, we introduced in Definition I.2.2.7 the *orbit monoid* S_x and the *orbit group* K_x as

$$S_x \;=\; \{w \in K;\; f(x) \neq 0 \text{ for some } f \in A_w\} \;\subseteq\; K, \qquad K_x \;=\; S_x - S_x \;\subseteq\; K.$$

Let $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the rational vector space associated to K. Given $w \in K$, we write again w for the element $w \otimes 1 \in K_{\mathbb{Q}}$. The *weight cone* of X is the convex polyhedral cone

$$\omega_X \ := \ \omega(A) \ = \ \mathrm{cone}(w \in K;\ A_w \neq \{0\}) \ \subseteq \ K_{\mathbb{Q}}.$$

DEFINITION 1.1.1. The *orbit cone* of a point $x \in X$ is the convex cone $\omega_x \subseteq K_{\mathbb{Q}}$ generated by the weight monoid $S_x \subseteq K$. For the set of all orbit cones we write

$$\Omega_X \ := \ \{\omega_x;\ x \in X\}.$$

If we want to specify the acting group H or even the variety X in the orbit data, then we write $K_{H,x}$ or $K_{H,X,x}$ etc..

REMARK 1.1.2. Let $Y \subseteq X$ be an H-invariant closed subvariety and let $x \in Y$. According to Proposition I. 2.2.5, one has

$$S_{H,Y,x} \ = \ S_{H,X,x}, \qquad K_{H,Y,x} \ = \ K_{H,X,x}, \qquad \omega_{H,Y,x} \ = \ \omega_{H,X,x}.$$

Considering for $x \in X$ its orbit closure $Y := \overline{H \cdot x}$, we infer that $\omega_{H,X,x} = \omega_{H,Y,x}$ equals the weight cone ω_Y and thus is polyhedral.

The following observation reduces the study of orbit cones of a quasitorus action to the case of a torus action.

REMARK 1.1.3. Let $K^t \subseteq K$ denote the torsion part, set $K^0 := K/K^t$ and let $\alpha \colon K \to K^0$ be the projection. Then we have the coarsened grading

$$A \ = \ \bigoplus_{u \in K^0} A_u, \qquad\qquad A_u \ = \ \bigoplus_{w \in \alpha^{-1}(u)} A_w.$$

This coarsened grading describes the action of the unit component $H^0 \subseteq H$ on X, and we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\ \alpha\ } & K^0 \\ \downarrow & & \downarrow \\ K_{\mathbb{Q}} & \xrightarrow[\alpha]{\cong} & K^0_{\mathbb{Q}} \end{array}$$

For every $x \in X$, the isomorphism $\alpha \colon K_{\mathbb{Q}} \to K^0_{\mathbb{Q}}$ maps the H-orbit cone $\omega_{H,x}$ onto the H^0-orbit cone $\omega_{H^0,x}$.

PROPOSITION 1.1.4. *Let $\nu \colon X' \to X$ be the H-equivariant normalization. Then, for every $x' \in X'$, we have $\omega_{x'} = \omega_{\nu(x')}$.*

PROOF. The inclusion $\omega_{\nu(x')} \subseteq \omega_{x'}$ is clear by equivariance. The reverse inclusion follows from considering equations of integral dependence for the homogeneous elements $f \in \Gamma(X', \mathcal{O})$ with $f(x') \neq 0$. □

We shall use the orbit cones to describe properties of orbit closures. The basic statement in this regard is the following one.

PROPOSITION 1.1.5. *Assume that H is a torus and let $x \in X$. The factor group H/H_x acts with a dense free orbit on the orbit closure $\overline{H \cdot x} \subseteq X$. Moreover, the orbit closure $\overline{H \cdot x}$ has the affine toric variety $\mathrm{Spec}(\mathbb{K}[\omega_x \cap K_x])$ as its (H/H_x)-equivariant normalization.*

PROOF. The first assertion is obvious, and the second one follows immediately from Proposition 1.1.4 and the fact that the algebra of global functions of $\overline{H \cdot x}$ is the semigroup algebra $\mathbb{K}[S_x]$ of the orbit monoid, see Proposition I. 2.2.9. □

The collection of orbits in a given orbit closure $\overline{H\cdot x}$ comes with a partial ordering: we write $H\cdot x_1 \le H\cdot x_2$ if $H\cdot x_1 \subseteq \overline{H\cdot x_2}$ holds.

Proposition 1.1.6. *Let $x \in X$. Then we have a commutative diagram of order preserving bijections*

$$\begin{array}{ccc} H\text{-orbits}(\overline{H \cdot x}) & \xrightarrow{\ H\cdot y \ \mapsto\ \omega_{H,y}\ } & \text{faces}(\omega_{H,x}) \\ \cong \updownarrow & & \updownarrow \cong \\ H^0\text{-orbits}(\overline{H^0 \cdot x}) & \xrightarrow[\ H^0\cdot y \ \mapsto\ \omega_{H^0,y}\ ]{} & \text{faces}(\omega_{H^0,x}) \end{array}$$

Moreover, for any homogeneous function $f \in A_w$ with $f(x) \neq 0$ and any point $y \in \overline{H\cdot x}$, we have $f(y) \neq 0 \Leftrightarrow w \in \omega_{H,x}$.

Proof. The torus $T := H^0/H^0_x$ acts on $Z := \overline{H^0 \cdot x}$, and the T-orbits of Z coincide with its H^0-orbits. According to Proposition 1.1.5, the T-equivariant normalization of Z is the affine toric variety $Z' = \operatorname{Spec}(\mathbb{K}[\omega_x \cap K_x])$. The T-orbits of Z' are in order reversing bijection with the faces of ω_x via $T \cdot z \mapsto \omega_z$, see Proposition II.1.2.2. The assertion follows from Proposition 1.1.4 and the fact that the normalization map $\nu\colon Z' \to Z$ induces an order preserving bijection between the sets of T-orbits of Z' and Z. $\square$

We collect some basic observations on the explicit computation of orbit data, when the algebra A is given in terms of homogeneous generators and relations. The following notions will be crucial.

Definition 1.1.7. Let K be a finitely generated abelian group, A a K-graded affine $\mathbb{K}$-algebra and $\mathfrak{F} = (f_1, \ldots, f_r)$ a system of homogeneous generators for A.

(i) The *projected cone associated to* $\mathfrak{F}$ is $(E \xrightarrow{Q} K, \gamma)$, where $E := \mathbb{Z}^r$, the homomorphism $Q\colon E \to K$ sends the i-th canonical basis vector $e_i \in E$ to $w_i := \deg(f_i) \in K$ and $\gamma \subseteq E_{\mathbb{Q}}$ is the convex cone generated by $e_1, \ldots, e_r$.
(ii) A face $\gamma_0 \preceq \gamma$ is called an $\mathfrak{F}$-*face* if the product over all f_i with $e_i \in \gamma_0$ does not belong to the ideal $\sqrt{\langle f_j;\ e_j \not\in \gamma_0\rangle} \subseteq A$.

Construction 1.1.8. Fix a system of homogeneous generators $\mathfrak{F} = (f_1, \ldots, f_r)$ for our K-graded affine $\mathbb{K}$-algebra A. Setting $\deg(T_i) := w_i := \deg(f_i)$ defines a K-grading on $\mathbb{K}[T_1, \ldots, T_r]$ and we have graded epimorphism

$$\mathbb{K}[T_1, \ldots, T_r] \ \to\ A, \qquad T_i \ \mapsto\ f_i.$$

On the geometric side, this gives a diagonal H-action on $\mathbb{K}^r$ via the characters $\chi^{w_1}, \ldots, \chi^{w_r}$ and an H-equivariant closed embedding

$$X \ \to\ \mathbb{K}^r, \qquad x \ \mapsto\ (f_1(x), \ldots, f_r(x)).$$

With $E = \mathbb{Z}^r$ and $\gamma = \operatorname{cone}(e_1, \ldots, e_r)$, we have $\mathbb{K}[T_1, \ldots, T_r] = \mathbb{K}[E \cap \gamma]$, where T_i is identified with χ^{e_i}. Thus, we may also regard $\mathbb{K}^r$ as the toric variety associated to the cone $\delta := \gamma^\vee \subseteq F_{\mathbb{Q}}$, where $F := \operatorname{Hom}(E, \mathbb{Z})$.

Proposition 1.1.9. *Situation as in Construction 1.1.8. Consider a face $\gamma_0 \preceq \gamma$ and its corresponding face $\delta_0 \preceq \delta$, i.e., we have $\delta_0 = \gamma_0^\perp \cap \delta$. Then the following statements are equivalent.*

(i) *The face $\gamma_0 \preceq \gamma$ is an $\mathfrak{F}$-face.*
(ii) *There is a point $z \in X$ with $z_i \neq 0 \Leftrightarrow e_i \in \gamma_0$ for all $1 \le i \le r$.*
(iii) *The toric orbit $\mathbb{T}^r \cdot z_{\delta_0} \subseteq \mathbb{K}^r$ corresponding to $\delta_0 \preceq \delta$ meets X.*

PROOF. The equivalence of (i) and (ii) is an immediate consequence of the following equivalence

$$\prod_{e_i \in \gamma_0} f_i \ \notin \ \sqrt{\langle f_j;\ e_j \not\in \gamma_0 \rangle} \quad \Longleftrightarrow \quad \bigcup_{e_i \in \gamma_0} V(X, f_i) \ \not\supseteq \ \bigcap_{e_j \not\in \gamma_0} V(X; f_j).$$

The equivalence of (ii) and (iii) is clear by the fact that $\mathbb{T}^r \cdot z_{\delta_0}$ consists exactly of the points $z \in \mathbb{K}^r$ with $z_i \neq 0 \Leftrightarrow e_i \in \gamma_0$ for all $1 \leq i \leq r$. □

PROPOSITION 1.1.10. *Situation as in Construction 1.1.8. Let $\gamma_0 \preceq \gamma$ be an $\mathfrak{F}$-face and let $\delta_0 \preceq \delta$ denote the corresponding face, i.e., we have $\delta_0 = \gamma_0^\perp \cap \delta$. Then, for every $x \in X \cap \mathbb{T}^r \cdot z_{\delta_0}$, the orbit data are given by*

$$S_{H,x} \ = \ Q(\gamma_0 \cap E), \qquad K_{H,x} \ = \ Q(\mathrm{lin}(\gamma_0) \cap E), \qquad \omega_{H,x} \ = \ Q(\gamma_0).$$

In particular, the orbit cones of the H-action on X are precisely the projected $\mathfrak{F}$-faces, i.e., we have

$$\{\omega_{H,x};\ x \in X\} \quad = \quad \{Q(\gamma_0);\ \gamma_0 \preceq \gamma \text{ is an } \mathfrak{F}\text{-face}\}.$$

PROOF. To obtain the first assertion, observe that the orbit monoid of $x \in X$ is the monoid in K generated by all w_i with $f_i(x) \neq 0$. The other two assertions follow directly. □

REMARK 1.1.11. Situation as in Construction 1.1.8. Let $g_1, \ldots, g_s$ generate the kernel of $\mathbb{K}[T_1, \ldots, T_r] \to A$ as an ideal. By an *$\mathfrak{F}$-set*, we mean a subset $I \subseteq \{1, \ldots, r\}$ with the property

$$\prod_{i \in I} T_i \ \notin \ \sqrt{\langle g_1^I, \ldots, g_s^I \rangle}, \qquad \text{with} \quad g_j^I \ := \ g_j(S_1, \ldots, S_r), \quad S_l \ := \ \begin{cases} T_l & l \in I, \\ 0 & l \notin I. \end{cases}$$

Then the $\mathfrak{F}$-faces are precisely the $\mathrm{cone}(e_i;\ i \in I)$, where I is an $\mathfrak{F}$-set, and the orbit cones are precisely the $\mathrm{cone}(w_i;\ i \in I)$, where $w_i = \deg(f_i)$ and I is an $\mathfrak{F}$-set.

For the case that $X \subseteq \mathbb{K}^r$ is defined by a single equation, we can easily figure out the $\mathfrak{F}$-faces in a purely combinatorial manner. Recall that the *Newton polytope* of a polynomial $g \in \mathbb{K}[T_1, \ldots, T_r]$ is defined as

$$N(g) \ := \ \mathrm{conv}(\nu;\ a_\nu \neq 0) \ \subseteq \ \mathbb{Q}^r, \qquad \text{where } g \ = \sum_{\nu \in \mathbb{Z}^r_{\geq 0}} a_\nu T^\nu.$$

PROPOSITION 1.1.12. *Situation as in Construction 1.1.8. Suppose that $X \subseteq \mathbb{K}^r$ is the zero set of $g \in \mathbb{K}[T_1, \ldots, T_r]$. Then, for every face $\gamma_0 \preceq \gamma$, the following statements are equivalent.*

(i) *The face $\gamma_0 \preceq \gamma$ is an $\mathfrak{F}$-face.*
(ii) *$N(g) \cap \gamma_0$ is empty or contains at least two points.*
(iii) *The number of vertices of $N(g)$ contained in γ_0 differs from one.*

PROOF. The equivalence of (ii) and (iii) is clear. For that of (i) and (ii) write first $g = \sum_{\nu \in \mathbb{Z}^r_{\geq 0}} a_\nu g_\nu$ with $g_\nu := T_1^{\nu_1} \cdots T_r^{\nu_r}$. Then, for any point $z \in \mathbb{K}^r$, we have

$$g_\nu(z) \neq 0 \quad \Longleftrightarrow \quad \nu_i \neq 0 \Rightarrow z_i \neq 0 \text{ holds for } 1 \leq i \leq r.$$

Moreover, if we have $g(z) = 0$, then the number of ν with $a_\nu g_\nu(z) \neq 0$ is different from one.

Now suppose that (i) holds and let $z \in X$ be as in Proposition 1.1.9 (ii). Then we have $g(z) = 0$. If all monomials g_ν with $a_\nu \neq 0$ vanish on z, then $N(g) \cap \gamma_0$ is empty. If not all monomials g_ν with $a_\nu \neq 0$ vanish on z, then there are at least two multi-indices ν, μ with $a_\nu \neq 0 \neq a_\mu$ and $g_\nu(z) \neq 0 \neq g_\mu(z)$. This implies

$$\nu_1 e_1 + \ldots + \nu_r e_r \ \in \ N(g) \cap \gamma_0, \qquad \mu_1 e_1 + \ldots + \mu_r e_r \ \in \ N(g) \cap \gamma_0.$$

Conversely, suppose that (ii) holds. Define $z \in \mathbb{K}^r$ by $z_i = 1$ if $e_i \in \gamma_0$ and $z_i = 0$ otherwise. If $N(g) \cap \gamma_0$ is empty, then we have $z \in X$ which gives (i). If $N(g) \cap \gamma_0$ contains at least two points, then γ_0 contains at least two vertices ν, μ of $N(g)$, and we find a point $z' \in \mathbb{T}^r \cdot z$ with $g(z') = 0$. □

1.2. Semistable quotients. Again, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let K be a finitely generated abelian group and consider an affine K-graded $\mathbb{K}$-algebra

$$A \quad = \quad \bigoplus_{w \in K} A_w.$$

Then the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ acts on the affine variety $X := \operatorname{Spec} A$. The following is Mumford's definition [**116**] of semistability specialized to the case of a "linearization of the trivial line bundle".

Definition 1.2.1. The *set of semistable points* associated to an element $w \in K_{\mathbb{Q}}$ is the H-invariant open subset

$$X^{ss}(w) \ := \ \{x \in X;\ f(x) \neq 0 \text{ for some } f \in A_{nw},\ n > 0\} \ \subseteq \ X.$$

Note that the set of semistable points $X^{ss}(w)$ is non-empty if and only if w belongs to the weight cone ω_X. The following two statements subsume the basic features of semistable sets.

Proposition 1.2.2. *For every $w \in \omega_X$, the H-action on $X^{ss}(w)$ admits a good quotient $\pi \colon X^{ss}(w) \to Y(w)$ with $Y(w)$ projective over $Y(0) = \operatorname{Spec} A_0 = X /\!\!/ H$.*

Proposition 1.2.3. *For any two $w_1, w_2 \in \omega_X$ with $X^{ss}(w_1) \subseteq X^{ss}(w_2)$ we have a commutative diagram*

$$\begin{array}{ccc} X^{ss}(w_1) & \subseteq & X^{ss}(w_2) \\ {\scriptstyle /\!\!/ H}\downarrow & & \downarrow{\scriptstyle /\!\!/ H} \\ Y(w_1) & \xrightarrow[\varphi^{w_1}_{w_2}]{} & Y(w_2) \end{array}$$

with a projective surjection $\varphi^{w_1}_{w_2}$. Moreover, we have $\varphi^{w_1}_{w_3} = \varphi^{w_2}_{w_3} \circ \varphi^{w_1}_{w_2}$ whenever composition is possible.

Both propositions are direct consequences of the following two constructions which we list separately for later use. The first one ensures existence of the quotient.

Construction 1.2.4. For a homogeneous $f \in A$, let $A_{(f)} \subseteq A_f$ denote the degree zero part. Given any two $f \in A_{nw}$ and $g \in A_{mw}$ with $m, n > 0$, we have a commutative diagram of K-graded affine algebras and the associated commutative diagram of affine H-varieties

$$\begin{array}{ccccc} A_f & \longrightarrow & A_{fg} & \longleftarrow & A_g \\ \uparrow & & \uparrow & & \uparrow \\ A_{(f)} & \longrightarrow & A_{(fg)} & \longleftarrow & A_{(g)} \end{array} \qquad\qquad \begin{array}{ccccc} X_f & \longleftarrow & X_{fg} & \longrightarrow & X_g \\ {\scriptstyle \pi_f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_g} \\ V_f & \longleftarrow & V_{fg} & \longrightarrow & V_g \end{array}$$

The morphism $V_{fg} \to V_f$ is an open embedding, and we have $X_{fg} = \pi_f^{-1}(V_{fg})$. Gluing the V_f gives a variety $Y(w)$ and the maps π_f glue together to a good quotient $\pi \colon X^{ss}(w) \to Y(w)$ for the action of H on $X^{ss}(w)$.

The second construction relates the first one to the Proj-construction, see Example 2.3.10. This yields projectivity of the quotient space.

Construction 1.2.5. In the situation of Construction 1.2.4, the weight $w \in K$ defines a Veronese subalgebra

$$A(w) \ := \ \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nw} \ \subseteq \ \bigoplus_{w' \in K} A_{w'} \ = \ A.$$

Proposition I. 1.2.2 ensures that $A(w)$ is finitely generated and thus defines a $\mathbb{K}^*$-variety $X(w) := \operatorname{Spec} A(w)$. For every homogeneous $f \in A(w)$, we have commutative diagrams, where the second one is obtained by applying the Spec functor:

$$\begin{array}{ccc} A(w)_f & \longrightarrow & A_f \\ \uparrow & & \uparrow \\ A(w)_{(f)} & \xrightarrow{\cong} & A_{(f)} \end{array} \qquad\qquad \begin{array}{ccc} X(w)_f & \longleftarrow & X_f \\ \downarrow & & \downarrow \\ U(w)_f & \xleftarrow{\cong} & V_f \end{array}$$

Gluing the $U(w)_f$ gives $\operatorname{Proj}(A(w))$ and the isomorphisms $V_f \to U(w)_f$ glue together to an isomorphism $Y(w) \to \operatorname{Proj}(A(w))$. In particular, $Y(w)$ is projective over $Y(0) = \operatorname{Spec} A_0$.

Recall that we defined the orbit cone of a point $x \in X$ to be the convex cone $\omega_x \subseteq K_{\mathbb{Q}}$ generated by all $w \in K$ that admit an $f \in A_w$ with $f(x) \neq 0$.

Definition 1.2.6. The *GIT-cone* of an element $w \in \omega_X$ is the (nonempty) intersection of all orbit cones containing it:

$$\lambda(w) \quad := \quad \bigcap_{\substack{x \in X, \\ w \in \omega_x}} \omega_x.$$

From finiteness of the number of orbit cones, see Proposition 1.1.10, we infer that the GIT-cones are polyhedral and moreover, that there are only finitely many of them.

Lemma 1.2.7. *For every $w \in \omega_X$, the associated set $X^{ss}(w) \subseteq X$ of semistable points is given by*

$$X^{ss}(w) \ = \ \{x \in X;\ w \in \omega_x\} \ = \ \{x \in X;\ \lambda(w) \subseteq \omega_x\}.$$

This description shows in particular, that there are only finitely many sets $X^{ss}(w)$ of semistable points. The following statement describes the collection of all sets $X^{ss}(w)$ of semistable points.

Theorem 1.2.8. *The collection $\Lambda(X) = \{\lambda(w);\ w \in \omega_X\}$ of all GIT-cones is a quasifan in $K_{\mathbb{Q}}$ having the weight cone ω_X as its support. Moreover, for any two $w_1, w_2 \in \omega_X$, we have*

$$\begin{array}{rclcrcl} \lambda(w_1) & \subseteq & \lambda(w_2) & \iff & X^{ss}(w_1) & \supseteq & X^{ss}(w_2), \\ \lambda(w_1) & = & \lambda(w_2) & \iff & X^{ss}(w_1) & = & X^{ss}(w_2). \end{array}$$

The collection $\Lambda(X)$, also denoted as $\Lambda(X,H)$ if we want to specify the quasitorus H, is called the *GIT-(quasi-)fan* of the H-variety X. In the proof of the Theorem we need the following.

Lemma 1.2.9. *Let $w \in \omega_X \cap K$. Then the associated GIT-cone $\lambda := \lambda(w) \in \Lambda(X)$ satisfies*

$$\lambda \quad = \quad \bigcap_{w \in \omega_x^\circ} \omega_x \quad = \quad \bigcap_{\lambda^\circ \subseteq \omega_x^\circ} \omega_x,$$

$$w \ \in \ \lambda^\circ \quad = \quad \bigcap_{w \in \omega_x^\circ} \omega_x^\circ \quad = \quad \bigcap_{\lambda^\circ \subseteq \omega_x^\circ} \omega_x^\circ.$$

Proof. For any orbit cone ω_x with $w \in \omega_x$, there is a unique minimal face $\omega \preceq \omega_x$ with $w \in \omega$. This face satisifies $w \in \omega^\circ$. According to Proposition 1.1.6, the face $\omega \preceq \omega_x$ is again an orbit cone. This gives the first formula. The second one follows from an elementary observation: if the intersection of the relative interiors of a finite number of convex polyhedral cones is nonempty, then it equals the relative interior of the intersection of the cones. □

Proof of Theorem 1.2.8. As mentioned, finiteness of the number of orbit cones ensures that $\Lambda(X)$ is a finite collection of convex polyhedral cones. The displayed equivalences are clear by Lemma 1.2.7. They allow us in particular to write

$$X^{ss}(\lambda) \;:=\; X^{ss}(w), \quad \text{where } \lambda \;=\; \lambda(w) \text{ for } w \in \omega_X.$$

The only thing we have to show is that $\Lambda(X)$ is a quasifan. This is done below by verifying several claims. For the sake of short notation, we set for the moment $\omega := \omega_x$ and $\Lambda := \Lambda(X)$.

Claim 1. Let $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \subseteq \lambda_2$. Then, for every $x_1 \in X^{ss}(\lambda_1)$ with $\lambda_1^\circ \subseteq \omega_{x_1}^\circ$, there exists an $x_2 \in X^{ss}(\lambda_2)$ with $\omega_{x_1} \preceq \omega_{x_2}$.

Let us verify the claim. We have $X^{ss}(\lambda_2) \subseteq X^{ss}(\lambda_1)$ and Proposition 1.2.3 provides us with a dominant, proper, hence surjective morphism $\varphi\colon Y(\lambda_2) \to Y(\lambda_1)$ of the quotient spaces fitting into the commutative diagram

$$\begin{array}{ccc} X^{ss}(\lambda_2) & \subseteq & X^{ss}(\lambda_1) \\ {\scriptstyle /\!\!/ H}\big\downarrow{\scriptstyle p_2} & & {\scriptstyle p_1}\big\downarrow{\scriptstyle /\!\!/ H} \\ Y(\lambda_2) & \xrightarrow[\varphi]{} & Y(\lambda_1) \end{array}$$

If a point $x_1 \in X^{ss}(\lambda_1)$ satisfies $\lambda_1^\circ \subseteq \omega_{x_1}^\circ$, then, by Lemma 1.2.7 and Proposition 1.1.6, its H-orbit is closed in $X^{ss}(\lambda_1)$. Corollary I. 2.3.7 thus tells us that $x_1 \in \overline{H \cdot x_2}$ holds for any point x_2 belonging to the (nonempty) intersection $X^{ss}(\lambda_2) \cap p_1^{-1}(p_1(x_1))$. Using once more Proposition 1.1.6 gives Claim 1.

Claim 2. Let $\lambda_1, \lambda_2 \in \Lambda$. Then $\lambda_1 \subseteq \lambda_2$ implies $\lambda_1 \preceq \lambda_2$.

For the verification, let $\tau_2 \preceq \lambda_2$ be the (unique) face with $\lambda_1^\circ \subseteq \tau_2^\circ$, and let $\omega_{1,1}, \ldots, \omega_{1,r}$ be the orbit cones with $\lambda_1^\circ \subseteq \omega_{1,i}^\circ$. Then we obtain, using Lemma 1.2.9 for the second observation,

$$\tau_2^\circ \cap \omega_{1,i}^\circ \;\neq\; \emptyset, \qquad \lambda_1 \;=\; \omega_{1,1} \cap \ldots \cap \omega_{1,r}.$$

By Claim 1, we have $\omega_{1,i} \preceq \omega_{2,i}$ with orbit cones $\omega_{2,i}$ satisfying $\lambda_2 \subseteq \omega_{2,i}$, and hence $\tau_2 \subseteq \omega_{2,i}$. The first of the displayed formulas implies $\tau_2 \subseteq \omega_{1,i}$, and the second one thus gives $\tau_2 = \lambda_1$. So, Claim 2 is verified.

Claim 3. Let $\lambda \in \Lambda$. Then every face $\lambda_0 \preceq \lambda$ belongs to Λ.

To see this, consider any $w \in \lambda_0^\circ$. Lemma 1.2.9 yields $w \in \lambda(w)^\circ$. By the definition of GIT-cones, we have $\lambda(w) \subseteq \lambda$. Claim 2 gives even $\lambda(w) \preceq \lambda$. Thus, we have two faces, λ_0 and $\lambda(w)$ of λ having a common point w in their relative interiors. This means $\lambda_0 = \lambda(w)$, and Claim 3 is verified.

Claim 4. Let $\lambda_1, \lambda_2 \in \Lambda$. Then $\lambda_1 \cap \lambda_2$ is a face of both, λ_1 and λ_2.

Let $\tau_i \preceq \lambda_i$ be the minimal face containing $\lambda_1 \cap \lambda_2$. Choose w in the relative interior of $\lambda_1 \cap \lambda_2$, and consider the GIT-cone $\lambda(w)$. By Lemma 1.2.9 and the definition of GIT-cones, we see

$$w \;\in\; \lambda(w)^\circ \cap \tau_i^\circ, \qquad \lambda(w) \;\subseteq\; \lambda_1 \cap \lambda_2 \;\subseteq\; \tau_i.$$

By Claim 2, the second relation implies in particular $\lambda(w) \preceq \lambda_i$. Hence, we can conclude $\lambda(w) = \tau_i$, and hence $\lambda_1 \cap \lambda_2$ is a face of both λ_i. Thus, Claim 4 is verified, and the properties of a quasifan are established for Λ. □

REMARK 1.2.10. Theorem 1.2.8 provides another proof for the fact that the GKZ-decomposition of a vector configuration $\mathcal{W} = (w_1, \ldots, w_r)$ in a rational vector space $K_\mathbb{Q}$ is a fan, see Theorem II. 2.2.2. Indeed, let $K \subseteq K_\mathbb{Q}$ be the lattice generated by $w_1, \ldots, w_r$ and consider the action of the torus $T = \operatorname{Spec} \mathbb{K}[K]$ on $\mathbb{K}^r$ given by $t{\cdot}z = (\chi^{w_1}(t)z_1, \ldots, \chi^{w_r}(t)z_r)$. Then the orbit cones of this action are precisely the $\mathcal{W}$-cones and thus Theorem 1.2.8 gives the result.

EXAMPLE 1.2.11 (A GIT-quasifan, which is not a fan). Consider the affine variety

$$X \quad := \quad V(\mathbb{K}^4;\ Z_1Z_2 + Z_3Z_4 - 1).$$

Then X is isomorphic to $\mathrm{SL}(2)$ and thus it is smooth, factorial, and $\Gamma(X, \mathcal{O})^* = \mathbb{K}^*$ holds. Moreover $\mathbb{K}^*$ acts on X via

$$t{\cdot}(z_1, z_2, z_3, z_4) \quad := \quad (tz_1, t^{-1}z_2, tz_3, t^{-1}z_4).$$

This action has only one orbit cone, which is the whole line $\mathbb{Q}$. In particular, the resulting GIT-quasifan does not consist of pointed cones.

EXAMPLE 1.2.12. Consider the affine space $X := \mathbb{K}^4$ with the action of the torus $H := \mathbb{K}^*$ given by

$$t \cdot (z_1, z_2, z_3 z_4) \quad := \quad (tz_1, tz_2, t^{-1}z_3, t^{-1}z_4).$$

This setting stems from the $\mathbb{Z}$-grading of $\mathbb{K}[Z_1, \ldots, Z_4]$ given by $\deg(Z_1) = \deg(Z_2) = 1$ and $\deg(Z_3) = \deg(Z_4) = -1$. The GIT-quasifan lives in $\mathbb{Q}$ and consists of the three cones

$$\lambda(-1) \ = \ \mathbb{Q}_{\le 0}, \qquad \lambda(0) \ = \ \{0\}, \qquad \lambda(1) \ = \ \mathbb{Q}_{\ge 0}.$$

As always, we have $X^{ss}(0) = X$. The two further sets of semistable points are $X^{ss}(-1) = \{z \in \mathbb{K}^4;\ z_3 \ne 0 \ne z_4\}$ and $X^{ss}(1) = \{z \in \mathbb{K}^4;\ z_1 \ne 0 \ne z_2\}$. The whole GIT-system looks as follows

$$\begin{array}{ccccc} X^{ss}(-1) & \subseteq & X^{ss}(0) & \supseteq & X^{ss}(1) \\ \downarrow & & \downarrow & & \downarrow \\ Y(-1) & \longrightarrow & Y(0) & \longleftarrow & Y(1) \end{array}$$

Note that $Y(0)$ is the affine cone $V(\mathbb{K}^4; T_1T_2 - T_3T_4)$ with the apex $y_0 = 0$, the projections $\varphi_0^i \colon Y(i) \to Y(0)$ are isomorphisms over $Y(0) \setminus \{y_0\}$, and the fibers over the apex y_0 are isomorphic to $\mathbb{P}_1$.

1.3. A_2-quotients. In the preceeding section, we constructed semiprojective quotients via semistable points. Here we construct more generally quotient spaces with the following property.

DEFINITION 1.3.1. We say that a prevariety X has the *A_2-property* , if any two points $x, x' \in X$ admit a common affine open neighborhood in X.

A prevariety with the A_2-property is necessarily separated, see [**115**, Proposition I.5.6], and thus we will just speak of A_2-varieties. Examples of A_2-varieties are the quasiprojective varieties. By [**160**, Theorem A], a normal variety X has the A_2-property if and only if it admits a closed embedding into a toric variety. In particular, toric varieties are A_2-varieties.

We fix the setup for the rest of this section. By K we denote a finitely generated abelian group and we consider an affine K-graded $\mathbb{K}$-algebra

$$A \quad = \quad \bigoplus_{w \in K} A_w.$$

Then the quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ acts on the affine variety $X := \operatorname{Spec} A$. The necessary data for our quotient construction are again given in terms of orbit cones.

Definition 1.3.2. Let Ω_X denote the collection of all orbit cones ω_x, where $x \in X$. A *bunch of orbit cones* is a nonempty collection $\Phi \subseteq \Omega_X$ such that

(i) given $\omega_1, \omega_2 \in \Phi$, one has $\omega_1^\circ \cap \omega_2^\circ \neq \emptyset$,
(ii) given $\omega \in \Phi$, every orbit cone $\omega_0 \in \Omega_X$ with $\omega^\circ \subseteq \omega_0^\circ$ belongs to Φ.

A *maximal bunch of orbit cones* is a bunch of orbit cones $\Phi \subseteq \Omega_X$ which cannot be enlarged by adding further orbit cones.

Definition 1.3.3. Let $\Phi, \Phi' \subseteq \Omega_X$ be bunches of orbit cones. We say that Φ *refines* Φ' (written $\Phi \le \Phi'$), if for any $\omega' \in \Phi'$ there is an $\omega \in \Phi$ with $\omega \subseteq \omega'$.

The following example shows that the above notions generalize the combinatorial concepts treated in Section II. 2.

Example 1.3.4. Let F be a lattice and $\delta \subseteq F_{\mathbb{Q}}$ a pointed simplicial cone of full dimension. Let $E := \operatorname{Hom}(F, \mathbb{Z})$ be the dual lattice, and $\gamma := \delta^\vee$ the dual cone. Given a homomorphism $Q \colon E \to K$ to a finitely generated abelian group K, we obtain a K-grading of $A := \mathbb{K}[E]$ via $\deg(\chi^e) = Q(e)$. Consider

$$H \ = \ \operatorname{Spec} \mathbb{K}[K], \qquad\qquad X_\delta \ = \ \operatorname{Spec} \mathbb{K}[\gamma \cap E].$$

The orbit cones of the H-action on X_δ are just the cones $Q(\gamma_0)$, where $\gamma_0 \preceq \gamma$. Moreover, for the primitive generators $e_1, \ldots, e_r$ of γ, set $w_i := Q(e_i) \in K_{\mathbb{Q}}$. Then the (maximal) bunches of orbit cones are precisely the (maximal) $\mathcal{W}$-bunches of the vector configuration $\mathcal{W} = (w_1, \ldots, w_r)$ in $K_{\mathbb{Q}}$.

Definition 1.3.5. To any collection of orbit cones $\Phi \subseteq \Omega_X$, we associate the following subset

$$U(\Phi) \ := \ \{x \in X;\ \omega_0 \preceq \omega_x \text{ for some } \omega_0 \in \Phi\} \ \subseteq \ X.$$

Example 1.3.6. Consider the GIT-fan $\Lambda(X) = \{\lambda(w);\ w \in \omega_X\}$. Every GIT-cone $\lambda = \lambda(w)$ defines a bunch of orbit cones

$$\Phi(w) \ := \ \{\omega_x \in \Omega_X;\ w \in \omega_x^\circ\} \ = \ \{\omega_x \in \Omega_X;\ \lambda^\circ \subseteq \omega_x^\circ\} \ =: \ \Phi(\lambda).$$

For any two $\lambda, \lambda' \in \omega_X$, we have $\Phi(\lambda) \le \Phi(\lambda')$ if and only if $\lambda \preceq \lambda'$ holds. Moreover, for the set associated to $\Phi(w)$ we have

$$U(\Phi(w)) \quad = \quad X^{ss}(w).$$

Construction 1.3.7. Let $\Phi \subseteq \Omega_X$ be a bunch of orbit cones and consider $x \in U(\Phi)$. Fix homogeneous $h_1, \ldots, h_r \in A$ such that $h_i(x) \neq 0$ holds and the orbit cone ω_x is generated by $\deg(h_i)$, where $1 \le i \le r$. For $u \in \mathbb{Z}^r_{>0}$, consider the H-invariant open set

$$U(x) \ := \ X_{f^u} \ \subseteq \ X, \qquad f^u \ := \ h_1^{u_1} \cdots h_r^{u_r}.$$

Then the sets $U(x)$ do not depend on the particular choice of $u \in \mathbb{Z}^r_{>0}$. Moreover we have $U(x) \subseteq U(\Phi)$ and for any $w \in \omega_x^\circ$, we find some u with

$$\deg(f^u) \ = \ u_1 \deg(h_1) + \ldots + u_r \deg(h_r) \ \in \ \mathbb{Q}_{>0} w.$$

Proof. We have to verify that every $x' \in X_{f^u}$ belongs to $U(\Phi)$. By construction, we have $\omega_x \subseteq \omega_{x'}$. Consider $\omega_0 \in \Phi$ with $\omega_0 \preceq \omega_x$. Then ω_0° is contained in the relative interior of some face $\omega_0' \preceq \omega_{x'}$. By saturatedness of Φ, we have $\omega_0' \in \Phi$, and hence $x' \in U(\Phi)$. $\square$

We are ready to list the basic properties of the assignment $\Phi \mapsto U(\Phi)$; The statements are analogous to Propositions 1.2.2 and 1.2.3. We say that a subset $U \subseteq X$ is *saturated* w.r.t. a map $p\colon X \to Y$ if $U = p^{-1}(p(U))$ holds.

Proposition 1.3.8. *Let $\Phi \subseteq \Omega_X$ be a bunch of orbit cones. Then $U(\Phi) \subseteq X$ is H-invariant, open and admits a good quotient $U(\Phi) \to Y(\Phi)$, where the quotient space $Y(\Phi)$ is an A_2-variety. Moreover, the sets $U(x) \subseteq U(\Phi)$ with $H \cdot x$ closed in $U(\Phi)$ are saturated w.r.t. $U(\Phi) \to Y(\Phi)$.*

Proposition 1.3.9. *For any two bunches of orbit cones $\Phi_1, \Phi_2 \subseteq \Omega_X$ with $\Phi_1 \geq \Phi_2$, we have $U(\Phi_1) \subseteq U(\Phi_2)$, and there is a commutative diagram*

$$\begin{array}{ccc} U(\Phi_1) & \subseteq & U(\Phi_2) \\ {\scriptstyle /\!\!/ H}\downarrow & & \downarrow{\scriptstyle /\!\!/ H} \\ Y(\Phi_1) & \xrightarrow[\varphi_{\Phi_2}^{\Phi_1}]{} & Y(\Phi_2) \end{array}$$

with a dominant morphism $\varphi_{\Phi_2}^{\Phi_1}$. Moreover, we have $\varphi_{\Phi_3}^{\Phi_1} = \varphi_{\Phi_3}^{\Phi_2} \circ \varphi_{\Phi_2}^{\Phi_1}$ whenever composition is possible.

Lemma 1.3.10. *Let $\Phi \subseteq \Omega_X$ satisfy 1.3.2 (i) and let $x \in U(\Phi)$. Then the orbit $H \cdot x$ is closed in $U(\Phi)$ if and only if $\omega_x \in \Phi$ holds.*

Proof. First let $H \cdot x$ be closed in $U(\Phi)$. By the definition of $U(\Phi)$, we have $\omega_0 \preceq \omega_x$ for some $\omega_0 \in \Phi$. Consider the closure $C_X(H \cdot x)$ of $H \cdot x$ taken in X, and choose $x_0 \in C_X(H \cdot x)$ with $\omega_{x_0} = \omega_0$. Again by the definition of $U(\Phi)$, we have $x_0 \in U(\Phi)$. Since $H \cdot x$ is closed in $U(\Phi)$, we obtain $x_0 \in H \cdot x$, and hence $\omega = \omega_0 \in \Phi$.

Now, let $\omega_x \in \Phi$. We have to show that any $x_0 \in C_X(H \cdot x) \cap U(\Phi)$ lies in $H \cdot x$. Clearly, $x_0 \in C_X(H \cdot x)$ implies $\omega_{x_0} \preceq \omega_x$. By the definition of $U(\Phi)$, we have $\omega_0 \preceq \omega_{x_0}$ for some $\omega_0 \in \Phi$. Since Φ satisfies 1.3.2 (i), we have $\omega_0^\circ \cap \omega_x^\circ \neq \emptyset$. Together with $\omega_0 \preceq \omega_x$ this implies $\omega_0 = \omega_{x_0} = \omega_x$, and hence $x_0 \in H \cdot x$. $\square$

Proof of Proposition 1.3.8. We regard $U(\Phi)$ as a union of sets $U(x)$ as provided in Construction 1.3.7, where $x \in U(\Phi)$ runs through those points that have a closed H-orbit in $U(\Phi)$; according to Lemma 1.3.10 these are precisely the points $x \in U(\Phi)$ with $\omega_x \in \Phi$.

First consider two such $x_1, x_2 \in U(\Phi)$. Then we have $\omega_{x_i} \in \Phi$, and we can choose homogeneous $f_1, f_2 \in A$ such that $\deg(f_1) = \deg(f_2)$ lies in $\omega_{x_1}^\circ \cap \omega_{x_2}^\circ$ and $U(x_i) = X_{f_i}$ holds. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} X_{f_1} & \longleftarrow & X_{f_1 f_2} & \longrightarrow & X_{f_2} \\ \downarrow{\scriptstyle /\!\!/ H} & & \downarrow{\scriptstyle /\!\!/ H} & & \downarrow{\scriptstyle /\!\!/ H} \\ Y_{f_1} & \longleftarrow & Y_{f_1 f_2} & \longrightarrow & Y_{f_2} \end{array}$$

where the upper horizontal maps are open embeddings, the downwards maps are good quotients for the respective affine H-varieties, and the lower horizontal arrows indicate the induced morphisms of the affine quotient spaces.

By the choice of f_1 and f_2, the quotient f_2/f_1 is an invariant function on X_{f_1}, and the inclusion $X_{f_1 f_2} \subseteq X_{f_1}$ is just the localization by f_2/f_1. Since f_2/f_1 is invariant, the latter holds as well for the quotient spaces; that means that the map $Y_{f_1 f_2} \to Y_{f_1}$ is localization by f_2/f_1.

Now, cover $U(\Phi)$ by sets $U(x_i)$ with $H \cdot x_i$ closed in $U(\Phi)$. The preceding consideration allows gluing of the maps $U(x_i) \to U(x_i)/\!\!/H$ along $U_{ij} \to U_{ij}/\!\!/H$, where $U_{ij} := U(x_i) \cap U(x_j)$. This gives a good quotient $U(\Phi) \to U(\Phi)/\!\!/H$. By construction, the open sets $U(x_i) \subseteq U(\Phi)$ are saturated with respect to the quotient map.

In order to see that $Y = U(\Phi)/\!\!/H$ is an A_2-variety (and thus in particular separated), consider $y_1, y_2 \in Y$. Then there are f_i as above with $y_i \in Y_{f_i}$. The union $Y_{f_1} \cup Y_{f_2}$ is quasiprojective, because, for example, the set $Y_{f_1} \setminus Y_{f_2}$ defines an ample divisor. It follows that there is a common affine neighbourhood of y_1, y_2 in $Y_{f_1} \cup Y_{f_2}$ and hence in Y. □

Proof of Proposition 1.3.9. We only have to show that $\Phi_1 \geq \Phi_2$ implies $U(\Phi_1) \subseteq U(\Phi_2)$. Consider $x \in U(\Phi_1)$. Then we have $\omega_1 \preceq \omega_x$ for some $\omega_1 \in \Phi_1$. Since $\Phi_1 \geq \Phi_2$ holds, there is an $\omega_2 \in \Phi_2$ with $\omega_2 \subseteq \omega_1$. Let $\omega_0 \preceq \omega_1$ be the face with $\omega_2^\circ \subseteq \omega_0^\circ$. By Property 1.3.2 (ii), the face ω_0 belongs to Φ_2. Because of $\omega_0 \preceq \omega_x$, we have $x \in U(\Phi_2)$. □

1.4. Quotients of H-factorial affine varieties. We consider a quasitorus action on a normal affine variety, where we assume that for every invariant divisor some multiple is principal. The aim is to show that the constructions presented in the preceding two sections provide basically all open subsets admitting quasiprojective or torically embeddable quotients. The results generalize Theorem II. 3.2.4 which settled linear representations of quasitori.

Definition 1.4.1. Let a reductive affine algebraic group G act on a variety X.

(i) By a *good G-set* in X we mean an open subset $U \subseteq X$ with a good quotient $U \to U/\!\!/G$.
(ii) We say that a subset $U' \subseteq U$ of a good G-set $U \subseteq X$ is *G-saturated* in U if it satisfies $U' = \pi^{-1}(\pi(U'))$, where $\pi\colon U \to U/\!\!/G$ is the good quotient.

Note that an open subset $U' \subseteq U$ of a good G-set U is G-saturated in U if and only if any orbit $G\cdot x \subseteq U'$ which is closed in U' is as well closed in U. Moreover, for every G-saturated open subset $U' \subseteq U$ of a good G-set $U \subseteq X$ with quotient $\pi\colon U \to U/\!\!/G$, the image $\pi(U') \subseteq U$ is open and $\pi\colon U' \to \pi(U')$ is a good quotient. The latter reduces the problem of describing all good G-sets to the description of the "maximal" ones. Here are the precise concepts.

Definition 1.4.2. Let a reductive affine algebraic group G act on a variety X.

(i) By a *qp-maximal subset* of X we mean a good G-set $U \subseteq X$ with $U/\!\!/G$ quasiprojective such that U is maximal w.r.t. G-saturated inclusion among all good G-sets $W \subseteq X$ with $W/\!\!/G$ quasiprojective.
(ii) By a *$(G,2)$-maximal subset* of X we mean a good G-set $U \subseteq X$ with $U/\!\!/G$ an A_2-variety such that U is maximal w.r.t. G-saturated inclusion among all good G-sets $W \subseteq X$ with $W/\!\!/G$ an A_2-variety.

We are ready to formulate the first result. Let K be a finitely generated abelian group and A a normal K-graded affine algebra. Then the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ acts on the normal affine variety $X = \operatorname{Spec} A$. For every GIT-cone $\lambda \subseteq K_{\mathbb{Q}}$, we define its set of semistable points to be

$$X^{ss}(\lambda) \;:=\; X^{ss}(w), \quad \text{where } w \in \lambda^\circ.$$

This does not depend on the particular choice of $w \in \lambda^\circ$. Moreover, to any H-invariant open subset $U \subseteq X$, we associate the cone

$$\lambda(U) \ := \ \bigcap_{x \in U} \omega_x \ \subseteq \ K_{\mathbb{Q}}.$$

Theorem 1.4.3. *Assume that X is normal and for every H-invariant divisor on X some positive multiple is principal. Then, with the GIT-fan $\Lambda(X,H)$ of the H-action on X, we have mutually inverse bijections*

$$\begin{array}{rcl} \Lambda(X,H) & \longleftrightarrow & \{\text{qp-maximal subsets of } X\} \\ \lambda & \mapsto & X^{ss}(\lambda) \\ \lambda(U) & \mapsfrom & U. \end{array}$$

These bijections are order-reversing maps of partially ordered sets in the sense that we always have

$$\lambda \ \preceq \ \lambda' \quad \Longleftrightarrow \quad X^{ss}(\lambda) \ \supseteq \ X^{ss}(\lambda').$$

Proof. First recall from Proposition 1.2.2 that all sets $X^{ss}(w)$ are good H-sets with a quasiprojective quotient space $X^{ss}(w) /\!\!/ H$. The assertion thus is a direct consequence of the following two claims.

Claim 1. If $U \subseteq X$ is a good H-set such that $U /\!\!/ H$ is quasiprojective, then U is H-saturated in some set $X^{ss}(w)$ of semistable points.

Set $Y := U /\!\!/ H$, and let $p \colon U \to Y$ be the quotient map. For a global section f of a divisor D, set for short

$$Z(f) \ := \ \operatorname{Supp}(\operatorname{div}_D(f)) \ = \ \operatorname{Supp}(D + \operatorname{div}(f)).$$

Choose an (effective) ample divisor E on Y allowing global sections $h_1, \ldots, h_r$ such that the sets $Y \setminus Z(h_i)$ form an affine cover of Y. Consider the pullback data $D' := p^*E$ and $f_i' := p^*(h_i)$. Let $D_1, \ldots, D_s$ be the prime divisors contained in $X \setminus U$. Since the complement $U \setminus Z(f_i')$ in X is of pure codimension one, we have

$$U \ \setminus \ Z(f_i') \ = \ X \setminus (D_1 \cup \ldots \cup D_s \cup \overline{Z(f_i')}).$$

Consequently, by closing the components of D' in X and adding a suitably big multiple of $D_1 + \ldots + D_s$, we obtain an H-invariant Weil divisor D on X allowing global sections $f_1, \ldots, f_r$ such that

$$D|_U \ = \ D', \quad f_i|_U \ = \ f_i', \quad X \setminus Z(f_i) \ = \ U \setminus Z(f_i') \ = \ p^{-1}(Y \setminus Z(h_i)).$$

Replacing D with a suitable positive multiple, we may assume that it is principal, say $D = \operatorname{div}(f)$. Since D is H^0-invariant, f must be H^0-homogeneous. Fix a splitting $H = H^0 \times H^1$ with the unit component $H^0 \subseteq H$ and a finite group $H^1 \subseteq H$ and consider

$$f' \ := \ \prod_{h \in H^1} h \cdot f, \qquad \text{where } (h \cdot f)(x) := f(h \cdot x).$$

Then f' is even H-homogeneous and its divisor is a multiple of D. So, we may even assume that f is H-homogeneous say of weight $w \in K$. Since the functions f_i are rational H-invariants, also all the ff_i are homogeneous of weight $w \in K$. We infer saturatedness of U in $X^{ss}(w)$ from

$$U \ = \ \bigcup_{i=1}^{r} (U \setminus Z(f_i')) \ = \ \bigcup_{i=1}^{r} X_{ff_i}.$$

Claim 2. For every $w \in \omega_X$, the associated set of semistable points $X^{ss}(w)$ is qp-maximal.

By Claim 1, it suffices to prove that $X^{ss}(w)$ is not contained as a proper H-saturated subset in some set $X^{ss}(w')$. Any H-saturated inclusion $X^{ss}(w) \subseteq X^{ss}(w')$ gives a commutative diagram

$$\begin{array}{ccc} Y(w) & \xrightarrow{\quad\varphi\quad} & Y(w') \\ & {\scriptstyle \pi}\searrow \qquad \swarrow{\scriptstyle \pi'} & \\ & Y(0) & \end{array}$$

where φ is an open embedding from the quotient space $Y(w)$ into $Y(w')$. Moreover, we know that $\varphi\colon Y(w) \to Y(w')$ is projective and thus φ is an isomorphism. This implies $X^{ss}(w) = X^{ss}(w')$. □

We prepare the second result. Still K is a finitely generated abelian group, A a normal K-graded affine algebra and we consider the action of the quasitorus $H = \operatorname{Spec} \mathbb{K}[K]$ on the affine variety $X = \operatorname{Spec} A$. To any collection of orbit cones $\Phi \subseteq \Omega_X$, we associated in Definition 1.3.5 the subset

$$U(\Phi) \ = \ \{x \in X;\ \omega_0 \preceq \omega_x \text{ for some } \omega_0 \in \Phi\} \ \subseteq \ X.$$

Conversely, to any H-invariant subset $U \subseteq X$, we associate the following collection of orbit cones

$$\Phi(U) \ := \ \{\omega_x;\ x \in U \text{ with } H \cdot x \text{ closed in } U\} \ \subseteq \ \Omega_X.$$

Theorem 1.4.4. *Assume that X is normal and for every H-invariant divisor on X some positive multiple is principal. Then we have mutually inverse bijections*

$$\begin{array}{rcl} \{\textit{maximal bunches of orbit cones in } \Omega_X\} & \longleftrightarrow & \{(H,2)\textit{-maximal subsets of } X\} \\ \Phi & \mapsto & U(\Phi) \\ \Phi(U) & \mapsfrom & U. \end{array}$$

These bijections are order-reversing maps of partially ordered sets in the sense that we always have

$$\Phi \ \le \ \Phi' \quad \Longleftrightarrow \quad U(\Phi) \ \supseteq \ U(\Phi').$$

The collections $\Phi(\lambda)$, where $\lambda \in \Lambda(X,H)$, are maximal bunches of orbit cones and they correspond to the qp-maximal subsets of X; in particular, the latter ones are $(H,2)$-maximal.

Proof. According to Proposition 1.3.8, for every maximal bunch of orbit cones Φ, the subset $U(\Phi)$ admits a good quotient with an A_2-variety as quotient space. The following claim gives the converse.

Claim 1. If the H-invariant open set $U \subseteq X$ admits a good quotient $U \to U /\!\!/ H$ with $U /\!\!/ H$ an A_2-variety, then the collection $\Phi(U)$ of orbit cones satisfies 1.3.2 (i).

By definition, the elements of $\Phi(U)$ are precisely the orbit cones ω_x, where $H \cdot x$ is a closed subset of U. We have to show that for any two cones $\omega_{x_i} \in \Phi(U)$, their relative interiors intersect nontrivially. Consider the quotient $\pi\colon U \to U /\!\!/ H$, and let $V \subseteq U /\!\!/ H$ be a common affine neighbourhood of $\pi(x_1)$ and $\pi(x_2)$. Then $\pi^{-1}(V)$ is again affine. Thus $X \setminus \pi^{-1}(V)$ is of pure codimension one and, by the assumption on X, it is the zero set of a homogeneous function $f \in A$. It follows that the degree of f lies in the relative interior of both cones, ω_{x_1} and ω_{x_2}.

Claim 2. For every collection $\Phi \subseteq \Omega_X$ satisfying 1.3.2 (i), we have $\Phi(U(\Phi)) = \Phi$.

Consider any $\omega \in \Phi(U(\Phi))$. By the definition of $\Phi(U(\Phi))$, we have $\omega = \omega_x$ for some $x \in U(\Phi)$ such that $H \cdot x$ is closed in $U(\Phi)$. According to Lemma 1.3.10, the latter implies $\omega \in \Phi$. Conversely, let $\omega \in \Phi$. Then we have $x \in U(\Phi)$, for any $x \in X$

with $\omega_x = \omega$. Moreover, Lemma 1.3.10 tells us that $H \cdot x$ is closed in $U(\Phi)$. This implies $\omega \in \Phi(U(\Phi))$.

Claim 3. Let $U \subseteq X$ admit a good quotient $U \to U /\!\!/ H$ with an A_2-variety $U /\!\!/ H$, and let $\Phi \subseteq \Omega_X$ be any bunch of orbit cones with $\Phi(U) \subseteq \Phi$. Then we have an H-saturated inclusion $U \subseteq U(\Phi)$.

First let us check that U is in fact a subset of $U(\Phi)$. Given $x \in U$, we may choose $x_0 \in C_X(H \cdot x)$ such that $H \cdot x_0$ is closed in U. By definition of $\Phi(U)$, we have $\omega_{x_0} \in \Phi(U)$, and hence $\omega_{x_0} \in \Phi$. Thus, $\omega_{x_0} \preceq \omega_x$ implies $x \in U(\Phi)$.

In order to see that the inclusion $U \subseteq U(\Phi)$ is H-saturated, let $x \in U$ with $H \cdot x$ closed in U. We have to show that any $x_0 \in C_X(H \cdot x)$ with $H \cdot x_0$ closed in $U(\Phi)$ belongs to $H \cdot x$. On the one hand, given such x_0, Claim 2 gives us

$$\omega_{x_0} \;\in\; \Phi(U(\Phi)) \;=\; \Phi.$$

On the other hand, the definition of $\Phi(U)$ yields $\omega_x \in \Phi$, and $x_0 \in C_X(H \cdot x)$ implies $\omega_{x_0} \preceq \omega_x$. Since Φ is a bunch of orbit cones, $\omega_{x_0}^\circ$ and ω_x° intersect nontrivially, and we obtain $\omega_{x_0} = \omega_x$. This gives $x_0 \in H \cdot x$ and Claim 3 is proved.

Now we turn to the assertions of the theorem. First we show that the assignment $\Phi \mapsto U(\Phi)$ is well defined, i.e., that $U(\Phi)$ is $(H,2)$-maximal. Consider any H-saturated inclusion $U(\Phi) \subseteq U$ with an $(H,2)$-set $U \subseteq X$. Using Claim 2, we obtain

$$\Phi \;=\; \Phi(U(\Phi)) \;\subseteq\; \Phi(U).$$

By maximality of Φ, this implies $\Phi = \Phi(U)$. Thus, we obtain $U(\Phi) = U(\Phi(U))$. By Claim 3, the latter set contains U as an H-saturated subset. This gives $U(\Phi) = U$ and, consequently, $U(\Phi)$ is $(H,2)$-maximal.

Thus, we have a well-defined map $\Phi \to U(\Phi)$ from the maximal connected collections in Ω_X to the $(H,2)$-maximal subsets of X. According to Claim 2, this map is injective. To see surjectivity, consider any $(H,2)$-maximal $U \subseteq X$. Choose a maximal conneceted collection Φ with $\Phi(U) \subseteq \Phi$. Claim 3 then shows $U = U(\Phi)$. The fact that $\Phi \mapsto U(\Phi)$ and $U \mapsto \Phi(U)$ are inverse to each other is then obvious.

Let us turn to the second statement of the assertion. The subset $U(\Phi')$ is contained in $U(\Phi)$ if and only if any closed H-orbit in $U(\Phi')$ is contained in $U(\Phi)$. By Lemma 1.3.10, the points with closed H-orbit in $U(\Phi')$ are precisely the points $x \in X$ with $\omega_x \in \Phi'$. By the definition of $U(\Phi)$, such a point x belongs to $U(\Phi)$ if and only if ω_x has a face contained in Φ.

Finally, for the third statement, we have to show that every set $X^{ss}(w)$ of semistable points is $(H,2)$-maximal. By what we proved so far, $X^{ss}(w)$ is an H-saturated subset of a set $U(\Phi)$ for some maximal connected collection $\Phi \subseteq \Omega_X$. Thus, we have a diagram of the associated quotient spaces

$$\begin{array}{ccc} Y(w) & \xrightarrow{\;\varphi\;} & Y(\Phi) \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & Y(0) & \end{array}$$

with an open embedding $\varphi \colon Y(w) \to Y(\Phi)$. Since $Y(w) \to Y(0)$ is projective, the morphism $\varphi \colon Y(w) \to Y(\Phi)$ is projective as well. Consequently it is an isomorphism. The claim follows. $\square$

Corollary 1.4.5. *Let $\Phi \subseteq \Omega_X$ be a saturated connected collection. Then the quotient space $U(\Phi) /\!\!/ H$ is quasiprojective if and only if we have*

$$\bigcap_{\omega \in \Phi} \omega^\circ \;\neq\; \emptyset.$$

As a further application of Theorem 1.4.4, we obtain a statement in the spirit of [**153**, Cor. 2.3].

Corollary 1.4.6. *Let a quasitorus H act on a normal affine variety X such that for every H-invariant divisor on X some positive multiple is principal. Moreover, let G be any algebraic group acting on X such that the actions of G and H commute. Then every $(H,2)$-maximal open subset of X is G-invariant.*

Proof. If $U \subseteq X$ is an $(H,2)$-maximal open subset, then Theorem 1.4.4 says that we have $U = U(\Phi)$ for some maximal connected collection Φ of H-orbit cones. Since the actions of H and G commute, the H-orbit cone is constant along G-orbits. Thus, $G \cdot U = U$ holds. □

2. Bunched rings

2.1. Bunched rings and their varieties. We present an explicit construction of varieties with prescribed Cox ring. The input is a factorially graded affine algebra R and a collection Φ of pairwise overlapping cones in the grading group K of R. The output variety X has R as its Cox ring and Φ fixes the isomorphy type of X among all varieties sharing R as their Cox ring. We formulate the construction in an elementary manner which turns out to be suitable for explicit applications and requires only minimal background knowledge; the proofs of the basic properties relie on the interpretation in terms of Geometric Invariant Theory and are given in Section 2.2.

Let K be a finitely generated abelian group and R a normal affine K-graded $\mathbb{K}$-algebra with $R^* = \mathbb{K}^*$. Recall from Definition I. 5.3.1 that a *K-prime* element in R is a homogeneous nonzero nonunit $f \in R$ such that $f|gh$ with homogeneous $g,h \in R$ always implies $f|g$ or $f|h$. Moreover, we say that R is *factorially graded* if and only if every nonzero homogeneous nonunit $f \in R$ is a product of K-primes. If R is factorially graded, then it admits a system $\mathfrak{F} = (f_1, \ldots, f_r)$ of pairwise nonassociated K-prime generators. By Definition 1.1.7, the *projected cone* associated to $\mathfrak{F}$ is $(E \xrightarrow{Q} K, \gamma)$, where $E := \mathbb{Z}^r$, the homomorphism $Q\colon E \to K$ sends the i-th canonical basis vector $e_i \in E$ to $w_i := \deg(f_i) \in K$ and $\gamma \subseteq E_{\mathbb{Q}}$ is the convex cone generated by $e_1, \ldots, e_r$. Moreover, we introduced in Definition 1.1.7 the notion of an *$\mathfrak{F}$-face*: this is a face $\gamma_0 \preceq \gamma$ such that the product over all f_i with $e_i \in \gamma_0$ does not belong to the radical of the ideal $\langle f_j;\ e_j \notin \gamma_0 \rangle \subseteq R$.

Definition 2.1.1. Let K be a finitely generated abelian group and R a factorially K-graded affine algebra with $R^* = \mathbb{K}^*$. Moreover, let $\mathfrak{F} = (f_1, \ldots, f_r)$ be a system of pairwise nonassociated K-prime generators for R and $(E \xrightarrow{Q} K, \gamma)$ the associated projected cone.

(i) We say that the K-grading of R is *almost free* if for every facet $\gamma_0 \preceq \gamma$ the image $Q(\gamma_0 \cap E)$ generates the abelian group K.
(ii) Let $\Omega_{\mathfrak{F}} = \{Q(\gamma_0);\ \gamma_0 \preceq \gamma\ \mathfrak{F}\text{-face}\}$ denote the collection of projected $\mathfrak{F}$-faces. An *$\mathfrak{F}$-bunch* is a nonempty subset $\Phi \subseteq \Omega_{\mathfrak{F}}$ such that
   (a) for any two $\tau_1, \tau_2 \in \Phi$, we have $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$,
   (b) if $\tau_1^\circ \subseteq \tau^\circ$ holds for $\tau_1 \in \Phi$ and $\tau \in \Omega_{\mathfrak{F}}$, then $\tau \in \Phi$ holds.
(iii) We say that an $\mathfrak{F}$-bunch Φ is *true* if for every facet $\gamma_0 \prec \gamma$ the image $Q(\gamma_0)$ belongs to Φ.

Definition 2.1.2. A *bunched ring* is a triple $(R, \mathfrak{F}, \Phi)$, where R is an almost freely factorially K-graded affine $\mathbb{K}$-algebra such that $R^* = \mathbb{K}^*$ holds, $\mathfrak{F}$ is a system of pairwise non-associated K-prime generators for R and Φ is a true $\mathfrak{F}$-bunch.

The following multigraded version of the Proj-construction associates to any bunched ring $(R, \mathfrak{F}, \Phi)$ a variety having R as its Cox ring.

Construction 2.1.3. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring and $(E \xrightarrow{Q} K, \gamma)$ its projected cone. The *collection of relevant faces* and the *covering collection* are

$$\begin{aligned} \mathrm{rlv}(\Phi) &:= \{\gamma_0 \preceq \gamma;\ \gamma_0 \text{ an } \mathfrak{F}\text{-face with } Q(\gamma_0) \in \Phi\}, \\ \mathrm{cov}(\Phi) &:= \{\gamma_0 \in \mathrm{rlv}(\Phi);\ \gamma_0 \text{ minimal}\}. \end{aligned}$$

Consider the action of the quasitorus $H := \mathrm{Spec}\,\mathbb{K}[K]$ on $\overline{X} := \mathrm{Spec}\,R$. We define the localization of $\overline{X}$ with respect to an $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$ to be

$$\overline{X}_{\gamma_0} \ := \ \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \text{ for some } (u_1, \ldots, u_r) \ \in \ \gamma_0^\circ.$$

This does not depend on the particular choice of $(u_1, \ldots, u_r) \in \gamma_0^\circ$. Moreover, we define an open H-invariant subset of $\overline{X}$ by

$$\widehat{X} \ := \ \widehat{X}(R, \mathfrak{F}, \Phi) \ := \ \bigcup_{\gamma_0 \in \mathrm{rlv}(\Phi)} \overline{X}_{\gamma_0} \ = \ \bigcup_{\gamma_0 \in \mathrm{cov}(\Phi)} \overline{X}_{\gamma_0} \ = \ \widehat{X}(\Phi),$$

where $\widehat{X}(\Phi) \subseteq \overline{X}$ is the set associated to the bunch $\Phi \subseteq \Omega_{\mathfrak{F}} = \Omega_{\overline{X}}$ of orbit cones, see Definition 1.3.5. Thus, the H-action on $\widehat{X}$ admits a good quotient; we set

$$X \ := \ X(R, \mathfrak{F}, \Phi) \ := \ \widehat{X}(R, \mathfrak{F}, \Phi) /\!\!/ H$$

and denote the quotient map by $p\colon \widehat{X} \to X$. The affine open subsets $\overline{X}_{\gamma_0} \subseteq \widehat{X}$, where $\gamma_0 \in \mathrm{rlv}(\Phi)$, are H-saturated and their images

$$X_{\gamma_0} \ := \ p(\overline{X}_{\gamma_0}) \ \subseteq \ X$$

form an affine cover of X. Moreover, every member f_i of $\mathfrak{F}$ defines a prime divisor $D_X^i := p(V(\widehat{X}, f_i))$ on X.

Theorem 2.1.4. *Let $\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$ arise from a bunched ring $(R, \mathfrak{F}, \Phi)$. Then X is a normal A_2-variety with*

$$\dim(X) \ = \ \dim(R) - \dim(K_{\mathbb{Q}}), \qquad \Gamma(X, \mathcal{O}^*) = \mathbb{K}^*,$$

there is an isomorphism $\mathrm{Cl}(X) \to K$ sending $[D_X^i]$ to $\deg(f_i)$, the map $p\colon \widehat{X} \to X$ is a characteristic space and the Cox ring $\mathcal{R}(X)$ is isomorphic to R.

Let us illustrate this with a couple of examples. The first one shows how toric varieties fit into the picture of bunched rings.

Example 2.1.5 (Bunched polynomial rings). Set $R := \mathbb{K}[T_1, \ldots, T_r]$, let K be a finitely generated abelian group and assume that R is almost freely K-graded via $\deg(T_i) = w_i$ with $w_1, \ldots, w_r \in K$. Then R is factorial and has $\mathfrak{F} := (T_1, \ldots, T_r)$ as a system of pairwise non-associated homogeneous K-prime generators. Every face $\gamma_0 \preceq \gamma$ is an $\mathfrak{F}$-face and thus a true $\mathfrak{F}$-bunch is nothing but a true $\mathcal{W}$-bunch for the vector configuration $\mathcal{W} = (w_1, \ldots, w_r)$ in $K_{\mathbb{Q}}$ in the sense of Definition II. 2.1.10. The variety associated to such a bunched ring $(R, \mathfrak{F}, \Phi)$ is the toric variety X associated to the $\mathcal{W}$-bunch Φ and its fan $\Sigma = \Phi^\sharp$ is obtained from Φ via Gale duality as described in Theorem II. 2.1.14. Note that the sets $X_{\gamma_0} \subseteq X$ are precisely the affine toric charts, where the cones $\gamma_0 \in \mathrm{cov}(\Phi)$ provide the maximal ones. Moreover, the divisors D_X^i are the toric prime divisors.

Example 2.1.6 (A singular del Pezzo surface). Set $K := \mathbb{Z}^2$ and consider the K-grading of $\mathbb{K}[T_1, \ldots, T_5]$ defined by $\deg(T_i) := w_i$, where w_i is the i-th column of the matrix

$$Q \ := \ \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix}$$

Then this K-grading descends to a K-grading of the following residue algebra which is factorial due to Proposition 2.4.1:

$$R \ := \ \mathbb{K}[T_1, \ldots, T_5] \,/\, \langle T_1 T_2 + T_3^2 + T_4 T_5 \rangle.$$

The classes $f_i \in R$ of $T_i \in \mathbb{K}[T_1, \ldots, T_5]$, where $1 \le i \le 5$, form a system $\mathfrak{F}$ of pairwise nonassociated K-prime generators of R. We have

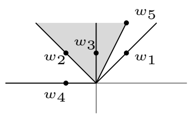
$$E \;=\; \mathbb{Z}^5, \qquad \gamma \;=\; \mathrm{cone}(e_1, \ldots, e_5)$$

and the K-grading is almost free. The $\mathfrak{F}$-faces can be directly computed using the definition, see also 2.4.6; writing $\gamma_{i_1,\ldots,i_k} := \mathrm{cone}(e_{i_1}, \ldots, e_{i_k})$, they are given as

$$\{0\},\ \gamma_1,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_{1,4},\ \gamma_{1,5},\ \gamma_{2,4},\ \gamma_{2,5},\ \gamma_{1,2,3},\ \gamma_{3,4,5},$$

$$\gamma_{1,2,3,4},\ \gamma_{1,2,3,5},\ \gamma_{1,2,4,5},\ \gamma_{1,3,4,5},\ \gamma_{2,3,4,5},\ \gamma_{1,2,3,4,5}.$$

In particular, we see that there is precisely one true maximal $\mathfrak{F}$-bunch Φ; it has $\tau := \mathrm{cone}(w_2, w_5)$ as its unique minimal cone.



Note that $\Phi = \Phi(w_3)$ is the bunch arising from $w_3 \in \tau^\circ$ as in Example 1.3.6. The collection of relevant faces and the covering collection are

$$\mathrm{rlv}(\Phi) \;=\; \{\gamma_{1,4}, \gamma_{2,5}, \gamma_{1,2,3}, \gamma_{3,4,5}, \gamma_{1,2,3,4}, \gamma_{1,2,3,5}, \gamma_{1,2,4,5}, \gamma_{1,3,4,5}, \gamma_{2,3,4,5}, \gamma_{1,2,3,4,5}\},$$

$$\mathrm{cov}(\Phi) \;=\; \{\gamma_{1,4}, \gamma_{2,5}, \gamma_{1,2,3}, \gamma_{3,4,5}\}.$$

The open set $\widehat{X}(R, \mathfrak{F}, \Phi)$ in $\overline{X} = V(\mathbb{K}^5; T_1T_2 + T_3^2 + T_4T_5)$ equals $\overline{X}^{ss}(w_3)$ and is the union of four affine charts:

$$\widehat{X}(R, \mathfrak{F}, \Phi) \;=\; \overline{X}_{f_1f_4} \,\cup\, \overline{X}_{f_2f_5} \,\cup\, \overline{X}_{f_1f_2f_3} \,\cup\, \overline{X}_{f_3f_4f_5}.$$

Since $\widehat{X}(R, \mathfrak{F}, \Phi)$ is a set of semistable points, the resulting variety $X = X(R, \mathfrak{F}, \Phi)$ is projective. Moreover, we have

$$\dim(X) \;=\; 2, \qquad \mathrm{Cl}(X) \;=\; K, \qquad \mathcal{R}(X) \;=\; R.$$

In fact, the methods presented later show that X is a $\mathbb{Q}$-factorial del Pezzo surface with one singularity, of type A_2.

Example 2.1.7 (The smooth del Pezzo surface of degree five). Consider the polynomial ring $A(2,5) := \mathbb{K}[T_{ij};\, 1 \le i < j \le 5]$ and the ideal $I(2,5) \subseteq A(2,5)$ generated by the Plücker relations:

$$T_{12}T_{34} - T_{13}T_{24} + T_{14}T_{23}, \qquad T_{12}T_{35} - T_{13}T_{25} + T_{15}T_{23}, \qquad T_{12}T_{45} - T_{14}T_{25} + T_{15}T_{24},$$

$$T_{13}T_{45} - T_{14}T_{35} + T_{15}T_{34}, \qquad T_{23}T_{45} - T_{24}T_{35} + T_{25}T_{34}.$$

The ring $R := A(2,5)/I(2,5)$ is factorial [**137**, Prop. 8.5], and the classes $f_{ij} \in R$ of T_{ij} define a system $\mathfrak{F} = (f_{ij})$ of pairwise nonassociated prime generators. The Plücker relations are homogeneous w.r.t. the grading by $K = \mathbb{Z}^5$ which associates to T_{ij} the ij-th column of the matrix

$$Q \;=\; \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{array}\right)$$

$$\begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array}$$

In particular, R is K-graded. These data describe the cone over the Grassmannian $\mathrm{Gr}\,(2,5)$ with an effective action of the five dimensional torus. The half sum $w = (3, -1, -1, -1, -1)$ of the columns of Q defines a true $\mathfrak{F}$-bunch $\Phi = \Phi(w)$ and hence we have a bunched ring $(R, \mathfrak{F}, \Phi)$. As we will see later, the associated variety $X(R, \mathfrak{F}, \Phi)$ is the smooth del Pezzo surface of degree five, i.e., the blow up of $\mathbb{P}_2$ in four points in general position; compare also [**143**, Prop. 3.2].

We say that a variety X is A_2-*maximal* if it is A_2 and admits no big open embedding $X \subsetneq X'$ into an A_2-variety X', where big means that $X' \setminus X$ is of codimension at least two in X'. Moreover, we call an $\mathfrak{F}$-bunch *maximal* if it cannot be enlarged by adding further projected $\mathfrak{F}$-faces.

Proposition 2.1.8. *Let X arise from a bunched ring $(R, \mathfrak{F}, \Phi)$. Then X is A_2-maximal if and only if Φ is maximal.*

The following statement shows in particular that every normal A_2-variety with finitely generated Cox ring can be realized as a big open subset in some A_2-maximal one and that the latter ones are obtained by Construction 2.1.3.

Theorem 2.1.9. *Let X be a normal A_2-variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$, finitely generated divisor class group $K := \mathrm{Cl}(X)$ and finitely generated Cox ring $R := \mathcal{R}(X)$. Suppose that $R^* = \mathbb{K}^*$ holds and let $\mathfrak{F}$ be any finite system of pairwise nonassociated K-prime generators for R.*

(i) *There exist a maximal $\mathfrak{F}$-bunch Φ and a big open embedding $X \to X(R, \mathfrak{F}, \Phi)$.*
(ii) *If X is A_2-maximal, then $X \cong X(R, \mathfrak{F}, \Phi)$ holds with some maximal $\mathfrak{F}$-bunch Φ.*

Corollary 2.1.10. *Let X be a normal A_2-maximal variety with $\Gamma(X, \mathcal{O}) = \mathbb{K}$, finitely generated divisor class group and finitely generated Cox ring. Then $X \cong X(R, \mathfrak{F}, \Phi)$ holds with some bunched ring $(R, \mathfrak{F}, \Phi)$.*

Corollary 2.1.11. *Let X be a normal complete A_2-variety (e.g. a normal projective one) with finitely generated divisor class group and finitely generated Cox ring. Then $X \cong X(R, \mathfrak{F}, \Phi)$ holds with some bunched ring $(R, \mathfrak{F}, \Phi)$.*

2.2. Proofs to Section 2.1. We enter the detailed discussion of Construction 2.1.3. First we consider almost free gradings and show that this notion is indeed a property of the grading and does not depend on the choice of a system $\mathfrak{F}$ of generators.

Construction 2.2.1. Let K be a finitely generated abelian group, R a factorially K-graded affine algebra with $R^* = \mathbb{K}^*$ and let $\mathfrak{F} = (f_1, \ldots, f_r)$ be any system of pairwise nonassociated K-prime generators for R. Consider the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} = \operatorname{Spec} R$ and set

$$\widehat{X}(\mathfrak{F}) \;:=\; \overline{X} \setminus \bigcup_{i \neq j} V(\overline{X}; f_i, f_j) \;=\; \bigcup_{i=1}^{r} \overline{X}_{f_1 \cdots f_{i-1} f_{i+1} \cdots f_r} \;\subseteq\; \overline{X}.$$

Then the subset $\widehat{X}(\mathfrak{F}) \subseteq \overline{X}$ is open, H-invariant and its complement $\overline{X} \setminus \widehat{X}(\mathfrak{F})$ is of codimension at least two in $\overline{X}$. In terms of Construction 2.1.3, the set $\widehat{X}(\mathfrak{F})$ is the union of the localizations $\overline{X}_{\gamma_0}$, where $\gamma_0 \preceq \gamma$ is a facet.

Proposition 2.2.2. *Let K be a finitely generated abelian group and R a factorially graded normal affine $\mathbb{K}$-algebra with $R^* = \mathbb{K}^*$. Moreover, let $\mathfrak{F} = (f_1, \ldots, f_r)$ be any system of pairwise nonassociated K-prime generators for R with associated projected cone $(E \xrightarrow{Q} K, \gamma)$. Then the following statements are equivalent.*

(i) *For every facet $\gamma_0 \preceq \gamma$, the image $Q(\gamma_0 \cap E)$ generates the abelian group K.*
(ii) *H acts freely on $\widehat{X}(\mathfrak{F})$.*
(iii) *H acts freely on an invariant open subset $W \subseteq \overline{X}$ with* $\mathrm{codim}(\overline{X} \setminus W) \geq 2$.

Proof. Suppose that (i) holds. By construction of $\widehat{X}(\mathfrak{F})$, we find for every $x \in \widehat{X}(\mathfrak{F})$ some $1 \leq i \leq r$ such that $f_j(x) \neq 0$ holds for all $j \neq i$. By (i) the weights $w_j = \deg(f_j)$, where $j \neq i$, generate the abelian group K. Thus, Proposition I. 2.2.8

tells us that the isotropy group H_x is trivial, which gives (ii). The implication "(ii)⇒(iii)" is obvious. Finally, suppose that (iii) holds. Since $W \subseteq \overline{X}$ is big and the f_i are pairwise non-associated K-primes, we find for every $1 \le i \le r$ a point $x \in W$ with $f_i(x) = 0$ but $f_j(x) \neq 0$ whenever $j \neq i$. Clearly, the weights w_j with $j \neq i$ generate the orbit group K_x. Since H_x is trivial, we infer $K_x = K$ from Proposition I. 2.2.8. □

We turn to $\mathfrak{F}$-bunches. The following observation enables us to apply Geometric Invariant Theory for quasitorus actions on affine varieties developed in the preceding section.

Remark 2.2.3. Let K be a finitely generated abelian group and R a factorially K-graded affine algebra with $R^* = \mathbb{K}^*$. Moreover, let $\mathfrak{F} = (f_1, \dots, f_r)$ be a system of pairwise nonassociated K-prime generators for R and $(E \xrightarrow{Q} K, \gamma)$ the associated projected cone. By Proposition 1.1.10, the projected $\mathfrak{F}$-faces are precisely the orbit cones of the action $H = \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} = \operatorname{Spec} R$. Thus, the (maximal) $\mathfrak{F}$-bunches are precisely the (maximal) bunches of orbit cones. According to Proposition 1.3.8, the $\mathfrak{F}$-bunch Φ determines a good H-set in $\overline{X}$; concretely this set is given as

$$\widehat{X}(\Phi) \ := \ \{x \in \overline{X};\ \omega_0 \in \Phi \text{ for some } \omega_0 \preceq \omega_x\} \ \subseteq \ \overline{X}.$$

According to Theorem 1.4.4, the set $\widehat{X}(\Phi)$ is $(H, 2)$-maximal if and only if Φ is maximal. Moreover, the closed H-orbits of $\widehat{X}(\Phi)$ are precisely the orbits $H \cdot x \subseteq \overline{X}$ with $\omega_x \in \Phi$.

Note that the interpretation in terms of orbit cones shows that an $\mathfrak{F}$-bunch does not depend on the particular choice of the system of generators $\mathfrak{F}$. The following two statements comprise in particular the assertions made in Construction 2.1.3.

Proposition 2.2.4. *Situation as in Construction 2.1.3. The good H-subset $\widehat{X}(\Phi) \subseteq \overline{X}$ associated to Φ satisfies*

$$\widehat{X}(\Phi) \ = \ \bigcup_{\gamma_0 \in \operatorname{rlv}(\Phi)} \overline{X}_{\gamma_0} \ = \ \widehat{X}(R, \mathfrak{F}, \Phi).$$

Moreover, the localizations $\overline{X}_{\gamma_0} \subseteq \overline{X}$, where $\gamma_0 \in \operatorname{rlv}(\Phi)$, are H-saturated subsets of $\widehat{X}(\Phi)$.

Proof. Consider $z \in \widehat{X}(\Phi)$ with $H \cdot z$ closed in $\widehat{X}(\Phi)$. By Lemma 1.3.10, the orbit cone ω_z belongs to Φ. In terms of $\mathfrak{F} = (f_1, \dots, f_r)$ and its projected cone $(E \xrightarrow{Q} K, \gamma)$ we have

$$\omega_z \ = \ \operatorname{cone}(\deg(f_i);\ f_i(z) \neq 0) \ = \ Q(\gamma_0), \quad \text{where } \gamma_0 \ := \ \operatorname{cone}(e_i;\ f_i(z) \neq 0).$$

This shows that $\overline{X}_{\gamma_0}$ is a neighbourhood $U(z)$ as considered in Construction 1.3.7. In particular, we have $\overline{X}_{\gamma_0} \subseteq \widehat{X}(\Phi)$ and Proposition 1.3.8 ensures that this inclusion is H-saturated. Going through the points with closed orbit in $\widehat{X}$, we see that $\widehat{X}(\Phi)$ is a union of certain H-saturated subsets $\overline{X}_{\gamma_0}$ with $\gamma_0 \in \operatorname{rlv}(\Phi)$.

To conclude the proof we have to show that in fact every $\overline{X}_{\gamma_0}$ with $\gamma_0 \in \operatorname{rlv}(\Phi)$ is an H-saturated subset of $\widehat{X}(\Phi)$. Given $\gamma_0 \in \operatorname{rlv}(\Phi)$, choose a point $z \in \overline{X}$ satisfying $f_i(z) \neq 0$ if and only if $e_i \in \gamma_0$. Then we have $\omega_z = Q(\gamma_0)$. This implies $z \in \widehat{X}(\Phi)$ and Lemma 1.3.10 says that $H \cdot z$ is closed in $\widehat{X}(\Phi)$. Thus, the preceding consideration shows that $\overline{X}(\gamma_0)$ is an H-saturated subset of $\widehat{X}(\Phi)$. □

Proposition 2.2.5. *Let K be a finitely generated abelian group, R an almost freely factorially graded normal affine $\mathbb{K}$-algebra with $R^* = \mathbb{K}^*$ and $\mathfrak{F} = (f_1, \dots, f_r)$ a system of pairwise nonassociated K-prime generators for R. Then, for any $\mathfrak{F}$-bunch Φ, the following statements are equivalent.*

(i) *The* $\mathfrak{F}$*-bunch* Φ *is true.*
(ii) *We have an* H*-saturated inclusion* $\widehat{X}(\mathfrak{F}) \subseteq \widehat{X}(\Phi)$.

Moreover, if $(R, \mathfrak{F}, \Phi)$ *is a bunched ring, then, in the notation of Construction 2.1.3, every* $D_X^i = p(V(\widehat{X}, f_i))$ *is a prime divisor on* X.

Proof. If Φ is a proper $\mathfrak{F}$-bunch, then Proposition 2.2.4 ensures that we have H-saturated inclusions $\overline{X}_{\gamma_0} \subseteq \widehat{X}(\Phi)$, where $\gamma_0 \subseteq \gamma$ is a facet. Thus, $\widehat{X}(\mathfrak{F}) \subseteq \widehat{X}(\Phi)$ is H-saturated. Conversely, if $\widehat{X}(\mathfrak{F}) \subseteq \widehat{X}(\Phi)$ is H-saturated, then we look at points $z_i \in \widehat{X}(\mathfrak{F})$ with $f_i(z_i) = 0$ and $f_j(z_i) \neq 0$ for all $j \neq i$. The orbits $H \cdot z_i$ are closed in $\widehat{X}(\Phi)$ and thus the corresponding orbit cones ω_i belong to Φ, see Theorem 1.4.4. In the setting of Definition 2.1.1, the orbit cones ω_i are exactly the Q-images of the facets of γ. It follows that the $\mathfrak{F}$-bunch Φ is proper. The supplement is then clear because the restriction $p\colon \widehat{X}(\mathfrak{F}) \to p(\widehat{X}(\mathfrak{F}))$ is a geometric quotient for a free H-action. □

Proof of Theorem 2.1.4. According to Proposition 2.2.4, the good quotient $p\colon \widehat{X} \to X$ exists and X is an A_2-variety. Proposition 2.2.2 tells us that H acts freely on $\widehat{X}(\mathfrak{F})$ and by Proposition 2.2.5 we have an H-saturated inclusion $\widehat{X}(\mathfrak{F}) \subseteq \widehat{X}$. Thus, the action of H on $\widehat{X}$ is strongly stable. We conclude that $X = \widehat{X} /\!\!/ H$ is of dimension $\dim(R) - \dim(H)$. Proposition I. 6.4.5 provides the desired isomorphism $\mathrm{Cl}(X) \cong K$ and Theorem I. 6.4.3 shows that $p\colon \widehat{X} \to X$ is a characteristic space. The latter implies $\mathcal{R}(X) \cong R$. □

Proof of Proposition 2.1.8. Let X be A_2-maximal. If Φ were not maximal, then we had $\Phi \subsetneq \Phi'$ with some $\mathfrak{F}$-bunch Φ'. This gives an H-saturated inclusion $\widehat{X}(\Phi) \subsetneq \widehat{X}(\Phi')$ and thus an open embedding $X \subsetneq X'$ of the quotient varieties. Since $\widehat{X}(\Phi)$ is big in $\widehat{X}(\Phi')$, also X is big in X'. A contradiction.

Now let Φ be maximal. Consider a big open embedding $X \subseteq X'$ into an A_2-variety. Replacing, if necessary, X' with its normalization, we may assume that X' is normal. Then X and X' share the same Cox ring R and thus occur as good quotients of open subsets $\widehat{X} \subseteq \widehat{X}'$ of their common total coordinate space $\overline{X}$. By $(H, 2)$-maximality of $\widehat{X} \subseteq \overline{X}$, we obtain $\widehat{X} = \widehat{X}'$ and thus $X = X'$. □

Proof of Theorem 2.1.9. By Theorem I. 5.3.4, the Cox ring R is factorially K-graded. Consider the corresponding total coordinate space $\overline{X} = \operatorname{Spec} R$ with its action of $H = \operatorname{Spec} \mathbb{K}[K]$ and the characteristic space $q_X : \widehat{X} \to X$ which is a good quotient for the action of H. By Proposition I. 6.1.6, we have a small complement $\overline{X} \setminus q_X^{-1}(X')$, where $X' \subseteq X$ denotes the set of smooth points, and H acts freely on $q_X^{-1}(X')$. Proposition 2.2.2 thus tells us that the K-grading is almost free. Next observe that $\widehat{X}$ is an H-saturated subset of some $(H, 2)$-maximal subset of $\overline{X}$. According to Theorem 1.4.4, the latter is of the form $\widehat{X}(\Phi)$ with a maximal $\mathfrak{F}$-bunch $\Phi \subseteq \Omega_{\mathfrak{F}} = \Omega_{\overline{X}}$. Propositions 2.2.2 and 2.2.5 show that Φ is true. Assertions (i) and (ii) of the theorem follow. □

2.3. Example: flag varieties. We show how flag varieties fit into the language of bunched rings. Let G be a connected linear algebraic group. Recall that a *Borel subgroup* $B \subseteq G$ is a maximal connected solvable subgroup of G and, more generally, a *parabolic subgroup* $P \subseteq G$ is a subgroup containing some Borel subgroup. The homogeneous space G/P is a smooth projective variety, called a *flag variety*. The following example explains this name.

EXAMPLE 2.3.1. Consider the special linear group SL_n. The subgroup $B_n \subseteq \mathrm{SL}_n$ of upper triangular matrices is a Borel subgroup; it is the stabilizer of the standard complete flag

$$\mathfrak{f}_n \;=\; \mathbb{K}^1 \subset \ldots \subset \mathbb{K}^{n-1} \;\in\; \mathrm{Gr}\,(1,n) \times \ldots \times \mathrm{Gr}\,(n-1,n)$$

under the diagonal SL_n-action on the product of Grassmannians. Thus, $\mathrm{SL}_n/B_n \cong \mathrm{SL}_n \cdot \mathfrak{f}_n$ is the set of all complete flags. Similarly, any sequence $0 < d_1 < d_2 < \ldots < d_s < n$, defines a parabolic $P_{d_1,\ldots,d_s} \subseteq \mathrm{SL}_n$, namely the stabilizer of the partial flag

$$\mathfrak{f}_{d_1,\ldots,d_s} \;=\; \mathbb{K}^{d_1} \subset \ldots \subset \mathbb{K}^{d_s} \;\in\; \mathrm{Gr}\,(d_1,n) \times \ldots \times \mathrm{Gr}\,(d_s,n)$$

and the possible flag varieties SL_n/P are precisely the SL_n-orbits $\mathrm{SL}_n \cdot \mathfrak{f}_{d_1,\ldots,d_s}$. If $s = 1$ and $d_1 = k$ holds, then $P_k \subseteq \mathrm{SL}_n$ is a maximal parabolic subgroup, and G/P_k is nothing but the Grassmannian $\mathrm{Gr}\,(k,n)$.

EXAMPLE 2.3.2. In SL_3 we have two maximal parabolic subgroups P_1 and P_2 with $B_3 = P_1 \cap P_2$, namely

$$P_1 \;=\; \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}, \qquad P_2 \;=\; \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

The flag varieties SL_3/P_1 and SL_3/P_2 both are isomorphic to $\mathbb{P}_2$, while the variety of complete flags SL_3/B_3 is given in $\mathbb{P}_2 \times \mathbb{P}_2$ as

$$\{([x_0,x_1,x_2],[y_0,y_1,y_2])\,;\; x_0y_0 + x_1y_1 + x_2y_2 = 0\}.$$

We determine the characteristic space of a flag variety G/P for simply connected semisimple G. The following observation is an important ingredient.

PROPOSITION 2.3.3. *Let a connected affine algebraic group G with $\mathbb{X}(G) = \{1\}$ act rationally by means of algebra automorphisms on an affine $\mathbb{K}$-algebra A with $A^* = \mathbb{K}^*$. If A is factorial, then the algebra of invariants A^G is factorial as well.*

PROOF. Given a non-zero non-unit $a \in A^G$, consider its decomposition into primes $a = a_1^{\nu_1} \cdots a_r^{\nu_r}$ in A. Then, for every $g \in G$, one has

$$a \;=\; g \cdot a \;=\; (g \cdot a_1)^{\nu_1} \cdots (g \cdot a_r)^{\nu_r}.$$

Using $A^* = \mathbb{K}^*$, we see that g permutes the a_i up to multiplication by a constant. Since G is connected and $\mathbb{X}(G) = \{1\}$ holds, we can conclude $a_1, \ldots, a_r \in A^G$. □

Now consider a parabolic subgroup $P \subseteq G$ and let $P' \subseteq P$ denote its commutator group. Then $H := P/P'$ is a torus and it acts freely via multiplication from the right on the homogeneous space G/P'. The canonical map $G/P' \to G/P$ is a geometric quotient for this H-action.

PROPOSITION 2.3.4. *Let G be a simply connected semisimple affine algebraic group and $P \subseteq G$ a parabolic subgroup. Then we have* $\mathrm{Cl}(G/P) \cong \mathbb{X}(H)$ *and* $G/P' \xrightarrow{/H} G/P$ *is a characteristic space.*

PROOF. By the assumptions on G, we have $\Gamma(G,\mathcal{O}^*) = \mathbb{K}^*$ and the algebra $\Gamma(G,\mathcal{O})$ is factorial; see [**96**, Prop. 1.2,] and [**95**, Prop. 4.6,]. Moreover, P' is connected and has a trivial character group. Consider the action of P' on G by multiplication from the right. Using Chevalley's Theorem [**88**, Theorem 11.2], we see that G/P' is a quasiaffine variety. Proposition 2.3.3 applied to the induced representation of P' on $\Gamma(G,\mathcal{O})$ shows that $\Gamma(G/P',\mathcal{O})$ is factorial. This gives $\mathrm{Cl}(G/P') = 0$. The assertion now follows from Proposition I. 6.4.5 and Theorem I. 6.4.3. □

Let us express this in terms of bunched rings. We first consider the case of a Borel subgroup $B \subseteq G$. Then the commutator $B' \subseteq B$ is a maximal unipotent subgroup $U \subseteq G$ and some maximal torus $T \subseteq G$ projects isomorphically onto $H = B/B'$. In particular, we have $B = TU$.

To proceed, we recall some basic representation theory; we refer to [**72**] for details. For every simple (finite-dimensional, rational) G-module V, the subspace V^U of U-invariant vectors is one-dimensional, and T acts on V^U by a character $\mu_V \in \mathbb{X}(T)$, called the *highest weight* of V. The set of the highest weights of the simple G-modules is a submonoid $X_+(G) \subseteq \mathbb{X}(T)$ and $V \mapsto \mu_V$ induces a bijection

$$\{\text{isomorphism classes of simple } G\text{-modules}\} \quad \longrightarrow \quad X_+(G).$$

The elements of $X_+(G)$ are called *dominant weights* of the group G (with respect to the pair (B,T)). The cone $C_+ \subseteq \mathbb{X}(T)_{\mathbb{Q}}$ generated by $X_+(G) \subseteq \mathbb{X}(T)$ is called the *positive Weyl chamber.* The intersection $C_+ \cap \mathbb{X}(T)$ coincides with $X_+(G)$. The monoid $X_+(G)$ has a unique system of free generators ϖ_i, $1 \le i \le s$, where s is the rank of the lattice $\mathbb{X}(T)$; the ϖ_i are called the *fundamental weights* of G.

The algebra $\Gamma(G/U, \mathcal{O})$ comes with the structure of a rational G-module induced from the left G-action on G/U. For every $\mu \in X_+(G)$, there is a unique simple G-submodule $V(\mu) \subseteq \Gamma(G/U, \mathcal{O})$ having μ as highest weight, see [**132**, Theorem 3.12]; in other words, we have the isotypic decomposition

$$\Gamma(G/U, \mathcal{O}) \quad = \quad \bigoplus_{\mu \in X_+(G)} V(\mu).$$

The T-action on $\Gamma(G/U, \mathcal{O})$ coming from right T-multiplication on G/U induces scalar multiplication on every submodule $V(\mu)$ defined by the highest weight μ^* of the dual G-module $V(\mu)^*$. Thus, the above isotypic decomposition is a G-equivariant $X_+(G)$-grading which we consider as a $\mathbb{X}(T)$-grading; note that $X_+(G)$ generates $\mathbb{X}(T)$ as a lattice.

We are ready to turn $R := \Gamma(G/U, \mathcal{O})$ into a bunched ring. It is factorial, graded by $K := \mathbb{X}(T)$ and we have $R^* = \mathbb{K}^*$. For every fundamental weight $\varpi_i \in X_+(G)$ fix a basis $\mathfrak{F}_i = (f_{ij}, 1 \le j \le s_i)$ of $V(\varpi_i)$ and put all these bases together to a family $\mathfrak{F} = (f_{ij})$. Finally, set $\Phi := \{C_+\}$.

Proposition 2.3.5. *The triple $(R, \mathfrak{F}, \Phi)$ is a bunched ring having the flag variety G/B as its associated variety.*

In the proof and also later, we make use of the following simple criterion for K-primality.

Lemma 2.3.6. *Let K be finitely generated abelian group and A be a factorially K-graded $\mathbb{K}$-algebra. If $w \in S(A)$ is indecomposable in $S(A)$, then every $0 \neq f \in A_w$ is K-prime.*

Proof of Proposition 2.3.5. First recall that the fundamental weights ϖ_i generate the weight monoid $X_+(G)$. Next note that for any two dominant weights $\mu, \mu' \in X_+(G)$ the G-module $V(\mu + \mu')$ is simple and thus we have a surjective multiplication map

$$V(\mu) \times V(\mu') \ \to \ V(\mu + \mu').$$

Consequently, the f_{ij} generate R. By Lemma 2.3.6, they are K-prime elements and clearly they are pairwise non-associated. Since each $V(\varpi_i)$ is of dimension at least two, Φ is a true $\mathfrak{F}$-bunch. Moreover, Φ is the only possible true $\mathfrak{F}$-bunch and thus the associated open subset of $\operatorname{Spec} R$ must be G/B'. $\square$

Remark 2.3.7. The total coordinate space $\operatorname{Spec} R$ of G/B admits an explicit realization as a G-orbit closure in a representation space, see [**76**, Section 5]:

$$\operatorname{Spec} \Gamma(G/U, \mathcal{O}) \;\cong\; \overline{G \cdot (v_{\varpi_1}, \ldots, v_{\varpi_r})} \;\subseteq\; V(\varpi_1) \oplus \ldots \oplus V(\varpi_r).$$

We turn to arbitrary flag varieties. Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Let $C \preceq C_+$ be a face of the positive Weyl chamber. Set $K_C := K \cap \operatorname{lin}(C)$ and consider the associated Veronese subalgebra

$$R_C \;=\; \bigoplus_{u \in K_C} R_u.$$

Let $\mathfrak{F}_C$ denote the system of generators obtained by putting together the bases $\mathfrak{F}_i$ with $\varpi_i \in C$. Moreover, set $\Phi_C := \{C\}$.

Proposition 2.3.8. *The triple $(R_C, \mathfrak{F}_C, \Phi_C)$ is a bunched ring with associated variety G/P_C, where $P_C \subseteq G$ is the parabolic subgroup defined by the index set $\{i; \varpi_i \in C\}$. Moreover, there is a commutative diagram*

$$\begin{array}{ccc}
\operatorname{Spec} R & \xrightarrow{/\widetilde{H}} & \operatorname{Spec} R_C \\
\uparrow & & \uparrow \\
G/B' & \longrightarrow & G/P'_C \\
{\scriptstyle /H}\downarrow & & \downarrow{\scriptstyle /H_C} \\
G/B & \longrightarrow & G/P_C
\end{array}$$

Finally, given any parabolic subgroup $P \subseteq G$, the associated flag variety G/P is isomorphic to some G/P_C.

Proof of Proposition 2.3.8. Using [**132**, Theorem 3.12], one verifies that R_C is the ring of functions of G/P'_C. Then the statement follows from Propositions 2.3.4 and 2.3.5. The supplement is due to the fact that any parabolic $P \subseteq G$ is conjugate to some P_C, see [**88**, Theorem 30.1]. □

Example 2.3.9. Consider $G = \mathrm{SL}_n$ and an extremal ray $C = \operatorname{cone}(\varpi_k)$ of the positive Weyl chamber C_+. Then $P_C = P_k$ holds, G/P_C is the Grassmannian $\mathrm{Gr}\,(k,n)$, and the total coordinate space $\operatorname{Spec} R_C$ is the affine cone over the Plücker embedding of $\mathrm{Gr}\,(k,n)$ with the standard $\mathbb{Z}_{\geq 0}$-grading.

Example 2.3.10. Consider again the special linear group SL_3 and the Borel subgroup $B_3 \subseteq \mathrm{SL}_3$. Then the commutator $U = B_3'$ and a maximal torus with $B = TU$ are

$$U \;=\; \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad T \;=\; \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix};\; t_1 t_2 t_3 = 1 \right\}.$$

In order to determine SL_3/U explicitly, consider the SL_3-module $\mathbb{K}^3 \times (\mathbb{K}^3)^*$, where SL_3 acts canonically on $\mathbb{K}^3$ and $(\mathbb{K}^3)^*$ is the dual module. The point (e_1, e_3^*) has U as its stabilizer, and its orbit closure is the affine quadric

$$Z \;:=\; \overline{\mathrm{SL}_3 \cdot (e_1, e_3^*)} \;=\; V(\mathbb{K}^3 \times (\mathbb{K}^3)^*;\; X_1 Y_1 + X_2 Y_2 + X_3 Y_3).$$

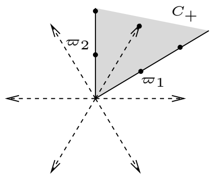
In fact, $\mathrm{SL}_3/U \;\cong\; \mathrm{SL}_3 \cdot (e_1, e_3^*)$ is obtained from Z by removing the coordinate subspaces $V(Z; X_1, X_2, X_3)$ and $V(Z; Y_1, Y_2, Y_3)$. The T-action on SL_3/U via multiplication from the right looks in coordinates of $\mathbb{K}^3 \times (\mathbb{K}^3)^*$ as

$$t \cdot ((x_1, x_2, x_3), (y_1, y_2, y_3)) \;=\; ((t_1 x_1, t_1 x_2, t_1 x_3), (t_1 t_2 y_1, t_1 t_2 y_2, t_1 t_2 y_3)),$$

and the orbit $\mathrm{SL}_3 \cdot (e_1, e_3^*)$ coincides with the set of semistable points $Z^{ss}(\chi)$, where the weight $\chi \in \mathbb{X}(T)$ may be taken as $\chi(t) = t_1^2 t_2$. Thus, we recover the characteristic space via the identifications

$$\begin{array}{ccc} Z^{ss}(\chi) & \xrightarrow{/T} & Z^{ss}(\chi)/T \\ \cong \updownarrow & & \updownarrow \cong \\ G/U & \xrightarrow[/T]{} & G/B \end{array}$$

Let $\chi_1, \chi_2 \in \mathbb{X}(T)$ denote the characters with $\chi_i(t) = t_i$. Then the fundamental weights are $\varpi_1 = \chi_1$ and $\varpi_2 = \chi_1 + \chi_2$. Thus, a weight $c_1\chi_1 + c_2\chi_2$ is dominant if and only if $c_1 \geq c_2$ holds.



The fundamental modules are $V(\varpi_1) \cong \mathbb{K}^3$ and $V(\varpi_2) \cong \Lambda^2\mathbb{K}^3 \cong (\mathbb{K}^3)^*$. Thus, we may take $\mathfrak{F}_1 = (X_1, X_2, X_3)$ as a basis for $V(\varpi_2)$ and $\mathfrak{F}_2 = (Y_1, Y_2, Y_3)$ as a basis for $V(\varpi_1)$. The ring $R = \Gamma(G/U, \mathcal{O})$ is then given as

$$R \quad = \quad \mathbb{K}[X_1, X_2, X_3, Y_1, Y_2, Y_3]/\langle X_1Y_1 + X_2Y_2 + X_3Y_3 \rangle.$$

with $\deg(X_i) = \varpi_1$ and $\deg(Y_i) = \varpi_2$. Finally, the Veronese subalgebras of R producing the bunched rings of the flag varieties G/P_1 and G/P_2 are $\mathbb{K}[X_1, X_2, X_3]$ and $\mathbb{K}[Y_1, Y_2, Y_3]$, respectively.

2.4. Example: quotients of quadrics. Here we consider bunched rings arising from a non-degenerate affine quadric. The resulting varieties are quotients of suitable quasitorus actions on the quadric; we call them *full intrinsic quadrics.*

We first give a guide to concrete examples of full intrinsic quadrics, and will later see that the general ones are isomorphic to these. For $m \in \mathbb{Z}_{\geq 1}$ consider the quadratic forms

$$g_{2m} \quad := \quad T_1T_2 + \ldots + T_{2m-1}T_{2m},$$
$$g_{2m+1} \quad := \quad T_1T_2 + \ldots + T_{2m-1}T_{2m} + T_{2m+1}^2.$$

Write $R(r) := \mathbb{K}[T_1, \ldots, T_r]/\langle g_r \rangle$ for the factor ring, let $f_i \in R(r)$ denote the class of the variable T_i, and set $\mathfrak{F}(r) = (f_1, \ldots, f_r)$. The following holds even for non algebraically closed fields $\mathbb{K}$, see [**137**, Theorem 8.2]; in our case the proof is simple.

Proposition 2.4.1 (Klein-Nagata). *For $r \geq 5$, the ring $R(r)$ is factorial and $\mathfrak{F}(r)$ is a system of pairwise non-associated prime generators.*

Proof. For $r \geq 5$, the polynomial g_{r-2} is irreducible, and thus we have an integral factor ring

$$\mathbb{K}[T_2, \ldots, T_r]\langle g_{r-2}(T_3, \ldots, T_r) \rangle \ \cong \ \mathbb{K}[T_1, \ldots, T_r]\langle T_1, g_r \rangle \ \cong \ R(r)/\langle f_1 \rangle.$$

In other words, f_1 is prime in $R(r)$. Moreover, localizing $R(r)$ by f_1 gives a factorial ring isomorphic to $\mathbb{K}[T_1^{\pm 1}, T_3, \ldots, T_r]$. Thus, $R(r)$ is factorial. $\square$

In order to find suitable gradings of $R(r)$, we first observe that there is a unique maximal grading keeping the variables homogeneous and any other grading keeping the variables homogeneous is a coarsening of this maximal one.

Construction 2.4.2 (Maximal diagonal grading). Consider any polynomial $g \in \mathbb{K}[T_1, \ldots, T_r]$ of the form

$$g \quad = \quad a_0 T_1^{l_{01}} \cdots T_r^{l_{0r}} \; + \; \ldots \; + \; a_k T_1^{l_{k1}} \cdots T_r^{l_{kr}}.$$

First, we build a $k \times r$ matrix P_g from the exponents l_{ij} of g. Define row vectors $l_i := (l_{i1}, \ldots, l_{ir})$ and set

$$P_g \quad = \quad \begin{pmatrix} l_1 - l_0 \\ \vdots \\ l_k - l_0 \end{pmatrix}$$

With the row lattice $M_g \subseteq \mathbb{Z}^r$ of P_g, we define the *gradiator* of g to be the projection $Q_g \colon \mathbb{Z}^r \to K_g := \mathbb{Z}^r / M_g$. It gives rise to a K_g-grading on $\mathbb{K}[T_1, \ldots, T_r]$ via

$$\deg(T_1) \;:=\; Q_g(e_1), \qquad \ldots, \qquad \deg(T_r) \;:=\; Q_g(e_r).$$

This grading is effective and $T_1, \ldots, T_r, g$ are homogeneous. Moreover, given any other such grading, say by an abelian group K, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^r & \xrightarrow{\;e_i \mapsto \deg T_i\;} & K \\ & {\scriptstyle Q_g} \searrow \quad \nearrow {\scriptstyle \alpha} & \\ & K_g & \end{array}$$

If, instead of one g, we have several $g_1, \ldots, g_s$, then, replacing P_g with the stack matrix of $P_{g_1}, \ldots, P_{g_s}$, we obtain a gradiator $Q_{g_1, \ldots, g_s}$ with the analogous property.

For our polynomials g_{2m} and g_{2m+1} the gradiators are obtained by a direct computation and are easy to write down.

Proposition 2.4.3. *For the polynomials g_{2m} and g_{2m+1}, the associated maximal grading groups are $K_{2m} = K_{2m+1} = \mathbb{Z}^{m+1}$ and gradiators are given by*

$$Q_{2m} \quad = \quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & \ldots & \ldots & 0 & 1 \end{pmatrix},$$

$$Q_{2m+1} \quad = \quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & \ldots & \ldots & 1 & 1 & 1 \end{pmatrix}.$$

From Definition 2.1.1, we directly extract the following (necessary and sufficient) conditions that a grading gives rise to a bunched ring.

Remark 2.4.4. Let R be a factorially K-graded affine $\mathbb{K}$-algebra with $R^* = \mathbb{K}^*$ and $\mathfrak{F} = (f_1, \ldots, f_r)$ a system of pairwise nonassociated K-prime generators with projected cone $(E \xrightarrow{Q} K, \gamma)$.

(i) The K-grading of R is almost free if and only if $Q(\gamma_0 \cap \mathbb{Z}^r)$ generates K as an abelian group for every facet $\gamma_0 \preceq \gamma$.

(ii) $\mathfrak{F} = (f_1, \ldots, f_r)$ admits a true $\mathfrak{F}$-bunch if and only if $Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ \neq \emptyset$ holds in $K_{\mathbb{Q}}$ for any two facets $\gamma_1, \gamma_2 \preceq \gamma$.

Let us illustrate this construction of bunched rings with an example in the case of six variables; the resulting variety has torsion in its divisor class group.

Example 2.4.5. For the polynomial $g = T_1T_2 + T_3T_4 + T_5T_6$ in $\mathbb{K}[T_1, \ldots, T_6]$, we have $K_g = \mathbb{Z}^4$. Consider $K := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and the coarsening

$$\alpha\colon K_g \to K, \qquad e_1, e_2, e_3 \mapsto (1, \overline{1}), \quad e_4 \mapsto (2, \overline{0}).$$

Then $Q := \alpha \circ Q_g\colon \mathbb{Z}^6 \to K$ fullfills the conditions of Remark 2.4.4; more explicitly, the K-grading of the factor ring $R = \mathbb{K}[T_1, \ldots, T_6]/\langle g \rangle$ is given by

$$\deg(f_1) = (1, \overline{1}), \quad \deg(f_2) = (1, \overline{2}), \quad \deg(f_3) = (1, \overline{1}),$$

$$\deg(f_4) = (1, \overline{2}), \quad \deg(f_5) = (1, \overline{1}), \quad \deg(f_6) = (1, \overline{2}),$$

where f_i is the class of T_i. For $\mathfrak{F} = (f_1, \ldots, f_6)$, there is one true $\mathfrak{F}$-bunch, namely $\Phi = \{\mathbb{Q}_{\geq 0}\}$. The variety $X = X(R, \mathfrak{F}, \Phi)$ is a $\mathbb{Q}$-factorial projective 4-fold.

In particular for the treatment of more advanced examples, explicit knowledge of the $\mathfrak{F}(r)$-faces may be useful. Here comes a simple recipe to determine them.

Remark 2.4.6. First let $r = 2m$ and arrange the set $\mathfrak{R} := \{1, \ldots, r\}$ according to the following scheme

$$\begin{array}{ccccc} 2 & 4 & \cdots & 2m-2 & 2m \\ \\ 1 & 3 & \cdots & 2m-3 & 2m-1 \end{array}$$

The *column sets* are $\{1, 2\}, \ldots, \{2m-1, 2m\}$. Consider subsets $U \cup V \subseteq \mathfrak{R}$, where U is located in the upper row and V in the lower one. We look for the types:

(i) the union $U \cup V$ contains no column set, for example

$$\begin{array}{lcccccc} U & \boxed{2 \quad 4} & 6 & 8 & \boxed{10} & 12 \\ V & 1 \quad 3 & \boxed{5 \quad 7} & & 9 & 11 \end{array}$$

(ii) the union $U \cup V$ contains at least two column sets, for example

$$\begin{array}{lcccc} U & \boxed{2 \quad 4 \quad 6} & 8 & \boxed{10 \quad 12} \\ V & 1 \quad 3 & \boxed{5 \quad 7 \quad 9} & 11 \end{array}$$

By Proposition 1.1.9, the possible $\mathfrak{F}$-faces for $R(2m)$ and $\mathfrak{F} = \mathfrak{F}(2m)$ are $\gamma_0 = \mathrm{cone}(e_i;\ i \in U \cup V) \preceq \gamma$ with $U \cup V$ of type (i) or (ii).

Now, let $r = 2m+1$. Then we arrange the set $\mathfrak{R} := \{1, \ldots, r\}$ according to the following scheme

$$\begin{array}{cccccc} 2 & 4 & \cdots & 2m-2 & 2m & 2m+1 \\ \\ 1 & 3 & \cdots & 2m-3 & 2m-1 & 2m+1 \end{array}$$

This time the column sets are $\{1, 2\}, \ldots, \{2m-1, 2m\}$ and $\{2m+1\}$. Again, we consider two types of $U \cup V \subseteq \mathfrak{R}$ with U in the upper row and V in the lower one:

(i) the union $U \cup V$ contains no column set,
(ii) the union $U \cup V$ contains at least two column sets.

Then, as before, the possible $\mathfrak{F}$-faces for $R(2m+1)$ and $\mathfrak{F} = \mathfrak{F}(2m+1)$ are $\gamma_0 = \mathrm{cone}(e_i;\ i \in U \cup V) \preceq \gamma$ with a constellation $U \cup V$ of type (i) or (ii).

By a *full intrisic quadric* we mean a variety X with Cox ring of the form $\mathcal{R}(X) \cong \mathbb{K}[T_1, \ldots, T_r]/\langle g \rangle$, where g is a quadratic form of rank r and the classes $f_i \in \mathcal{R}(X)$ of the variables T_i are $\mathrm{Cl}(X)$-homogeneous. The following statement shows that any full intrinsic quadric is isomorphic to the variety arising from a bunched ring $(R(r), \mathfrak{F}(r), \Phi)$ as discussed before.

PROPOSITION 2.4.7. *Let $g \in \mathbb{K}[T_1, \dots, T_r]$ be a quadratic form of rank r. Consider an effective grading of $\mathbb{K}[T_1, \dots, T_r]$ by an abelian group K such that $T_1, \dots, T_r$ and g are K-homogeneous. Then there exist linearly independent K-homogeneous linear forms $S_1, \dots, S_r$ in $T_1, \dots, T_r$ with $g(T_1, \dots, T_r) = g_r(S_1, \dots, S_r)$.*

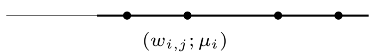
PROOF. We may assume that the group K is finitely generated. The K-grading on $\mathbb{K}[T_1, \dots, T_r]$ is given by a linear action of a quasitorus $H := \operatorname{Spec} \mathbb{K}[K]$ on $\mathbb{K}^r$. Let $\mathrm{O}(g)$ be the subgroup of $\mathrm{GL}_r(\mathbb{K})$ consisting of linear transformations preserving the quadratic form g and $\mathrm{EO}(g)$ be the extension of $\mathrm{O}(g)$ by the subgroup of scalar matrices.

By assumption, H is a subgroup of $\mathrm{EO}(g)$ consisting of diagonal matrices. Since g is of rank r, the subgroup $\mathrm{EO}(g)$ is conjugate to $\mathrm{EO}(g_r)$, and this conjugation sends H to a subgroup H' of $\mathrm{EO}(g_r)$. Since H' is diagonalizable, it is contained in a maximal torus of $\mathrm{EO}(g_r)$, see [**88**, Section 22.3, Cor. B].

On the other hand, the intersection of $\mathrm{EO}(g_r)$ with the subgroup of diagonal matrices is a maximal torus of $\mathrm{EO}(g_r)$. Any two maximal tori of an affine algebraic group are conjugate, and we may assume that H' is a subgroup of $\mathrm{EO}(g_r)$ consisting of diagonal matrices. So the conjugation sends H to H'. This means that every new basis vector is a linear combination of the old ones having the same degree, and the assertion follows. □

We conclude the section with two classification results on full intrinsic quadrics with small divisor class groups from [**31**].

PROPOSITION 2.4.8. *Let X be a full intrinsic quadric with $\mathrm{Cl}(X) \cong \mathbb{Z}$. Then X arises from a bunched ring $(R, \mathfrak{F}, \Phi)$ with Φ given by*



with positive integers $w_{i,j}$, where $0 \le i \le n$ and $1 \le j \le \mu_i$, such that $w_{i,1} < w_{i+1,1}$, $w_{ij} = w_{ik}$ and $w_{i,1} + w_{n-i,1} = w$ holds for some fixed $w \in \mathbb{N}$, and the μ_i satisfy

$$\mu_i \ge 1, \qquad \mu_i = \mu_{n-i}, \qquad \mu_0 + \ldots + \mu_n \ge 5.$$

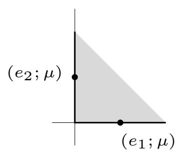
The variety X is always $\mathbb{Q}$-factorial, projective, and $\mathbb{Q}$-Fano, and for its dimension, we have

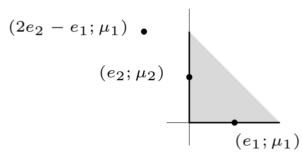
$$\dim(X) \;=\; \mu_1 + \ldots + \mu_n - 2.$$

Moreover, X is smooth if and only if $n = 0$ and $w_{01} = 1$ hold, and in this case it is a smooth projective quadric.

The second result is the "intrinsic quadrics version" of Kleinschmidt's classification [**93**] of smooth complete toric varieties with Picard number two, compare also Example 4.2.10.

THEOREM 2.4.9. *Let X be a smooth full intrinsic quadric with $\mathrm{Cl}(X) \cong \mathbb{Z}^2$. Then X arises from a bunched ring $(R, \mathfrak{F}, \Phi)$ with an $\mathfrak{F}$-bunch Φ given by one of the following figures:*





where in the left hand side case, $\mu \ge 3$ holds, and in the right hand side case, one has $\mu_i \ge 1$ and $2\mu_1 + \mu_2 \ge 5$. Any such X is projective, and its dimension is given by

$$\dim(X) = 2\mu - 3, \quad \text{or} \quad \dim(X) = 2\mu_1 + \mu_2 - 3,$$

where the first equation corresponds to the l.h.s. case, and the second one to the r.h.s. case. The variety X is Fano if and only if Φ belongs to the l.h.s. case. Moreover, different figures define non-isomorphic varieties.

Example 2.4.10. In the setting of Theorem 2.4.9, consider the bunch arising frome the l.h.s. picture with $\mu = 3$. Then R is defined by a quadratic form of rank six and X equals the flag variety SL_3/B_3 discussed in Example 2.3.10

2.5. The canonical toric embedding. As we will see here, every variety defined by a bunched ring allows a closed embedding into a toric variety with nice properties. For a fixed bunched ring, we give a canonical construction in 2.5.3. The toric ambient variety arising from this construction may be (non-canonically) completed, this is discussed in 2.5.5 and 2.5.6.

Remark 2.5.1. Consider a morphism $\varphi\colon X \to Z$ from a normal variety X to a toric variety Z with acting torus T and base point $z_0 \in Z$. Let $D_Z^i = \overline{T\cdot z_i}$, where $1 \le i \le r$, be the T-invariant prime divisors and set

$$Z' \quad := \quad T\cdot z_0 \ \cup \ T\cdot z_1 \ \cup \ \ldots \ \cup \ T\cdot z_r.$$

Suppose that $\varphi^{-1}(D_Z^i)$, where $1 \le i \le r$, are pairwise different irreducible hypersurfaces in X. Then the complement $X \setminus \varphi^{-1}(Z')$ is of codimension at least two in X, and we have a canonical pullback homomorphism

$$\mathrm{WDiv}^T(Z) = \mathrm{CDiv}^T(Z') \xrightarrow{\quad \varphi^* \quad} \mathrm{CDiv}(\varphi^{-1}(Z')) \subseteq \mathrm{WDiv}(X)$$

It sends principal divisors to principal divisors and consequently induces a pullback homomorphism $\varphi^*\colon\ \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ on the level of divisor class groups.

Definition 2.5.2. Let X be a normal variety and Z a toric variety with acting torus T and invariant prime divisors $D_Z^i = \overline{T\cdot z_i}$, where $1 \le i \le r$. A *neat embedding* of X into Z is a closed embedding $\imath\colon X \to Z$ such that $\imath^{-1}(D_Z^i)$, where $1 \le i \le r$, are pairwise different irreducible hypersurfaces in X and the pull back homomorphism $\imath^*\colon\ \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism.

Construction 2.5.3. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring and $(E \xrightarrow{Q} K, \gamma)$ the associated projected cone. Then, with $M := \ker(Q)$, we have the mutually dual exact sequences

$$0 \longrightarrow L \longrightarrow F \xrightarrow{P} N$$

$$0 \longleftarrow K \xleftarrow[Q]{} E \longleftarrow M \longleftarrow 0$$

The *envelope* of the collection $\mathrm{rlv}(\Phi)$ of relevant $\mathfrak{F}$-faces is the saturated Q-connected γ-collection

$$\mathrm{Env}(\Phi) \quad := \quad \{\gamma_0 \preceq \gamma;\ \gamma_1 \preceq \gamma_0 \text{ and } Q(\gamma_1)^\circ \subseteq Q(\gamma_0)^\circ \text{ for some } \gamma_1 \in \mathrm{rlv}(\Phi)\}.$$

Let $\delta \subseteq F_{\mathbb{Q}}$ denote the dual cone of $\gamma \subseteq E_{\mathbb{Q}}$ and for $\gamma_0 \preceq \gamma$ let $\gamma_0^* = \gamma_0^\perp \cap \delta$ be the corresponding face. Then one has fans in the lattices F and N:

$$\widehat{\Sigma} \quad := \quad \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in \mathrm{Env}(\Phi)\},$$
$$\Sigma \quad = \quad \{P(\gamma_0^*);\ \gamma_0 \in \mathrm{Env}(\Phi)\}.$$

Consider the action of $H := \mathrm{Spec}\,\mathbb{K}[K]$ on $\overline{X} := \mathrm{Spec}\,R$. Set $\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Let $\overline{Z} = \mathbb{K}^r$ denote the toric variety associated to the cone δ in F. The system $\mathfrak{F} = (f_1, \ldots, f_r)$ of generators of R defines a closed embedding

$$\bar\imath\colon \overline{X} \ \to \ \overline{Z}, \qquad z \ \mapsto \ (f_1(z), \ldots, f_r(z)),$$

which becomes H-equivariant if we endow $\overline{Z}$ with the diagonal H-action given by the characters $\chi^{w_1}, \ldots, \chi^{w_r}$, where $w_i = \deg(f_i) \in K$. Denoting by $\widehat{Z}$ and Z toric varieties associated to the fans $\widehat{\Sigma}$ and Σ, we obtain a commutative diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\ \overline{\imath}\ } & \overline{Z} \\ \uparrow & & \uparrow \\ \widehat{X} & \xrightarrow{\ \widehat{\imath}\ } & \widehat{Z} \\ {\scriptstyle /\!\!/ H}\ \downarrow {\scriptstyle p_X} & & {\scriptstyle p_Z} \downarrow\ {\scriptstyle /\!\!/ H} \\ X & \xrightarrow[\ \imath\ ]{} & Z \end{array}$$

where the map $\widehat{\imath}$ is the restriction of $\overline{\imath}$ and the toric morphism $p_Z \colon \widehat{Z} \to Z$ arises from $P \colon F \to N$. We call the induced map of quotients $\imath \colon X \to Z$ the *canonical toric embedding* associated to the bunched $(R, \mathfrak{F}, \Phi)$.

PROPOSITION 2.5.4. *In the setting of Construction 2.5.3, the following statements hold.*

(i) *The quotient morphism* $p_Z \colon \widehat{Z} \to Z$ *is a toric characteristic space, we have* $\widehat{X} = \overline{\imath}^{-1}(\widehat{Z})$ *and* $\imath \colon X \to Z$ *is a closed embedding.*
(ii) *For any* $\gamma_0 \in \mathrm{rlv}(\Phi)$ *and the associated toric affine chart* $Z_{P(\gamma_0^*)} \subseteq Z$ *we have* $X_{\gamma_0} = \imath^{-1}(Z_{P(\gamma_0^*)})$.
(iii) *The prime divisors* $D_X^i = p_X(V(\widehat{X}, f_i))$ *and* $D_Z^i = p_Z(V(\widehat{Z}, T_i))$ *satisfy* $D_X^i = \imath^*(D_Z^i)$ *and we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Cl}(X) & \xleftarrow[\cong]{\ \imath^*\ } & \mathrm{Cl}(Z) \\ {\scriptstyle [D_X^i] \mapsto \deg(f_i)}\ \downarrow \cong & & \cong \downarrow\ {\scriptstyle [D_Z^i] \mapsto \deg(T_i)} \\ K & = & K \end{array}$$

In particular, the embedding $\imath \colon X \to Z$ *of the quotient varieties is a neat embedding.*
(iv) *The maximal cones of the fan* Σ *are precisely the cones* $P(\gamma_0^*) \in \Sigma$, *where* $\gamma_0 \in \mathrm{cov}(\Phi)$.
(v) *The image* $\imath(X) \subseteq Z$ *intersects every closed toric orbit of* Z *nontrivially.*

PROOF OF CONSTRUCTION 2.5.3 AND PROPOSITION 2.5.4. First note that by Proposition II. 2.3.5, the collection of cones Σ is indeed a fan. Moreover, by Theorem II. 1.3.1, the toric morphism $p_Z \colon \widehat{Z} \to Z$ is a characteristic space. For each $\gamma_0 \in \mathrm{rlv}(\Phi)$, the affine toric chart of $\widehat{Z}$ corresponding to $\gamma_0^* \in \widehat{\Sigma}$ is the localization $\overline{Z}_{\gamma_0} = \overline{Z}_{T^u}$, where $u \in \gamma_0^\circ$ and $T^u = T_1^{u_1} \cdots T_r^{u_r}$. Thus, we obtain

$$\overline{\imath}^{-1}(\overline{Z}_{\gamma_0}) \;=\; \overline{\imath}^{-1}(\overline{Z}_{T_1^{u_1} \cdots T_r^{u_r}}) \;=\; \overline{X}_{f_1^{u_1} \cdots f_r^{u_r}} \;=\; \overline{X}_{\gamma_0}.$$

Since $\widehat{Z}$ and $\widehat{X}$ are the union of the localizations of $\overline{Z}$ and $\overline{X}$ by faces $\gamma_0 \in \mathrm{rlv}(\Phi)$, we conclude $\widehat{X} = \overline{\imath}^{-1}(\widehat{Z})$. By Theorem I. 2.3.6, the induced map $\imath \colon X \to Z$ is a closed embedding. This establishes the construction and the first two items of the proposition.

We turn to the third assertion of the proposition. By the commutative diagram given in the construction, we have

$$\imath^{-1}(D_Z^i) \;=\; p_X(\overline{\imath}^{-1}(p_Z^{-1}(D_Z^i))) \;=\; p_X(V(\widehat{X}; f_i)) \;=\; D_X^i.$$

Thus, denoting by T the acting torus of Z, we have a well defined pullback homomorphism $\imath^* \colon \mathrm{WDiv}^T(Z) \to \mathrm{WDiv}(X)$. It satisfies $\imath^*(D_z^i) = D_X^i$ because of

$$\overline{\imath}^*(p_Z^*(D_Z^i)) \;=\; \overline{\imath}^*(\mathrm{div}(T_i)) \;=\; \mathrm{div}(f_i) \;=\; p_X^*(D_X^i).$$

As a consequence, we obtain that the diagram of (iii) is commutative and thus the embedding $\imath\colon X \to Z$ is neat.

We show (iv) and (v) of the proposition. By definition, the envelope $\mathrm{Env}(\Phi)$ has the covering collection $\mathrm{cov}(\Phi)$ as its collection of minimal cones. Consequently, the maximal cones of $\widehat{\Sigma}$ and Σ are given by

$$\widehat{\Sigma}^{\max} \;=\; \{\gamma_0^*;\ \gamma_0 \in \mathrm{cov}(\Phi)\}, \qquad\qquad \Sigma^{\max} \;=\; \{P(\gamma_0^*);\ \gamma_0 \in \mathrm{cov}(\Phi)\}.$$

This verifies in particular the fourth assertion. The last one is then a simple consequence. □

Note that, if the variety X associated to the bunched ring $(R, \mathfrak{F}, \Phi)$ is complete (projective), then the canonical ambient toric variety Z need not be complete (projective). Passing to completions of Z, means to give up the last two properties of the above proposition. However, the following construction preserves the first three properties.

Construction 2.5.5. Situation as in Construction 2.5.3. Let $\mathfrak{B} \subseteq \mathrm{faces}(\gamma)$ be any saturated Q-connected collection comprising $\mathrm{Env}(\Phi)$. Then $\mathfrak{B}$ defines fans in F in N:

$$\begin{array}{rcl} \widehat{\Sigma}_1 & := & \{\delta_0 \preceq \delta;\ \delta_0 \preceq \gamma_0^* \text{ for some } \gamma_0 \in \mathfrak{B}\}, \\ \Sigma_1 & = & \{P(\gamma_0^*);\ \gamma_0 \in \Theta\}. \end{array}$$

The fans $\widehat{\Sigma}$ and Σ defined by the envelope $\mathrm{Env}(\Phi)$ are subfans of $\widehat{\Sigma}_1$ and Σ_1 respectively. With the toric varieties $\widehat{Z}_1$ and Z_1, associated to $\widehat{\Sigma}_1$ and Σ_1, we obtain a commutative diagram

$$\begin{array}{ccccc} \overline{X} & \xrightarrow{\ \bar{\imath}\ } & \overline{Z} & \xrightarrow{\ \mathrm{id}\ } & \overline{Z} \\ \uparrow & & \uparrow & & \uparrow \\ \widehat{X} & \xrightarrow{\ \widehat{\imath}\ } & \widehat{Z} & \xrightarrow{\ \widehat{\imath}_1\ } & \widehat{Z}_1 \\ {\scriptstyle p_X}\downarrow{\scriptstyle /\!\!/ H} & & {\scriptstyle p_Z}\downarrow{\scriptstyle /\!\!/ H} & & {\scriptstyle p_{Z_1}}\downarrow{\scriptstyle /\!\!/ H} \\ X & \xrightarrow[\ \imath\ ]{} & Z & \xrightarrow[\ \imath_1\ ]{} & Z_1 \end{array}$$

where $\widehat{\imath}_1$ and $\imath_1$ are open embeddings, the compositions $\bar{\imath}_1 \circ \bar{\imath}$, $\widehat{\imath}_1 \circ \widehat{\imath}$ and $\imath_1 \circ \imath$ are closed embeddings satisfying the assertions (i), (ii) and (iii) of Proposition 2.5.4. In particular, $\imath_1 \circ \imath\colon X \to Z_1$ is a neat embedding.

Proof. Since $\widehat{X} \subseteq \overline{X}$ is $(H,2)$-maximal, we obtain $\widehat{X} = (\bar{\imath}_1 \circ \bar{\imath})^{-1}(\widehat{Z}_1)$. The remaining assertions are then obvious. □

The following special case of the above construction provides projective toric ambient varieties for the case that our variety arising from the bunched ring is projective.

Construction 2.5.6. Situation as in Construction 2.5.3. Let $\Lambda(\overline{X}, H)$ and $\Lambda(\overline{Z}, H)$ denote the GIT-fans of the actions of H on $\overline{X}$ and $\overline{Z}$ respectively. Suppose that the $\mathfrak{F}$-bunch Φ arises from a GIT-cone $\lambda \in \Lambda(\overline{X}, H)$ that means that we have

$$\Phi \;=\; \{Q(\gamma_0);\ \gamma_0 \preceq \gamma\ \mathfrak{F}\text{-face with } \lambda^\circ \subseteq Q(\gamma_0)\}.$$

Then $X = \overline{X}^{ss}(\lambda)$ holds. Moreover, for any GIT-cone $\eta_1 \in \Lambda(\overline{Z}, H)$ with $\eta_1^\circ \subseteq \lambda^\circ$, Construction 2.5.5 provides a neat embedding $X \to Z_1$ into the projective toric variety $Z_1 = \overline{Z}^{ss}(\eta_1) /\!\!/ H$ associated to bunch of cones arising from η_1.

Example 2.5.7. Consider again the bunched ring $(R, \mathfrak{F}, \Phi)$ with $K = \mathbb{Z}^2$ from 2.1.6. The graded ring is given by

$$R = \mathbb{K}[T_1, \ldots, T_5] \,/\, \langle T_1T_2 + T_3^2 + T_4T_5 \rangle, \qquad Q = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix},$$

where $\deg(T_i) = w_i$ with the i-th column w_i of Q. The classes f_i of T_i give $\mathfrak{F} = (f_1, \ldots, f_5)$ and the $\mathfrak{F}$-bunch Φ is

$$\Phi = \{\tau\}, \qquad \tau = \mathrm{cone}(w_2, w_5)$$

In the associated projected cone $(E \xrightarrow{Q} K, \gamma)$ we have $E = \mathbb{Z}^5$ and $\gamma = \mathrm{cone}(e_1, \ldots, e_5)$. The covering collection is

$$\mathrm{cov}(\Phi) = \{\gamma_{1,4}, \gamma_{2,5}, \gamma_{1,2,3}, \gamma_{3,4,5}\},$$

where $\gamma_{i_1,\ldots,i_k} := \mathrm{cone}(e_{i_1}, \ldots, e_{i_k})$. A Gale dual map $P\colon F \to N$ for $Q\colon E \to K$ is given by the matrix

$$P = \begin{pmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The maximal cones of the fan Σ constructed via $\mathrm{Env}(\Phi)$ in 2.5.3 correspond to the members of the covering collection; in terms the $v_1, \ldots, v_r$ of P they are given as

$$\mathrm{cone}(v_2, v_3, v_5), \quad \mathrm{cone}(v_1, v_3, v_4), \quad \mathrm{cone}(v_4, v_5), \quad \mathrm{cone}(v_1, v_2).$$

In particular, we see that the canonical toric ambient variety Z of X determined by $(R, \mathfrak{F}, \Phi)$ is not complete. Completions of Z are obtained as in 2.5.6. Recall that

$$X = X(R, \mathfrak{F}, \Phi) = \overline{X}^{ss}(w_3) /\!\!/ H$$

holds with the action of $H = \mathrm{Spec}\,\mathbb{K}[K]$ on $\overline{X} = \mathrm{Spec}\,R$. The two GIT-fans $\Lambda(\overline{X}, H)$ and $\Lambda(\overline{Z}, H)$ are

$\Lambda(\overline{X}, H)$ $\qquad$ $\Lambda(\overline{Z}, H)$

In $\Lambda(\overline{X}, H)$, the weight w_3 belongs to the GIT-cone $\lambda = \mathrm{cone}(w_2, w_5)$. In $\Lambda(\overline{Z}, H)$, we have three choices:

$$\eta_1 := \mathrm{cone}(w_2, w_3), \qquad \eta_{12} := \mathrm{cone}(w_3), \qquad \eta_2 := \mathrm{cone}(w_3, w_5).$$

The GIT-cones η_1, η_2 provide $\mathbb{Q}$-factorial projective toric completions Z_1, Z_2 of Z, whereas η_{12} gives a projective toric completion with non-$\mathbb{Q}$-factorial singularities.

3. Geometry via defining data

3.1. Stratification and local properties. We observe that the variety arising from a bunched ring comes with a decomposition into locally closed subvarieties; these turn out to be the intersections with the toric orbits of the canonical toric ambient variety. We show that the local divisor class groups are constant along the pieces and conclude some local properties from this.

Construction 3.1.1. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring, consider the action of $H := \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} := \operatorname{Spec} R$ and set

$$\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi), \qquad X := X(R, \mathfrak{F}, \Phi).$$

Let $(E \xrightarrow{Q} K, \gamma)$ be the projected cone associated to $\mathfrak{F} = (f_1, \ldots, f_r)$. To any $\mathfrak{F}$-face $\gamma_0 \preceq \gamma$, we associate a locally closed subset

$$\overline{X}(\gamma_0) \ := \ \{z \in \overline{X};\ f_i(z) \neq 0 \Leftrightarrow e_i \in \gamma_0 \text{ for } 1 \le i \le r\} \ \subseteq \ \overline{X}.$$

These sets are pairwise disjoint and cover the whole $\overline{X}$. Taking the pieces defined by relevant $\mathfrak{F}$-faces, one obtains a constructible subset

$$\widetilde{X} \ := \ \bigcup_{\gamma_0 \in \operatorname{rlv}(\Phi)} \overline{X}(\gamma_0) \ \subseteq \ \widehat{X}.$$

Note that $\widetilde{X}$ is the union of all closed H-orbits of $\widehat{X}$. The images of the pieces inside $\widetilde{X}$ form a decomposition of X into pairwise disjoint locally closed pieces:

$$X \ = \ \bigcup_{\gamma_0 \in \operatorname{rlv}(\Phi)} X(\gamma_0), \qquad \text{where } X(\gamma_0) \ := \ p_X(\overline{X}(\gamma_0)).$$

Example 3.1.2. If we have $R = \mathbb{K}[T_1, \ldots, T_r]$ and $\mathfrak{F} = \{T_1, \ldots, T_r\}$, then X is the toric variety arising from the image fan Σ associated to $\operatorname{rlv}(\Phi)$, and for any $\gamma_0 \in \operatorname{rlv}(\Phi)$, the piece $X(\gamma_0) \subseteq X$ is precisely the toric orbit corresponding to the cone $P(\gamma_0^*) \in \Sigma$.

Proposition 3.1.3. *Situation as in Construction 3.1.1. For every* $\gamma_0 \in \operatorname{rlv}(\Phi)$*, the associated piece* $X(\gamma_0) \subseteq X$ *has the following descriptions.*

(i) *In terms of the embedding* $X \subseteq Z$ *constructed in 2.5.3, the piece* $X(\gamma_0)$ *is the intersection of* X *with the toric orbit of* Z *given by* $P(\gamma_0^*) \in \Sigma$.

(ii) *In terms of the prime divisors* $D_X^i \subseteq X$ *defined in 2.1.3 via the generators* $f_i \in \mathfrak{F}$*, the piece* $X(\gamma_0)$ *is given as*

$$X(\gamma_0) \quad = \quad \bigcap_{e_i \notin \gamma_0} D_X^i \ \setminus \ \bigcup_{e_j \in \gamma_0} D_X^j$$

(iii) *In terms of the open subsets* $X_{\gamma_1} \subseteq X$ *and* $\overline{X}_{\gamma_1} \subseteq \overline{X}$ *defined in 2.1.3 via relevant faces* $\gamma_1 \in \operatorname{rlv}(\Phi)$*, we have*

$$X(\gamma_0) \quad = \quad X_{\gamma_0} \ \setminus \bigcup_{\gamma_0 \prec \gamma_1 \in \operatorname{rlv}(\Phi)} X_{\gamma_1},$$

$$p_X^{-1}(X(\gamma_0)) \quad = \quad \overline{X}_{\gamma_0} \ \setminus \bigcup_{\gamma_0 \prec \gamma_1 \in \operatorname{rlv}(\Phi)} \overline{X}_{\gamma_1}.$$

Proof of Construction 3.1.1 and Proposition 3.1.3. Obviously, $\overline{X}$ is the union of the locally closed pieces $\overline{X}(\gamma_0)$, where $\gamma_0 \preceq \gamma$ runs through the $\mathfrak{F}$-faces. Proposition 2.2.4 tells us that $\widehat{X} \subseteq \overline{X}$ is the $(H, 2)$-maximal subset given by the bunch of orbit cones

$$\Phi \quad = \quad \{\omega_z;\ H \cdot z \text{ closed in } \widehat{X}\}.$$

Moreover, it says that the closed orbits of $\widehat{X}$ are precisely the orbits $H \cdot z \subseteq \overline{X}$ with $\omega_z \in \Phi$. Given $z \in \widetilde{X}$, we have $\omega_z = Q(\gamma_0)$ for some $\gamma_0 \in \operatorname{rlv}(\Phi)$ and thus $H \cdot z$ is closed in $\widehat{X}$. Conversely, if $H \cdot z$ is closed in $\widehat{X}$, consider the $\mathfrak{F}$-face

$$\gamma_0 \ := \ \operatorname{cone}(e_i;\ f_i(z) \neq 0) \ \preceq \ \gamma.$$

Then we have $z \in \overline{X}(\gamma_0)$ and $Q(\gamma_0) = \omega_z \in \Phi$. The latter shows that γ_0 is a relevant face. This implies $z \in \widetilde{X}$.

All further statements are most easily seen by means of a neat embedding $X \subseteq Z$ as constructed in 2.5.3. Let $\overline{X} \subseteq \overline{Z} = \mathbb{K}^r$ denote the closed H-equivariant embedding arising from $\mathfrak{F}$. Then, $\overline{X}$ intersects precisely the $\overline{Z}(\gamma_0)$, where γ_0 is an $\mathfrak{F}$-face, and in these cases we have

$$\overline{X}(\gamma_0) \quad = \quad \overline{Z}(\gamma_0) \cap \overline{X}.$$

As mentioned in Example 3.1.2, the images $Z(\gamma_0) = p_Z(\overline{Z}(\gamma_0))$, where $\gamma_0 \in \mathrm{rlv}(\Phi)$, are precisely the toric orbits of Z. Moreover, we have

$$\widehat{X} \ = \ \widehat{Z} \cap \overline{X}, \qquad\qquad \widetilde{X} \ = \ \widetilde{Z} \cap \overline{X}.$$

Since p_Z separates H-orbits along $\widetilde{Z}$, we obtain $X(\gamma_0) = Z(\gamma_0) \cap X$ for every $\gamma_0 \in \mathrm{rlv}(\Phi)$. Consequently, the $X(\gamma_0)$, where $\gamma_0 \in \mathrm{rlv}(\Phi)$, are pairwise disjoint and form a decomposition of X into locally closed pieces. Finally, using $X(\gamma_0) = Z(\gamma_0) \cap X$ and $D_X^i = \imath^*(D_Z^i)$, we obtain assertions 3.1.3 (ii) and (iii) directly from the corresponding representations of the toric orbit $Z(\gamma_0)$. □

We use the decomposition into pieces to study local properties of the variety associated to a bunched ring. First recall from Proposition I. 6.2.1 that we associated to any point $x \in X$ of a variety X with characteristic space $q_X \colon \widehat{X} \to X$ a submonoid of the divisor class group as follows. Let $\widehat{x} \in q_X^{-1}(x)$ be a point with closed H_X-orbit and set

$$S_x \ := \ \{[D] \in \mathrm{Cl}(X);\ f(\widehat{x}) \neq 0 \text{ for some } f \in \Gamma(X, \mathcal{R}_{[D]})\} \ \subseteq \ \mathrm{Cl}(X).$$

A first task is to is to express this for a variety X arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ in terms of the defining data. For this we use the isomorphism $\mathrm{Cl}(X) \to K$ of Theorem 2.1.4 sending the class of the prime divisor $D_X^i \subseteq X$ defined by $f_i \in \mathfrak{F}$ to the degree $\deg(f_i) \in K$.

Proposition 3.1.4. *Situation as in Construction 3.1.1. Let $\gamma_0 \in \mathrm{rlv}(\Phi)$ and $x \in X(\gamma_0)$. Then, under the isomorphism $\mathrm{Cl}(X) \to K$ of Theorem 2.1.4, the monoid S_x corresponds to $Q(\gamma_0 \cap E)$.*

Proof. According to Theorem I. 6.4.3 and Remark I. 6.4.7, we may identify the quasitorus $H = \mathrm{Spec}\, \mathbb{K}[K]$ with $H_X = \mathrm{Spec}\, \mathbb{K}[\mathrm{Cl}(X)]$ and the characteristic space $p \colon \widehat{X}(R, \mathfrak{F}, \Phi) \to X$ with $q_X \colon \widehat{X} \to X$ constructed from a Cox sheaf. Let $z \in \overline{X}(\gamma_0)$. Then z is a point with closed H-orbit in $\widehat{X}$ and thus, for $x = p(z)$, we see that S_x equals $Q(\gamma_0 \cap E)$. □

As an application, we compute the local class group $\mathrm{Cl}(X, x)$; recall that this is the group of Weil divisors $\mathrm{WDiv}(X)$ modulo the subgroup of all divisors being principal on a neighbourhood of $x \in X$.

Proposition 3.1.5. *Situation as in Construction 3.1.1. Let $\gamma_0 \in \mathrm{rlv}(\Phi)$ and $x \in X(\gamma_0)$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Cl}(X) & \longrightarrow & \mathrm{Cl}(X, x) \\ \cong \updownarrow & & \updownarrow \cong \\ K & \longrightarrow & K/Q(\mathrm{lin}(\gamma_0) \cap E)) \end{array}$$

In particular, the local divisor class groups are constant along the pieces $X(\gamma_0)$, where $\gamma_0 \in \mathrm{rlv}(\Phi)$.

Proof. By Proposition I. 6.2.1, the kernel of $\mathrm{Cl}(X) \to \mathrm{Cl}(X, x)$ is the subgroup of $\mathrm{Cl}(X)$ generated by S_x. Thus, Proposition 3.1.4 gives the assertion. □

Corollary 3.1.6. *Situation as in Construction 3.1.1. Inside the divisor class group* $\mathrm{Cl}(X) \cong K$*, the Picard group of* X *is given by*

$$\mathrm{Pic}(X) \quad \cong \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\mathrm{lin}(\gamma_0) \cap E).$$

Proof. The divisor class given by $w \in K$ stems from a Cartier divisor if and only if it defines the zero class in $\mathrm{Cl}(X, x)$ for any $x \in X$. According to Proposition 3.1.5, the latter is equivalent to $w \in Q(\mathrm{lin}(\gamma_0) \cap E)$ for all $\gamma_0 \in \mathrm{rlv}(\Phi)$. Since we have $\mathrm{cov}(\Phi) \subseteq \mathrm{rlv}(\Phi)$, and for any $\gamma_0 \in \mathrm{rlv}(\Phi)$, there is a $\gamma_1 \in \mathrm{cov}(\Phi)$ with $\gamma_1 \preceq \gamma_0$, it suffices to take the $\gamma_0 \in \mathrm{cov}(\Phi)$. □

Corollary 3.1.7. *Let* X *be the variety associated to a bunched ring and* $X \subseteq Z$ *the associated canonical toric embedding as provided in 2.5.3.*

(i) *For every* $x \in X$ *we have an isomorphism* $\mathrm{Cl}(X, x) \cong \mathrm{Cl}(Z, x)$.
(ii) *We have an isomorphism* $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z)$.

In particular, if the ambient variety Z *has a toric fixed point, then the Picard group* $\mathrm{Pic}(X)$ *is torsion free.*

Proof. The two items are clear. For the supplement, recall that a toric variety with toric fixed point has free Picard group; see for example [**70**, Sec. 3.4]. □

A point $x \in X$ is factorial ($\mathbb{Q}$-factorial) if and only if every Weil divisor is Cartier ($\mathbb{Q}$-Cartier) at x. Thus, Proposition 3.1.5 has the following application to singularities.

Corollary 3.1.8. *Situation as in Construction 3.1.1. Consider a relevant face* $\gamma_0 \in \mathrm{rlv}(\Phi)$ *and point* $x \in X(\gamma_0)$.

(i) *The point* x *is factorial if and only if* Q *maps* $\mathrm{lin}(\gamma_0) \cap E$ *onto* K.
(ii) *The point* x *is* $\mathbb{Q}$*-factorial if and only if* $Q(\gamma_0)$ *is of full dimension.*

Corollary 3.1.9. *The variety* X *arising from a bunched ring* $(R, \mathfrak{F}, \Phi)$ *is* $\mathbb{Q}$*-factorial if and only if* Φ *consists of cones of full dimension.*

Whereas local factoriality admits a simple combinatorial characterization, smoothness is difficult in general. Nevertheless, we have the following statement.

Proposition 3.1.10. *Situation as in Construction 3.1.1. Suppose that* $\widehat{X}$ *is smooth, let* $\gamma_0 \in \mathrm{rlv}(\Phi)$*, and* $x \in X(\gamma_0)$*. Then* x *is a smooth point if and only if* Q *maps* $\mathrm{lin}(\gamma_0) \cap E$ *onto* K.

Proof. The "only if" part is clear by Corollary 3.1.8. Conversely, if Q maps $\mathrm{lin}(\gamma_0) \cap E$ onto K, then Propositions I. 2.2.8 and 1.1.10 say that the fibre $p^{-1}(x)$ consists of a single free H-orbit. Consequently, H acts freely over an open neighbourhood $U \subseteq X$ of x. Thus x is smooth. □

Corollary 3.1.11. *Let* X *be the variety associated to a bunched ring and* $X \subseteq Z$ *the embedding into the toric variety* Z *constructed in 2.5.3. Moreover, let* $x \in X$.

(i) *The point* x *is a factorial* ($\mathbb{Q}$*-factorial*) *point of* X *if and only if it is a smooth* ($\mathbb{Q}$*-factorial*) *point of* Z.
(ii) *If* $\widehat{X}$ *is smooth, then* x *is a smooth point of* X *if and only if it is a smooth point of* Z.

As an immediate consequence of general results on quotient singularities, see [**41**] and [**86**], one obtains the following.

Proposition 3.1.12. *Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. Set $X = X(R, \mathfrak{F}, \Phi)$ and $\widehat{X} = \widehat{X}(R, \mathfrak{F}, \Phi)$. Suppose that $\widehat{X}$ is smooth. Then X has at most rational singularities. In particular, X is Cohen-Macaulay.*

Example 3.1.13. Consider the surface $X = X(R, \mathfrak{F}, \Phi)$ from 2.1.6. Recall that R is graded by $K = \mathbb{Z}^2$ and this is given as

$$R \;=\; \mathbb{K}[T_1, \dots, T_5] \,/\, \langle T_1T_2 + T_3^2 + T_4T_5 \rangle,$$
$$Q \;=\; \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix},$$

where $\deg(T_i) = w_i$ with the i-th column w_i of Q. The system $\mathfrak{F}$ consists of the classes of $T_1, \dots, T_5$ and the $\mathfrak{F}$-bunch is

$$\Phi \;=\; \{\tau\}, \qquad\qquad \tau = \mathrm{cone}(w_2, w_5).$$

In the projected cone $(E \xrightarrow{Q} K, \gamma)$, we have $E = \mathbb{Z}^5$ and $\gamma = \mathrm{cone}(e_1, \dots, e_5)$. With $\gamma_{i_1,\dots,i_k} := \mathrm{cone}(e_{i_1}, \dots, e_{i_k})$, the collection of relevant faces is

$$\mathrm{rlv}(\Phi) \;=\; \{\gamma_{1,4}, \gamma_{2,5}, \gamma_{1,2,3}, \gamma_{3,4,5}, \gamma_{1,2,3,4}, \gamma_{1,2,3,5}, \gamma_{1,2,4,5}, \gamma_{1,3,4,5}, \gamma_{2,3,4,5}, \gamma_{1,2,3,4,5}\}.$$

The stratum $X(\gamma_{1,2,3,4,5}) \subseteq X$ is open. The facets of γ define open sets of the prime divisors D_X^i divisors and we have four strata, each consisting of one point:

$$X(\gamma_{1,2,3,4}) \;=\; D_X^5, \qquad \dots, \qquad X(\gamma_{2,3,4,5}) \;=\; D_X^1,$$
$$X(\gamma_{1,4}), \quad X(\gamma_{2,5}), \quad X(\gamma_{1,2,3}), \quad X(\gamma_{3,4,5}).$$

In order to determine the local class groups, note that for the relevant faces $\gamma_0 \preceq \gamma$ we obtain

$$Q(\gamma_0 \cap E) \;=\; \begin{cases} \mathbb{Z} \cdot (-1, 1) + \mathbb{Z} \cdot (0, 3) & \gamma_0 = \gamma_{2,5}, \\ \mathbb{Z}^2 & \text{else.} \end{cases}$$

Thus, all points different from the point $x_0 \in X(\gamma_{2,5})$ have trivial local class group. Moreover,

$$\mathrm{Cl}(X, x_0) \;=\; \mathbb{Z}/3\mathbb{Z}, \qquad \mathrm{Pic}(X) \;=\; \mathbb{Z} \cdot (-1, 1) + \mathbb{Z} \cdot (0, 3) \;\subseteq\; \mathbb{Z}^2 \;=\; \mathrm{Cl}(X).$$

Clearly, X is $\mathbb{Q}$-factorial. Since $\widehat{X}(R, \mathfrak{F}, \Phi)$ is smooth, the singular locus is determined by the combinatorial part of the data; it is $X(\gamma_{2,5}) = \{x_0\}$.

3.2. Base loci and cones of divisors. We first provide general descriptions of (stable) base loci as well as the cones of effective, movable, semiample and ample divisor classes on a variety in terms of its Cox ring. Then we interprete the results in the language of bunched rings.

Recall that a Weil divisor $D \in \mathrm{WDiv}(X)$ on a normal prevariety X is effective if its multiplicities are all non-negative. The *effective cone* is the cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ generated by the classes of effective divisors. Note that $\mathrm{Eff}(X)$ is convex, and, given $D \in \mathrm{WDiv}(X)$, we have $[D] \in \mathrm{Eff}(X)$ if and only if there is a non-zero $f \in \Gamma(X, \mathcal{O}_X(nD))$ for some $n > 0$.

Proposition 3.2.1. *Let X be a normal prevariety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and finitely generated divisor class group $\mathrm{Cl}(X)$. Let f_i, $i \in I$, be any system of non-zero homogeneous generators of the Cox ring $\mathcal{R}(X)$. Then the cone of effective divisor classes of X is given by*

$$\mathrm{Eff}(X) \;=\; \mathrm{cone}(\deg(f_i);\ i \in I).$$

Proof. Clearly each $\deg(f_i)$ is the class of an effective divisor and thus the cone on the right hand side is contained in $\mathrm{Eff}(X)$. Conversely, if $[D]$ belongs to $\mathrm{Eff}(X)$, then some multiple $n[D]$ is represented by an effective $nD \in \mathrm{WDiv}(X)$ and the canonical section 1_{nD} defines an non-zero element in $\Gamma(X, \mathcal{R}_{[nD]})$ which is a polynomial in the f_i. Consequently, $[D]$ is a non-negative linear combination of the classes $\deg(f_i)$. □

For a Weil divisor D on a normal prevariety X and a section $f \in \Gamma(X, \mathcal{O}_X(D))$ we introduced the D-divisor as $\mathrm{div}_D(f) = \mathrm{div}(f) + D$. The *base locus* and the *stable base locus* of D are defined as

$$\mathrm{Bs}(D) \ := \bigcap_{f \in \Gamma(X, \mathcal{O}_X(D))} \mathrm{Supp}(\mathrm{div}_D(f)), \qquad \mathbf{Bs}(D) \ := \bigcap_{n \in \mathbb{Z}_{\geq 1}} \mathrm{Bs}(nD).$$

REMARK 3.2.2. The base locus and the stable base locus of a Weil divisor D on a normal prevariety X only depend on the class $[D] \in \mathrm{Cl}(X)$. Moreover, if we have $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and $\mathrm{Cl}(X)$ is finitely generated, then we may write the base locus in terms of the Cox ring as

$$\mathrm{Bs}(D) \quad = \bigcap_{f \in \Gamma(X, \mathcal{R}_{[D]})} \mathrm{Supp}(\mathrm{div}_{[D]}(f)).$$

A Weil divisor $D \in \mathrm{WDiv}(X)$ is called *movable*, if its stable base locus is of codimension at least two in X. The *moving cone* $\mathrm{Mov}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ is the cone consisting of the classes of movable divisors. Note that $\mathrm{Mov}(X)$ is convex.

PROPOSITION 3.2.3. *Let X be a normal complete variety with finitely generated divisor class group $\mathrm{Cl}(X)$ and let f_i, $i \in I$, be any system of pairwise non-associated $\mathrm{Cl}(X)$-prime generators for the Cox ring $\mathcal{R}(X)$. Then the moving cone is given as*

$$\mathrm{Mov}(X) \quad = \quad \bigcap_{i \in I} \mathrm{cone}(\deg(f_j);\ j \in I \setminus \{i\}).$$

LEMMA 3.2.4. *Let X be a normal complete variety with $\mathrm{Cl}(X)$ finitely generated and let $w \in \mathrm{Cl}(X)$ be effective. Then the following two statements are equivalent.*

(i) *The stable base locus of the class $w \in \mathrm{Cl}(X)$ contains a divisor.*
(ii) *There exist an $w_0 \in \mathrm{Cl}(X)$ with $\dim \Gamma(X, \mathcal{R}_{nw_0}) = 1$ for any $n \in \mathbb{Z}_{\geq 0}$ and an $f_0 \in \Gamma(X, \mathcal{R}_{w_0})$ such that for any $m \in \mathbb{Z}_{\geq 1}$ and $f \in \Gamma(X, \mathcal{R}_{mw})$ one has $f = f' f_0$ with some $f' \in \Gamma(X, \mathcal{R}_{mw - w_0})$.*

PROOF. The implication "(ii)⇒(i)" is obvious. So, assume that (i) holds. The class $w \in \mathrm{Cl}(X)$ is represented by some non-negative divisor D. Let D_0 be a prime component of D which occurs in base locus of any positive multiple of D, and let $w_0 \in \mathrm{Cl}(X)$ be the class of D_0. Then the canonical section of D_0 defines an element $f_0 \in \Gamma(X, \mathcal{R}_{w_0})$ which by Proposition I. 5.3.9 divides any $f \in \Gamma(X, \mathcal{R}_{mw})$, where $m \in \mathbb{Z}_{\geq 1}$. Note that $\Gamma(X, \mathcal{R}_{nw_0})$ is of dimension one for every $n \in \mathbb{Z}_{\geq 1}$, because otherwise $\Gamma(X, \mathcal{R}_{na_0w_0})$ where $a_0 > 0$ is the multiplicity of D_0 in D, would provide enough sections in $\Gamma(X, \mathcal{R}_{nw})$ to move na_0D_0. □

PROOF OF PROPOSITION 3.2.3. Set $w_i := \deg(f_i)$ and denote by $I_0 \subseteq I$ the set of indices with $\dim \Gamma(X, \mathcal{R}_{nw_0}) \leq 1$ for all $n \in \mathbb{N}$. Let $w \in \mathrm{Mov}(X)$. Then Lemma 3.2.4 tells us that for any $i \in I_0$, there must be a monomial of the form $\prod_{j \neq i} f_j^{n_j}$ in some $\Gamma(X, \mathcal{R}_{nw})$. Consequently, w lies in the cone of the right hand side. Conversely, consider an element w of the cone of the right hand side. Then, for every $i \in I_0$, a product $\prod_{j \neq i} f_j^{n_j}$ belongs to some $\Gamma(X, \mathcal{R}_{nw})$. Hence none of the f_i, $i \in I_0$, divides all elements of $\Gamma(X, \mathcal{R}_{nw})$. Again by Lemma 3.2.4, we conclude $w \in \mathrm{Mov}(X)$. □

Now suppose that we are in the situation of Construction I. 6.1.3. That means that X is of affine intersection, $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ holds, $\mathrm{Cl}(X)$ is finitely generated and the Cox sheaf $\mathcal{R}$ is locally of finite type. Then we have a characteristic space

$q_X\colon \widehat{X} \to X$, where $\widehat{X} = \operatorname{Spec}_X \mathcal{R}$, which is a good quotient for the action of $H_X = \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$. For $x \in X$, we already considered the submonoid

$$S_x \;=\; \{[D] \in \operatorname{Cl}(X);\; f(\widehat{x}) \neq 0 \text{ for some } f \in \Gamma(X, \mathcal{R}_{[D]})\} \;\subseteq\; \operatorname{Cl}(X),$$

where $\widehat{x} \in q_X^{-1}(x)$ is a point with closed H_X-orbit. Let $\omega_x \subseteq \operatorname{Cl}_{\mathbb{Q}}(X)$ denote the convex cone generated by S_x.

Proposition 3.2.5. *Situation as in Construction I. 6.1.3. Then the base locus and the stable base locus of a Weil divisor D on X are given as*

$$\operatorname{Bs}(D) \;=\; \{x \in X;\; [D] \notin S_x\}, \qquad \mathbf{Bs}(D) \;=\; \{x \in X;\; [D] \notin \omega_x\}.$$

Proof. The assertions immediately follow from the description of the monoid S_x in terms of the $[D]$-divisors provided by Corollary I. 6.1.8:

$$\begin{aligned} S_x &= \{[D] \in \operatorname{Cl}(X);\; x \notin \operatorname{div}_{[D]}(f) \text{ for some } f \in \Gamma(X, \mathcal{R}_{[D]})\} \\ &= \{[D] \in \operatorname{Cl}(X);\; D \geq 0,\; x \notin \operatorname{Supp}(D)\}. \end{aligned}$$

□

Recall that a Weil divisor D in a normal prevariety X is said to be *semiample* if its stable base locus is empty. The cone $\operatorname{SAmple}(X)$ in $\operatorname{Cl}_{\mathbb{Q}}(X)$ generated by the classes of semiample divisors is convex. Moreover, D is *ample* if X is covered by affine sets $X_{nD,f}$ for some $n \in \mathbb{Z}_{\geq 1}$. The classes of ample divisors generate a cone $\operatorname{Ample}(X)$ in $\operatorname{Cl}_{\mathbb{Q}}(X)$ which is again convex.

Proposition 3.2.6. *Situation as in Construction I. 6.1.3. The the cones of semiample and ample divisor classes in $\operatorname{Cl}_{\mathbb{Q}}(X)$ are given as*

$$\operatorname{SAmple}(X) \;=\; \bigcap_{x \in X} \omega_x, \qquad \operatorname{Ample}(X) \;=\; \bigcap_{x \in X} \omega_x^{\circ}.$$

Proof. The statement on the semiample cone follows directly from the description of the stable base locus given in Proposition 3.2.5.

We turn to the ample cone. Surely, $\operatorname{Ample}(X)$ is the intersection of all cones $\omega_x^a \subseteq \operatorname{Cl}_{\mathbb{Q}}(X)$ generated by the submonoid

$$\begin{aligned} S_x^a &:= \{[D] \in \operatorname{Cl}(X);\; x \in X_{[D],f} \text{ for an } f \in \Gamma(X, \mathcal{R}_{[D]}) \text{ with } X_{[D],f} \text{ affine}\} \\ &\subseteq \operatorname{Cl}(X). \end{aligned}$$

Thus, we have to verify $\omega_x^a = \omega_x^{\circ}$ for any $x \in X$. We first show $\omega_x^a \supseteq \omega_x^{\circ}$. Let $[E] \in \omega_x^{\circ}$. Then $[E]$ admits a neighbourhood in ω_x° of the form

$$[E] \;\in\; \operatorname{cone}([E_1], \ldots, [E_k]) \;\subseteq\; \omega_x^{\circ}.$$

Here, we may assume that there are $g_i \in \Gamma(X, \mathcal{R}_{[E_i]})$ with $x \in X_{[E_i],g_i}$. Take any $[D] \in \omega_x^a$ and $f \in \Gamma(X, \mathcal{R}_{[D]})$ such that $X_{[D],f}$ is affine. Then we have

$$q_X^{-1}(x) \;\subseteq\; \widehat{X}_{fg_1 \cdots g_r} \subseteq \widehat{X}_f \;=\; q_X^{-1}(X_{[D],f}).$$

Since the restricted quotient map $q_X\colon \widehat{X}_f \to X_{[D],f}$ separates the disjoint closed H_X-invariant subsets $\widehat{X}_f \setminus \widehat{X}_{fg_1 \cdots g_r}$, we find a function $h \in \Gamma(X_{[D],f}, \mathcal{O}_X)$ with

$$h(x) \;\neq\; 0, \qquad h_{|q_X(\widehat{X}_f \setminus \widehat{X}_{fg_1 \cdots g_r})} \;=\; 0.$$

Fix $n \in \mathbb{Z}_{\geq 1}$ with $f' := hf^n \in \Gamma(X, \mathcal{R}_{[nD]})$. For any choice $a_1, \ldots, a_k$ of positive integers, we obtain a q_X-saturated affine open neighbourhood of the fiber over x:

$$q_X^{-1}(x) \;\subseteq\; \widehat{X}_{f' g_1^{a_1} \cdots g_r^{a_r}} \;\subseteq\; \widehat{X}_f \;=\; q_X^{-1}(X_{[D],f}).$$

Thus, $X_{n[D]+a_1[E_1]+\ldots+a_k[E_k], f' g_1^{a_1} \cdots g_r^{a_r}}$ is an affine neighbourhood of x. Consequently, all $n[D] + a_1[E_1] + \ldots + a_k[E_k]$ belong to ω_x^a. We conclude $[E] \in \omega_x^a$.

To show $\omega_x^a \subseteq \omega_x^\circ$, let $[D] \in \omega_x^a$. Replacing D with a suitable positive multiple, we may assume that $\widehat{X}_f$ is an affine neighbourhood of $q_X^{-1}(x)$ for some $f \in \Gamma(X, \mathcal{R}_{[D]})$. This enables us to choose an H-equivariant affine closure $\widehat{X} \subseteq Z$ with $\widehat{X}_f = Z_f$. Let $H_X \cdot \widehat{x}$ be the closed orbit of the fiber $q_X^{-1}(x)$. Then we have

$$[D] \ \in \ \omega_{Z,\widehat{x}} \ \subseteq \ \omega_x.$$

Let $\omega \preceq \omega_{Z,\widehat{x}}$ be the face with $[D] \in \omega^\circ$. Then $\omega = \omega_{Z,z}$ holds for some $z \in \overline{H_X \cdot \widehat{x}}$, where the closure is taken with respect to Z. Proposition 1.1.6 tells us $f(z) \neq 0$. Since $q^{-1}(x)$ and hence $H_X \cdot \widehat{x}$ are closed in $\widehat{X}_f$ and f vanishes along $Z \setminus \widehat{X}_f$, we obtain $z \in H_X \cdot \widehat{x}$ which implies $\omega = \omega_{Z,\widehat{x}}$ and $[D] \in \omega_{Z,\widehat{x}}^\circ \subseteq \omega_x^\circ$. □

Now we consider the variety X arising from a bunched ring $(R, \mathfrak{F}, \Phi)$. We first take a close look at the canonical isomorphism $K \cong \mathrm{Cl}(X)$ provided by Theorem 2.1.4. Set

$$E(R) \ := \ \bigcup_{w \in K} E(R)_w, \quad E(R)_w \ := \ \left\{ \frac{g}{h};\ g, h \in R \text{ homog.},\ \deg(g) - \deg(h) = w \right\}.$$

Then the vector space $E(R)_w$ contains precisely the homogeneous rational functions of weight w on $\overline{X}$; use $R^* = \mathbb{K}^*$ to see this. In the actual context, Proposition I. 6.4.5 tells us the following.

Proposition 3.2.7. *Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. Set $\widehat{X} := \widehat{X}(R, \mathfrak{F}, \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Then there is an epimorphism of abelian groups*

$$\delta\colon E(R) \ \to \ \mathrm{WDiv}(X), \qquad f \ \mapsto \ p_*(\mathrm{div}(f)).$$

We have $\mathrm{div}(f) = p^*(p_*(\mathrm{div}(f)))$ for every $f \in E(R)$. The epimorphism δ induces a well-defined isomorphism

$$K \ \to \ \mathrm{Cl}(X), \qquad w \ \mapsto \ [\delta(f)], \quad \textit{with any } f \in E(R)_w.$$

Fix $f \in E(R)_w$ and set $D := \delta(f)$. Then, for every open set $U \subseteq X$, we have an isomorphism of $\Gamma(U, \mathcal{O})$-modules

$$\Gamma(U, \mathcal{O}_X(D)) \ \to \ \Gamma(p^{-1}(U), \mathcal{O}_{\widehat{X}})_w, \qquad g \ \mapsto \ f p^*(g).$$

Moreover, for any section $g = h/f \in \Gamma(X, \mathcal{O}_X(D))$, the corresponding D-divisor satisfies

$$\mathrm{div}_D(g) \ = \ p_*(\mathrm{div}(h)), \qquad p^*(\mathrm{div}_D(g)) \ = \ \mathrm{div}(h).$$

If in this situation the open subset $X_{D,g} \subseteq X$ is affine, then its inverse image is given as $p^{-1}(X_{D,g}) = \overline{X}_h$.

Proposition 3.2.8. *Situation as in Construction 2.1.3. Then, for every $w \in K \cong \mathrm{Cl}(X)$, the base locus and the stable base locus are given as*

$$\mathrm{Bs}(w) \ = \ \bigcup_{\substack{\gamma_0 \in \mathrm{rlv}(\Phi) \\ w \notin Q(\gamma_0 \cap E)}} X(\gamma_0), \qquad \mathbf{Bs}(w) \ = \ \bigcup_{\substack{\gamma_0 \in \mathrm{rlv}(\Phi) \\ w \notin Q(\gamma_0)}} X(\gamma_0).$$

Proof. Proposition 3.1.4 tells us that, for every $\gamma_0 \in \mathrm{rlv}(\Phi)$ and $x \in X(\gamma_0)$, the monoid $S_x \subseteq \mathrm{Cl}(X)$ corresponds to $Q(\gamma_0 \cap E)$. Thus, Proposition 3.2.5 gives the desired statements. □

Proposition 3.2.9. *Situation as in Construction 2.1.3. The cones of effective, movable, semiample and ample divisor classes of X in $\mathrm{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}}$ are given as*

$$\mathrm{Eff}(X) \ = \ Q(\gamma), \qquad \mathrm{Mov}(X) \ = \ \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0),$$

$$\mathrm{SAmple}(X) \ = \ \bigcap_{\tau \in \Phi} \tau, \qquad \mathrm{Ample}(X) \ = \ \bigcap_{\tau \in \Phi} \tau^\circ.$$

Moreover, if $X \subseteq Z$ is the canonical toric embedding constructed in 2.5.3, then the cones of effective, movable, semiample and ample divisor classes in $\mathrm{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(Z)$ *coincide for X and Z.*

Proof. The descriptions of the effective and the moving cone are clear by Propositions 3.2.1 and 3.2.3. For the semiample and the ample cone, note as before that $S_x \subseteq \mathrm{Cl}(X)$ corresponds to $Q(\gamma_0 \cap E) \subseteq K$ and apply Proposition 3.2.6. □

Example 3.2.10. We continue the study of the surface $X = X(R, \mathfrak{F}, \Phi)$ from 2.1.6. The ring R and its grading by $K = \mathbb{Z}^2$ are given as

$$R = \mathbb{K}[T_1, \ldots, T_5] \,/\, \langle T_1T_2 + T_3^2 + T_4T_5 \rangle,$$
$$Q = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix},$$

where $\deg(T_i) = w_i$ with the i-th column w_i of Q. The system $\mathfrak{F}$ consists of the classes of $T_1, \ldots, T_5$ and the $\mathfrak{F}$-bunch is

$$\Phi = \{\tau\}, \qquad \tau = \mathrm{cone}(w_2, w_5).$$

The cones of effective, movable and semiample divisor classes in $\mathrm{Cl}_{\mathbb{Q}}(X) = \mathbb{Q}^2$ are given by

$$\mathrm{Eff}(X) = \mathrm{cone}(w_1, w_4), \qquad \mathrm{Mov}(X) = \mathrm{SAmple}(X) = \mathrm{cone}(w_2 w_5).$$

In the projected cone $(E \xrightarrow{Q} K, \gamma)$, we have $E = \mathbb{Z}^5$ and $\gamma = \mathrm{cone}(e_1, \ldots, e_5)$. With $\gamma_{i_1,\ldots,i_k} := \mathrm{cone}(e_{i_1}, \ldots, e_{i_k})$, the collection of relevant faces is

$$\mathrm{rlv}(\Phi) = \{\gamma_{1,4}, \gamma_{2,5}, \gamma_{1,2,3}, \gamma_{3,4,5}, \gamma_{1,2,3,4}, \gamma_{1,2,3,5}, \gamma_{1,2,4,5}, \gamma_{1,3,4,5}, \gamma_{2,3,4,5}, \gamma_{1,2,3,4,5}\}.$$

Recall that $X(\gamma_{2,5})$ consists of the singular point of X. For the (stable) base loci of $w := (0, 1)$, we obtain

$$\mathrm{Bs}(w) = \mathrm{Bs}(2w) = X(\gamma_{2,5}), \qquad \mathrm{Bs}(3w) = \mathbf{Bs}(w) = \emptyset.$$

3.3. Complete intersections. We consider a bunched ring $(R, \mathfrak{F}, \Phi)$, where R is a homogeneous complete intersection in the sense defined below. In this situation, there is a simple description of the canonical divisor and intersection numbers can easily be computed.

Definition 3.3.1. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring with grading group K and $\mathfrak{F} = (f_1, \ldots, f_r)$. We say that $(R, \mathfrak{F}, \Phi)$ is a *complete intersection*, if the kernel $I(\mathfrak{F})$ of the epimorphism

$$\mathbb{K}[T_1, \ldots, T_r] \to R, \qquad T_i \mapsto f_i.$$

is generated by K-homogeneous polynomials $g_1, \ldots, g_d \in \mathbb{K}[T_1, \ldots, T_r]$, where $d = r - \dim R$. In this situation, we call $(w_1, \ldots, w_r)$, where $w_i := \deg(f_i) \in K$ and $(u_1, \ldots, u_d)$, where $u_i := \deg(g_i) \in K$, *degree vectors* for $(R, \mathfrak{F}, \Phi)$.

Proposition 3.3.2. *Let the bunched ring $(R, \mathfrak{F}, \Phi)$ be a complete intersection with degree vectors $(w_1, \ldots, w_r)$ and $(u_1, \ldots, u_d)$. Then the canonical divisor class of $X = X(R, \mathfrak{F}, \Phi)$ is given in* $\mathrm{Cl}(X) = K$ *by*

$$w_X^{\mathrm{can}} = \sum_{j=1}^{d} u_j - \sum_{i=1}^{r} w_i.$$

Proof. Consider the embedding $X \to Z$ into a toric variety Z of X as constructed in 2.5.3. Then Proposition 3.1.11 tells us that the respective embeddings of the smooth loci fit into a commutative diagram

$$\begin{array}{ccc} X_{\mathrm{reg}} & \longrightarrow & Z_{\mathrm{reg}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Note that all maps induce isomorphisms on the respective divisor class groups. Moreover, the restrictions of the canonical divisors K_X and K_Z of X and Z give the canonical divisors of X_{reg} and Z_{reg}, respectively, see for example [**108**, p. 164]. Thus, we may assume that X and its ambient toric variety Z are smooth.

Let $\mathcal{I} \subset \mathcal{O}_Z$ be the ideal sheaf of X. Then the normal sheaf of X in Z is the rank d locally free sheaf $\mathcal{N}_X := (\mathcal{I}/\mathcal{I}^2)^*$, and a canonical bundle on X can be obtained as follows, see for example [**78**, Prop. II.8.20]:

$$\mathcal{K}_X \quad = \quad \mathcal{K}_Z|_X \otimes \Big(\bigwedge\nolimits^d \mathcal{N}_X\Big).$$

Choose a cover of Z by open subsets U_i such that $\mathcal{I}/\mathcal{I}^2$ is free over U_i. Then the g_l generate the relations of the f_j over $\widehat{U}_i := p_Z^{-1}(U_i)$. Thus, after suitably refining the cover, we find functions $h_{il} \in \mathcal{O}^*(\widehat{U}_i)$ of degree $\deg(g_l)$ such that $\mathcal{I}/\mathcal{I}^2(U_i)$ is generated by $g_1/h_{i1}, \ldots, g_d/h_{id}$. Therefore over U_i, the d-th exterior power of $\mathcal{I}/\mathcal{I}^2$ is generated by the function

$$\frac{g_1}{h_{i1}} \wedge \cdots \wedge \frac{g_d}{h_{id}}.$$

Proposition 3.2.7 tells us that the class of $\bigwedge^d \mathcal{I}/\mathcal{I}^2$ in K is minus the sum of the degrees of the g_j. As $\bigwedge^d \mathcal{N}_X$ is the dual sheaf, its class is $\deg(g_1) + \ldots + \deg(g_l)$. Furthermore, from II. 4.2.8 we know that the class of the canonical divisor of Z in K is given by $-(w_1 + \ldots + w_r)$. Putting all together we arrive at the assertion. □

A variety is called *($\mathbb{Q}$-)Gorenstein* if (some multiple of) its anticanonical divisor is Cartier. Moreover, it is called *($\mathbb{Q}$-)Fano* if (some multiple of) its anticanonical class is an ample Cartier divisor.

Corollary 3.3.3. *Let the bunched ring $(R, \mathfrak{F}, \Phi)$ be a complete intersection with degree vectors $(w_1, \ldots, w_r)$ and $(u_1, \ldots, u_d)$ and let $X = X(R, \mathfrak{F}, \Phi)$ be the associated variety.*

(i) *X is $\mathbb{Q}$-Gorenstein if and only if*

$$\sum_{i=1}^{r} w_i \ - \ \sum_{j=1}^{d} u_j \quad \in \quad \bigcap_{\tau \in \Phi} \operatorname{lin}(\tau),$$

(ii) *X is Gorenstein if and only if*

$$\sum_{i=1}^{r} w_i \ - \ \sum_{j=1}^{d} u_j \quad \in \quad \bigcap_{\gamma_0 \in \operatorname{cov}(\Phi)} Q(\operatorname{lin}(\gamma_0) \cap E).$$

(iii) *X is $\mathbb{Q}$-Fano if and only if we have*

$$\sum_{i=1}^{r} w_i \ - \ \sum_{j=1}^{d} u_j \quad \in \quad \bigcap_{\tau \in \Phi} \tau^\circ,$$

(iv) *X is Fano if and only if we have*

$$\sum_{i=1}^{r} w_i \;-\; \sum_{j=1}^{d} u_j \;\in\; \bigcap_{\tau \in \Phi} \tau^\circ \;\cap\; \bigcap_{\gamma_0 \in \mathrm{cov}(\Phi)} Q(\mathrm{lin}(\gamma_0) \cap E).$$

CONSTRUCTION 3.3.4 (Computing intersection numbers). Let the bunched ring $(R, \mathfrak{F}, \Phi)$ be a complete intersection with degree vectors $(w_1, \ldots, w_r)$ and $(u_1, \ldots, u_d)$ and suppose that $\Phi = \Phi(\lambda)$ holds with a full-dimensional GIT-cone $\lambda \in \Lambda(\overline{X}, H)$. Fix a full-dimensional $\eta \in \Lambda(\overline{Z}, H)$ with $\eta^\circ \subseteq \lambda^\circ$. For $w_{i_1}, \ldots, w_{i_{n+d}}$ let $w_{j_1}, \ldots, w_{j_{r-n-d}}$ denote the complementary weights and set

$$\begin{aligned} \tau(w_{i_1}, \ldots, w_{i_{n+d}}) &:= \mathrm{cone}(w_{j_1}, \ldots, w_{j_{r-n-d}}), \\ \mu(w_{i_1}, \ldots, w_{i_{n+d}}) &= [K : \langle w_{j_1}, \ldots, w_{j_{r-n-d}} \rangle]. \end{aligned}$$

Then the intersection product $K_{\mathbb{Q}}^{n+d} \to \mathbb{Q}$ of the ($\mathbb{Q}$-factorial) toric variety Z_1 associated to $\Phi(\eta)$ is determined by the values

$$w_{i_1} \cdots w_{i_{n+d}} \;=\; \begin{cases} \mu(w_{i_1}, \ldots, w_{i_{n+d}})^{-1}, & \eta \subseteq \tau(w_{i_1}, \ldots, w_{i_{n+d}}), \\ 0, & \eta \not\subseteq \tau(w_{i_1}, \ldots, w_{i_{n+d}}). \end{cases}$$

As a complete intersection, $X \subseteq Z_1$ inherits intersection theory. For a tuple $D_X^{i_1}, \ldots, D_X^{i_n}$ on X, its intersection number can be computed by

$$D_X^{i_1} \cdots D_X^{i_n} \;=\; w_{i_1} \cdots w_{i_n} \cdot u_1 \cdots u_d.$$

Note that the intersection number $D_X^{i_1} \cdots D_X^{i_n}$ vanishes if $\mathrm{cone}(e_{i_1}, \ldots, e_{i_n})$ does not belong to $\mathrm{rlv}(\Phi)$.

EXAMPLE 3.3.5. We continue the study of the surface $X = X(R, \mathfrak{F}, \Phi)$ from 2.1.6. The ring R and its grading by $K = \mathbb{Z}^2$ are given as

$$\begin{aligned} R &= \mathbb{K}[T_1, \ldots, T_5] \,/\, \langle T_1T_2 + T_3^2 + T_4T_5 \rangle, \\ Q &= \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix}, \end{aligned}$$

where $\deg(T_i) = w_i$ with the i-th column w_i of Q. The system $\mathfrak{F}$ consists of the classes of $T_1, \ldots, T_5$ and the $\mathfrak{F}$-bunch is

$$\Phi \;=\; \{\tau\}, \qquad\qquad \tau = \mathrm{cone}(w_2, w_5).$$

The degree of the defining relation is $\deg(T_1T_2 + T_3^2 + T_4T_5) = 2w_3$ and thus the canonical class of X is given as

$$w_X^c \;=\; 2w_3 - (w_1 + w_2 + w_3 + w_4 + w_5) \;=\; -3w_3.$$

In particular, we see that the anticanonical class is ample and thus X is a singular del Pezzo surface. The self intersection number of the canonical class is

$$(-3w_3)^2 \;=\; \frac{9(w_1 + w_2)(w_4 + w_5)}{4} \;=\; \frac{9}{4}\,(w_1 \cdot w_4 + w_2 \cdot w_4 + w_1 \cdot w_5 + w_2 \cdot w_5)$$

The $w_i \cdot w_j$ equal the toric intersection numbers $2w_i \cdot w_j \cdot w_3$. In order to compute these numbers, let w_{ij}^1, w_{ij}^2 denote the weights in $\{w_1, \ldots, w_5\} \setminus \{w_i, w_j, w_3\}$. Then we have

$$w_i \cdot w_j \cdot w_3 \;=\; \begin{cases} \mu(w_i, w_j, w_3)^{-1} & \tau \subseteq \mathrm{cone}(w_{ij}^1, w_{ij}^2), \\ 0 & \tau \not\subseteq \mathrm{cone}(w_{ij}^1, w_{ij}^2), \end{cases}$$

where the multiplicity $\mu(w_i, w_j, w_3)^{-1}$ is the absolute value of $\det(w_{ij}^1, w_{ij}^2)$. Thus, we can proceed in the computation:

$$(-3w_3)^2 \;=\; \frac{9 \cdot 2}{4}\left(|\det(w_2, w_5)|^{-1} + |\det(w_1, w_4)|^{-1}\right) \;=\; \frac{9}{2} \cdot \frac{4}{3} \;=\; 6.$$

Finally, we apply the techniques for complete intersections to complete d-dimensional varieties X with divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$. The Cox ring $\mathcal{R}(X)$ is finitely generated and the total coordinate space $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$ is a factorial affine variety coming with an action of $\mathbb{K}^*$ defined by the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$. Choose a system $f_1, \ldots, f_\nu$ of homogeneous pairwise nonassociated prime generators for $\mathcal{R}(X)$. This provides an $\mathbb{K}^*$-equivariant embedding

$$\overline{X} \ \to \ \mathbb{K}^\nu, \qquad \overline{x} \ \mapsto \ (f_1(\overline{x}), \ldots, f_\nu(\overline{x})).$$

where $\mathbb{K}^*$ acts diagonally with the weights $w_i = \deg(f_i) \in \mathrm{Cl}(X) \cong \mathbb{Z}$ on $\mathbb{K}^\nu$. Moreover, X is the geometric $\mathbb{K}^*$-quotient of $\widehat{X} := \overline{X} \setminus \{0\}$, and the quotient map $p \colon \widehat{X} \to X$ is a characteristic space.

Proposition 3.3.6. *For any* $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_\nu) \in \widehat{X}$ *the local divisor class group* $\mathrm{Cl}(X, x)$ *of* $x := p(\overline{x})$ *is finite of order* $\gcd(w_i;\ \overline{x}_i \neq 0)$. *The index of the Picard group* $\mathrm{Pic}(X)$ *in* $\mathrm{Cl}(X)$ *is given by*

$$[\mathrm{Cl}(X) : \mathrm{Pic}(X)] \quad = \quad \mathrm{lcm}_{x \in X}(|\,\mathrm{Cl}(X, x)|).$$

Suppose that the ideal of $\overline{X} \subseteq \mathbb{K}^\nu$ *is generated by* $\mathrm{Cl}(X)$*-homogeneous polynomials* $g_1, \ldots, g_{\nu-d-1}$ *of degree* $\gamma_j := \deg(g_j)$. *Then one obtains*

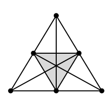
$$-\mathcal{K}_X \ = \ \sum_{i=1}^{\nu} w_i - \sum_{j=1}^{\nu-d-1} \gamma_j, \qquad (-\mathcal{K}_X)^d \ = \ \left(\sum_{i=1}^{\nu} w_i - \sum_{j=1}^{\nu-d-1} \gamma_j \right)^d \frac{\gamma_1 \cdots \gamma_{\nu-d-1}}{w_1 \cdots w_\nu}$$

for the anticanonical class $-\mathcal{K}_X \in \mathrm{Cl}(X) \cong \mathbb{Z}$. *In particular,* X *is a* $\mathbb{Q}$*-Fano variety if and only if the following inequality holds*

$$\sum_{j=1}^{\nu-d-1} \gamma_j \ < \ \sum_{i=1}^{\nu} w_i.$$

3.4. Mori dream spaces. We take a closer look to $\mathbb{Q}$-factorial projective varieties with a finitely generated Cox ring; Hu and Keel [**87**] called them *Mori dream spaces*. The Mori dream spaces sharing a given Cox ring fit into a nice picture provided by the GIT-fan; by the *moving cone* of the K-graded algebra R we mean here the intersection $\mathrm{Mov}(R)$ over all $\mathrm{cone}(w_1, \ldots, \widehat{w_i}, \ldots, w_r)$, where the w_i are the degrees of any system of pairwise nonassociated homogeneous K-prime generators for R.

Remark 3.4.1. Let $R = \oplus_K R_w$ be an almost freely factorially graded affine algebra with $R_0 = \mathbb{K}$ and consider the GIT-fan $\Lambda(\overline{X}, H)$ of the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} = \operatorname{Spec} R$.



Then every GIT-cone $\lambda \in \Lambda(\overline{X}, H)$ defines a projective variety $X(\lambda) := \overline{X}^{ss}(\lambda) /\!\!/ H$. If $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$ holds, then $X(\lambda)$ is the variety associated to the bunched ring $(R, \mathfrak{F}, \Phi(\lambda))$ with $\Phi(\lambda)$ defined as in 1.3.6. In particular, in this case we have

$$\mathrm{Cl}(X(\lambda)) \ = \ K, \qquad \mathcal{R}(X(\lambda)) \ = \ R,$$

$$\mathrm{Mov}(X(\lambda)) \ = \ \mathrm{Mov}(R), \qquad \mathrm{SAmple}(X(\lambda)) \ = \ \lambda.$$

All projective varieties with Cox ring R are isomorphic to some $X(\lambda)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$ and the Mori dream spaces among them are precisely those arising from a full dimensional λ.

Let X be the variety arising from a bunched ring $(R, \mathfrak{F}, \Phi)$. Every Weil divisor D on X defines a positively graded sheaf

$$\mathcal{S}^+ \; := \; \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{S}_n^+, \qquad \qquad \mathcal{S}_n^+ \; := \; \mathcal{O}_X(nD).$$

Using Propositions I. 1.2.2, I. 6.1.1 and I. 6.1.4, we see that this sheaf is locally of finite type, and thus we obtain a rational map

$$\varphi(D)\colon X \;\rightarrow\; X(D), \qquad X(D) \;:=\; \mathrm{Proj}(\Gamma(X, \mathcal{S}^+)).$$

Note that $X(D)$ is explicitly given as the closure of the image of the rational map $X \to \mathbb{P}^m$ determined by the linear system of a sufficiently big multiple nD.

Remark 3.4.2. Consider the GIT-fan $\Lambda(\overline{X}, H)$ of the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} = \operatorname{Spec} R$. Let $\lambda \in \Lambda(\overline{X}, H)$ be the cone with $[D] \in \lambda^\circ$ and $W \subseteq \overline{X}$ the open subset obtained by removing the zero sets of the generators $f_1, \ldots, f_r \in R$. Then we obtain a commutative diagram

$$\begin{array}{ccccc}
\widehat{X} & \supseteq & W & \subseteq & \overline{X}^{ss}(\lambda) \\
{\scriptstyle /\!\!/ H}\downarrow & & \downarrow & & \downarrow{\scriptstyle /\!\!/ H} \\
X & \supseteq & W/H & \longrightarrow & X(\lambda) \\
 & \searrow & & & \Vert \\
 & & {\scriptstyle \varphi(D)} & \dashrightarrow & X(D)
\end{array}$$

We list basic properties of the maps $\varphi(D)$; by a *small birational map* we mean a rational map defining an isomorphism of open subsets with complement of codimension two.

Remark 3.4.3. Let $D \in \mathrm{WDiv}(X)$ be any Weil divisor, and denote by $[D] \in \mathrm{Cl}(X)$ its class. Then the assocciated rational map $\varphi(D)\colon X \to X(D)$ is

(i) birational if and only if $[D] \in \mathrm{Eff}(X)^\circ$ holds.
(ii) small birational if and only if $[D] \in \mathrm{Mov}(X)^\circ$ holds.
(iii) a morphism if and only if $[D] \in \mathrm{SAmple}(X)$ holds.
(iv) an isomorphism if and only if $[D] \in \mathrm{Ample}(X)$ holds.

Recall that two Weil divisors $D, D' \in \mathrm{WDiv}(X)$ are said to be *Mori equivalent*, if there is a commutative diagram of rational maps

$$\begin{array}{ccc}
 & X & \\
{\scriptstyle \varphi(D)}\swarrow & & \searrow{\scriptstyle \varphi(D')} \\
X(D) & \xleftrightarrow[\cong]{} & X(D')
\end{array}$$

where the horizontal arrow stands for an isomorphism of varieties. The Mori equivalence is described by the GIT-fan of the characteristic quasitorus action on the total coordinate space.

Proposition 3.4.4. *Let $X = X(R, \mathfrak{F}, \Phi)$ be the variety arising from a bunched ring and let $\Lambda(\overline{X}, H)$ be the GIT-fan of the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on $\overline{X} = \operatorname{Spec} R$. Then for any two $D, D' \in \mathrm{WDiv}(X)$, the following statements are equivalent.*

(i) *The divisors D and D' are Mori equivalent.*
(ii) *One has $[D], [D'] \in \lambda^\circ$ for some GIT-chamber $\lambda \in \Lambda(\overline{X}, H)$.*

Proof. The assertion follows immediately from the observation that $X(D)$ is the GIT-quotient associated to the chamber $\lambda \in \Lambda(\overline{X}, H)$ with $[D] \in \lambda^\circ$. $\square$

We are ready to prove the characterization of Mori dream spaces given by Hu and Keel; their statement extends to arbitrary normal complete varieties and says the following.

Theorem 3.4.5. *Let X be a normal complete variety with finitely generated divisor class group. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*
(ii) *There are small birational maps $\pi_i \colon X \to X_i$, where $i = 1, \dots, r$, such that each semiample cone $\mathrm{SAmple}(X_i) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ is polyhedral and*

$$\mathrm{Mov}(X) \quad = \quad \pi_1^*(\mathrm{SAmple}(X_1)) \ \cup \ \dots \ \cup \ \pi_r^*(\mathrm{SAmple}(X_r)).$$

Moreover, if one of these two statements holds, then X_i from (ii) can be taken $\mathbb{Q}$-factorial and projective.

Lemma 3.4.6. *Let X be a normal complete variety with $\mathrm{Cl}(X)$ finitely generated. Then $\mathrm{Mov}(X)$ is of full dimension in the rational divisor class group $\mathrm{Cl}_{\mathbb{Q}}(X)$.*

Proof. Using Chow's Lemma and resolution of singularities, we obtain a birational morphism $\pi \colon X' \to X$ with a smooth projective variety X'. Let $D_1, \dots, D_r \in \mathrm{WDiv}(X)$ be prime divisors generating $\mathrm{Cl}(X)$, and consider their proper transforms $D_1', \dots, D_r' \in \mathrm{WDiv}(X')$. Moreover, let $E' \in \mathrm{WDiv}(X')$ be very ample such that all $E' + D_i'$ are also very ample, and denote by $E \in \mathrm{WDiv}(X)$ its push-forward. Then the classes E and $E + D_i$ generate a fulldimensional cone $\tau \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ and, since E' and the $E' + D_i'$ are movable, we have $\tau \subseteq \mathrm{Mov}(X)$. □

Lemma 3.4.7. *Let X be a normal complete variety with finitely generated divisor class group $\mathrm{Cl}(X)$. If $\mathrm{Mov}(X)$ is polyhedral then also $\mathrm{Eff}(X)$ is polyhedral.*

Proof. In the situation of Proposition 3.2.3, set $w_i := \deg(f_i)$. Since $\mathrm{Mov}(X)$ is polyhedral, it has only finitely many facets and these are cut out by hyperplanes $H_1, \dots, H_m$. Let H_k^+ denote the closed half space bounded by H_k which comprises $\mathrm{Mov}(X)$. We claim that for every k, there is at most one w_i with $w_i \notin H_k^+$. Otherwise, we have two $w_i, w_j \notin H_k^+$. Let $\sigma_k = \mathrm{Mov}(X) \cap H_k$ be the facet of $\mathrm{Mov}(X)$ cut out by H_k. Then the cones

$$\tau_i \ := \ \mathbb{Q}_{\ge 0} \cdot w_i + \sigma_k, \qquad\qquad \tau_j \ := \ \mathbb{Q}_{\ge 0} \cdot w_j + \sigma_k,$$

are of full dimension and their relative interiors intersect nontrivially. Consider any $w \in \tau_i^\circ \cap \tau_j^\circ$. Then, by the description of $\mathrm{Mov}(X)$ given in Proposition 3.2.3, we have $w \in \mathrm{Mov}(X)$. On the other hand, we have $w \notin H_k^+$, a contradiction. This proves our claim, i.e., each H_k^+ has at most the generator w_i in its complement. Thus, besides the generators of $\mathrm{Mov}(X)$, only finitely many w_i are needed to generate $\mathrm{Eff}(X)$. □

Proof of Theorem 3.4.5. Set $K := \mathrm{Cl}(X)$ and $R := \mathcal{R}(X)$. Let $\mathfrak{F} = (f_1, \dots, f_r)$ be a system of pairwise nonassociated homogeneous prime generators of R and set $w_i := \deg(f_i)$. By Proposition I. 6.1.6 and Construction I. 6.3.1, the group $H := \mathrm{Spec}\,\mathbb{K}[K]$ acts freely on an open subset $W \subseteq \overline{X}$ of $\overline{X} = \mathrm{Spec}\,R$ such that $\overline{X} \setminus W$ is of codimension at least two in $\overline{X}$. Proposition 2.2.2 then tells us that the corresponding K-grading of R is almost free. Moreover, by Lemma 3.4.6, the moving cone of X is of full dimension, and by Proposition 3.2.3, it is given as

$$\mathrm{Mov}(X) \quad = \quad \bigcap_{i=1}^{r} \mathrm{cone}(w_j;\ j \ne i).$$

Thus, we are in the setting of Remark 3.4.1. That means that $\mathrm{Mov}(X)$ is a union of fulldimensional GIT-chambers $\lambda_1, \dots, \lambda_r$, the relative interiors of which are contained in the relative interior of $\mathrm{Mov}(X)$ and the associated projective varieties

$X_i := \widehat{X}_i /\!\!/ H$, where $\widehat{X}_i := \overline{X}^{ss}(\lambda_i)$, are $\mathbb{Q}$-factorial, have $\mathcal{R}(X)$ as their Cox ring and λ_i as their semiample cone.

Moreover, if $q\colon \widehat{X} \to X$ and $q_i\colon \widehat{X}_i \to X_i$ denote the associated characteristic spaces, then the desired small birational maps $\pi_i\colon X \to X_i$ are obtained as follows. Let $X' \subseteq X$ and $X_i' \subseteq X_i$ be the respective sets of smooth points. Then, by Proposition I. 6.1.6, the sets $q^{-1}(X')$ and $q_i^{-1}(X_i')$ have a small complement in $\overline{X}$ and thus we obtain open embeddings with a small complement

$$X \longleftarrow (q^{-1}(X') \cap q_i^{-1}(X_i'))/\!\!/ H \longrightarrow X_i.$$

Now suppose that (ii) holds. Then $\mathrm{Mov}(X)$ is polyhedral and hence, by Lemma 3.4.7, also $\mathrm{Eff}(X)$ is polyhedral. Let $w_1, \ldots, w_d \in \mathrm{Eff}(X)$ be those primitive generators of extremal rays of $\mathrm{Eff}(X)$ that satisfy $\dim \Gamma(X, \mathcal{R}_{nw_i}) \le 1$ for any $n \in \mathbb{Z}_{\ge 0}$ and fix $0 \ne f_i \in \mathcal{R}(X)_{n_i w_i}$ with n_i minimal. Then we have

$$\bigoplus_{n \in \mathbb{Z}_{\ge 0}} \mathcal{R}(X)_{nw_i} \quad = \quad \mathbb{K}[f_i].$$

Set $\lambda_i := \pi_i^*(\mathrm{SAmple}(X_i))$. Then, by Gordon's Lemma and [**87**, Lemma 1.8], we have another finitely generated subalgebra of the Cox ring, namely

$$\mathcal{S}(X) \;:=\; \bigoplus_{w \in \mathrm{Mov}(X)} \mathcal{R}(X)_w \;=\; \sum_{i=1}^{r} \left(\bigoplus_{w \in \lambda_i} \mathcal{R}(X)_w \right).$$

We show that $\mathcal{R}(X)$ is generated by $\mathcal{S}(X)$ and the $f_i \in \mathcal{R}(X)_{n_i w_i}$. Consider any $0 \ne f \in \mathcal{R}(X)_w$ with $w \notin \mathrm{Mov}(X)$. Then, by Lemma 3.2.4, we have $f = f^{(1)} f_i$ for some $1 \le i \le d$ and some $f^{(1)} \in \mathcal{R}(X)$ homogeneous of degree $w(1) := w - n_i w_i$. If $w(1) \notin \mathrm{Mov}(X)$ holds, then we repeat this procedure with $f^{(1)}$ and obtain $f = f^{(2)} f_i f_j$ with $f^{(2)}$ homogeneous of degree $w(2)$. At some point, we must end with $w(n) = \deg(f^{(n)}) \in \mathrm{Mov}(X)$, because otherwise the sequence of the $w(n)$'s would leave the effective cone. □

Theorem 3.4.8. *Let X be a normal complete surface with finitely generated divisor class group $\mathrm{Cl}(X)$. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*
(ii) *One has $\mathrm{Mov}(X) = \mathrm{SAmple}(X)$ and this cone is polyhedral.*

Moreover, if one of these two statements holds, then the surface X is $\mathbb{Q}$-factorial and projective.

Proof. For "(i)⇒(ii)", we only have to show that the moving cone coincides with the semiample cone. Clearly, we have $\mathrm{SAmple}(X) \subseteq \mathrm{Mov}(X)$. Suppose that $\mathrm{SAmple}(X) \ne \mathrm{Mov}(X)$ holds. Then $\mathrm{Mov}(X)$ is properly subdivided into GIT-chambers, see Remark 3.4.1. In particular, we find two chambers λ' and λ both intersecting the relative interior of $\mathrm{Mov}(X)$ such that λ' is a proper face of λ. The associated GIT-quotients Y' and Y of the total coordinate space $\overline{X}$ have λ' and λ as their respective semiample cones. Moreover, the inclusion $\lambda' \subseteq \lambda$ gives rise to a proper morphism $Y \to Y'$, which is an isomorphism in codimension one. As Y and Y' are normal surfaces, we obtain $Y \cong Y'$, which contradicts the fact that the semiample cones of Y and Y' are of different dimension.

The verification of "(ii)⇒(i)" runs as in the preceding proof; this time one uses the finitely generated subalgebra

$$\mathcal{S}(X) \;:=\; \bigoplus_{w \in \mathrm{Mov}(X)} \mathcal{R}(X)_w \;=\; \bigoplus_{w \in \mathrm{SAmple}(X)} \mathcal{R}(X)_w.$$

Moreover, by Theorem 3.4.5, there is a small birational map $X \to X'$ with X' projective and $\mathbb{Q}$-factorial. As X and X' are complete surfaces, this map already defines an isomorphism. □

In the case of a $\mathbb{Q}$-factorial surface X, we obtain the following simpler characterization involving the cone $\mathrm{Nef}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ of numerically effective divisor classes; note that the implication "(ii)⇒(i)" was obtained for smooth surfaces in [**74**, Cor. 1].

Corollary 3.4.9. *Let X be a $\mathbb{Q}$-factorial projective surface with finitely generated divisor class group $\mathrm{Cl}(X)$. Then the following statements are equivalent.*

(i) *The Cox ring $\mathcal{R}(X)$ is finitely generated.*
(ii) *The effective cone $\mathrm{Eff}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ is polyhedral and $\mathrm{Nef}(X) = \mathrm{SAmple}(X)$ holds.*

Proof. If (i) holds, then we infer from [**29**, Cor. 7.4] that the semiample cone and the nef cone of X coincide. Now suppose that (ii) holds. From

$$\mathrm{SAmple}(X) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Nef}(X)$$

we then conclude $\mathrm{Mov}(X) = \mathrm{Nef}(X)$. Moreover, since $\mathrm{Eff}(X)$ is polyhedral, $\mathrm{Nef}(X)$ is given by a finite number of inequalities and hence is also polyhedral. Thus, we can apply Theorem 3.4.8. □

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