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AFFINE EMBEDDINGS WITH A FINITE NUMBER OF ORBITS

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Abstract. Let G be a reductive algebraic group and let H be a reductive subgroup of G. We describe all pairs (G, H) such that, for any affine G-variety X with a dense G-orbit isomorphic to G/H, the number of G-orbits in X is finite.

1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero and let H be an algebraic subgroup of G. Recall that a pointed irreducible algebraic G-variety X is called *embedding* of the homogeneous space G/H if the base point of X has the dense orbit and stabilizer H. We denote this by $G/H \hookrightarrow X$.

Let B be a Borel subgroup of G. By definition, the *complexity* $c(X)$ of a G-variety X is the codimension of a generic B-orbit in X for the restricted action $B : X$, see [V1] and [LV]. By Rosenlicht's theorem, $c(X)$ is equal to the transcendence degree of the field $\Bbbk(X)^B$ of rational B-invariant functions on X. A normal G-variety X is called *spherical* if $c(X) = 0$, or, equivalently, $\Bbbk(X)^B = \Bbbk$. A homogeneous space G/H and a subgroup $H \subseteq G$ are said to be spherical if G/H is a spherical G-variety with respect to the natural G-action.

Theorem 1. (Servedio [Se], Luna–Vust [LV], Akhiezer [Akh]) *A space G/H is spherical if and only if each embedding of G/H has finitely many G-orbits.*

To be more precise, F. J. Servedio proved that any affine spherical variety contains finitely many G-orbits, D. Luna, Th. Vust and D. N. Akhiezer extended this result to an arbitrary spherical variety, and D. N. Akhiezer constructed a projective embedding with infinitely many G-orbits for any homogeneous space of positive complexity.

Let us say that an embedding $G/H \hookrightarrow X$ is *affine* if the variety X is affine. In many problems of invariant theory, representation theory and other branches of mathematics, only affine embeddings of homogeneous spaces are considered. Hence for a homogeneous space G/H it is natural to ask: does there exist an affine embedding $G/H \hookrightarrow X$ with infinitely many G-orbits?

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Note that a given homogeneous space G/H admits an affine embedding if and only if G/H is quasi-affine (as an algebraic variety), see [PV, Th. 1.6]. In this situation, the subgroup H is said to be *observable* in G. For a description of observable subgroups, see [Su], [PV, Th. 4.18]. By Matsushima's criterion, G/H is affine iff H is reductive. (For a simple proof, see [L1, §2].) In particular, any reductive subgroup is observable. In the sequel, we suppose that H is an observable subgroup of G. In this paper, we are concerned with the following problem: characterize all quasi-affine homogeneous spaces G/H of a reductive group G with the property:

(AF) *For any affine embedding $G/H \hookrightarrow X$, the number of G-orbits in X is finite.*

Example 1. For any spherical quasi-affine homogeneous space, property (AF) holds (Theorem 1).

Example 2. ([Po]) Property (AF) holds for any homogeneous space of the group $SL(2)$. In fact, here $\dim X \leq 3$, and only a one-parameter family of one-dimensional orbits can appear in $X \setminus (G/H)$. But $SL(2)$ contains no two-dimensional observable subgroups.

Example 3. Let T be a maximal torus in G and let V be a finite-dimensional G-module. Suppose that a vector $v \in V$ is T-fixed. Then the orbit Gv is closed in V, see [Ko], [L2]. This shows that property (AF) holds for any subgroup H such that $T \subseteq H$.

Definition 1. An affine homogeneous space G/H is called *affinely closed* if it admits only one affine embedding $X = G/H$.

Homogeneous spaces G/H of Example 3 are affinely closed. Denote by $N_G(H)$ the normalizer of H in G. The following theorem generalizes Example 3:

Theorem 2. (Luna [L2]) *Let H be a reductive subgroup of a reductive group G. The homogeneous space G/H is affinely closed if and only if the group $N_G(H)/H$ is finite.*

This theorem provides many examples of homogeneous spaces with property (AF). Let us note that the complexity of the space G/T can be arbitrarily large, whence property (AF) cannot be characterized only in terms of complexity.

In this paper, we show that the union of two conditions—the sphericity and the finiteness of $N_G(H)/H$—is very close to characterizing all affine homogeneous spaces of a reductive group G with property (AF). Our main result is

Theorem 3. *For a reductive subgroup $H \subseteq G$,* (AF) *holds iff either $N_G(H)/H$ is finite or any extension of H by a one-dimensional torus in $N_G(H)$ is spherical in G.*

Corollary 1. *For an affine homogeneous space G/H of complexity > 1,* (AF) *holds iff G/H is affinely closed.*

Corollary 2. *An affine homogeneous space G/H of complexity 1 has* (AF) *iff either $N_G(H)/H$ is finite or $\operatorname{rk} N_G(H)/H = 1$ and $N_G(H)$ is spherical.*

The proofs of Theorem 3 and its corollaries are given in Sections 2, 4.

For simple G, there is a list of all affine homogeneous spaces of complexity one [Pa]. We immediately deduce from this list and Corollary 2 that for simple G, there exists only one series of affine homogeneous spaces of complexity one that admit affine embeddings with infinitely many G-orbits. Namely, $G = SL(n)$, $n > 4$, and $H^0 = SL(n-2) \times \Bbbk^*$, where $SL(n-2)$ is embedded in $SL(n)$ as the stabilizer of the first two basis vectors e_1 and e_2 in the tautological representation of $SL(n)$, and $\Bbbk^*$ acts on e_1 and e_2 with

weights α_1 and α_2 such that $\alpha_1 + \alpha_2 = 2 - n$, $\alpha_1 \neq \alpha_2$, and acts on $\langle e_3, \ldots, e_n \rangle$ by scalar multiplications.

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Notation and conventions.

G is a connected reductive group;
H is an observable subgroup of G;
$T \subseteq B$ are a maximal torus and a Borel subgroup of G;
U is the maximal unipotent subgroup of B;
$N_G(H)$ is the normalizer of H in G;
$W(H)$ is the quotient group $N_G(H)/H$;
$\gamma : N_G(H) \to W(H)$ is the quotient homomorphism;
$\Bbbk^*$ is the multiplicative group of nonzero elements of the base field $\Bbbk$;
L^0 is the identity component of an algebraic group L;
$Z(L)$ is the center of L, $\mathfrak{z}(L)$ is its Lie algebra;
X^L is the set of L-fixed points in an L-variety X;
L_x is the isotropy subgroup of $x \in X$;
$\Xi(G)_+$ is the semigroup of all dominant weights of G;
V_μ is an irreducible G-module with highest weight μ.

$\Bbbk[X]$ is the algebra of regular functions and $\Bbbk(X)$ is the field of rational functions on an algebraic variety X. $\operatorname{Spec} A$ is the affine variety corresponding to a finitely generated algebra A without nilpotent elements. Algebraic groups are denoted by uppercase Latin letters and their Lie algebras by the respective lower case Gothic letters.

2. Embeddings with infinitely many orbits

Theorem 4. *Let H be an observable subgroup in a reductive group G. Suppose that there is a non-trivial one-parameter subgroup $\lambda : \Bbbk^* \to W(H)$ such that the subgroup $H_1 = \gamma^{-1}(\lambda(\Bbbk^*))$ is not spherical in G. Then there exists an affine embedding $G/H \hookrightarrow X$ with infinitely many G-orbits.*

We shall prove this theorem in the next section. The idea of the proof is to apply Akhiezer's construction for the nonspherical homogeneous space G/H_1 and to consider the affine cone over a projective embedding of G/H_1 with infinitely many G-orbits.

Proof of Corollary 1. The assertion follows from Theorem 2 and Theorem 4, which is a part of Theorem 3 (for reductive H). Indeed, reductivity of H implies reductivity of $W(H)$ [L2]. If $W(H)$ is not finite, then it contains a non-trivial one-parameter subgroup $\lambda(\Bbbk^*)$. For $H_1 = \gamma^{-1}(\lambda(\Bbbk^*))$, we have $c(G/H_1) \geq 1$ whenever $c(G/H) > 1$.

Corollary 3. *Let G be a reductive group with infinite center $Z(G)$ and let H be an observable subgroup in G that does not contain $Z(G)^0$. Then property* (AF) *holds for for G/H if and only if H is a spherical subgroup of G.*

Proof. As H does not contain $Z(G)^0$, there exists a non-trivial one-parameter subgroup $\lambda(\Bbbk^*)$ in $Z(G)$ with finite intersection with H. The corresponding extension H_1 is spherical iff H is spherical in G.

Corollary 4. *Let H be a connected reductive subgroup in a reductive group G. Suppose that there exists a reductive nonspherical subgroup H_1 in G such that $H \subset H_1$ and $\dim H_1 = \dim H + 1$. Then property* (AF) *does not hold for G/H.*

Proof. Under these assumptions, there exists a non-trivial one-parameter subgroup of H_1 with finite intersection with H which normalizes (and even centralizes) H.

3. Proof of Theorem 4

Lemma 1. *If property* (AF) *holds for a homogeneous space G/H, then it holds for any homogeneous space G/H', where H' is an overgroup of H with $(H')^0 = H^0$.*

Proof. Suppose that there exists an affine embedding $G/H' \hookrightarrow X$ with infinitely many G-orbits. Consider the morphism $G/H \to G/H'$. The dual algebra homomorphism yields an embedding $\Bbbk[G/H'] \subseteq \Bbbk[G/H]$. Let A be the integral closure of the subalgebra $\Bbbk[X] \subseteq \Bbbk[G/H']$ in the field of rational functions $\Bbbk(G/H)$. We have the following commutative diagrams:

$$\begin{array}{ccccc} A & \hookrightarrow & \Bbbk[G/H] & \hookrightarrow & \Bbbk(G/H) \\ \uparrow & & \uparrow & & \uparrow \\ \Bbbk[X] & \hookrightarrow & \Bbbk[G/H'] & \hookrightarrow & \Bbbk(G/H') \end{array} \quad , \qquad \begin{array}{ccc} \operatorname{Spec} A & \hookleftarrow & G/H \\ \downarrow & & \downarrow \\ X & \hookleftarrow & G/H' \end{array} \quad .$$

The affine variety $\operatorname{Spec} A$ with a natural G-action can be considered as an affine embedding of G/H. The embedding $\Bbbk[X] \subseteq A$ defines a finite (surjective) morphism $\operatorname{Spec} A \to X$, and therefore $\operatorname{Spec} A$ contains infinitely many G-orbits. This contradiction completes the proof.

Remark 1. The converse statement does not hold. Indeed, set $G = SL(3)$ and $H = (t, t, t^{-2}) \subset T \subset SL(3)$. We can extend H by a one-parameter subgroup $(t, t^{-1}, 1)$. Then $H_1 = T$ is not a spherical subgroup in $SL(3)$ and, by Theorem 4, property (AF) does not hold here. On the other hand, one can extend H to H' by a finite non-cyclic subgroup of $W(H) \cong PSL(2)$. The group $W(H')$ is finite and, by Theorem 2, property (AF) holds for G/H'.

Lemma 2. (a) *Let $H \subseteq G$ be an observable subgroup and H_1 the extension of H by a one-dimensional torus $\lambda(\Bbbk^*) \subseteq W(H)$. Then there exists a finite-dimensional G-module V and an H_1-eigenvector $v \in V$ such that*

(1) *the orbit $G\langle v\rangle$ of the line $\langle v\rangle$ in the projective space $\mathbb{P}(V)$ is isomorphic to G/H_1;*
(2) *H fixes v;*
(3) *H_1 acts transitively on $\Bbbk^* v$.*

(b) *If H_1 is not spherical in G, then a couple (V, v) in* (a) *may be chosen so that*
(4) *the closure of $G\langle v\rangle$ in $\mathbb{P}(V)$ contains infinitely many G-orbits.*

(c) *If H is reductive, then one may suppose that $G_v = H$.*

Proof. (a) By Chevalley's Theorem, there exists a G-module V' and a vector $v' \in V'$ having property (1). Let us denote by χ the character of H at v'. Since H is observable in G, every finite-dimensional H-module can be embedded in a finite-dimensional G-module [BHM]. In particular, there exists a finite-dimensional G-module V'' containing

H-eigenvectors of character $-\chi$. Choose among them a H_1-eigenvector v'' and put $V = V' \otimes V''$ and $v = v' \otimes v''$. Properties (1) and (2) are satisfied.

If condition (3) also holds, then we are done. Otherwise, consider any G-module W having a vector with stabilizer H. Take an H_1-eigenvector $w \in W^H$ with non-trivial character, and replace V by $V \otimes W$ and v by $v \otimes w$. Now properties (1)–(3) are satisfied.

(b) Since H_1 is not spherical in G, by a result due to Akhiezer [Akh], we may choose (V', v') in (a) so that properties (1) and (4) are satisfied. Then we proceed as in (a) to obtain the couple (V, v). The closure $\overline{G\langle v\rangle} \subseteq \mathbb{P}(V)$ is contained in the image of the Segre embedding $\mathbb{P}(V') \times \mathbb{P}(V'') \hookrightarrow \mathbb{P}(V)$ or $\mathbb{P}(V') \times \mathbb{P}(V'') \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$, and projects G-equivariantly onto $\overline{G\langle v'\rangle} \subseteq \mathbb{P}(V')$. This implies (4) for (V, v).

(c) Let ω be a fundamental weight of the group H_1/H. Suppose that H_1/H acts at the vector v constructed above by a character $k\omega$. Since H_1 is reductive (and, in particular, is observable), there exists a G-module W' and an H_1-eigenvector $w' \in W'^H$ with weight $(1-k)\omega$ [BHM]. It remains to replace V by $V \otimes W'$ and v by $v \otimes w'$.

Remark 2. For an arbitrary observable subgroup, statement (c) Lemma 2 does not hold. For example, let G be the group $SL(3)$ and $H = U$ be a maximal unipotent subgroup normalized by T. Consider the subtorus $T' = \mathrm{diag}(t^2, t, t^{-3})$ in T as a one-parameter subgroup $\lambda(\Bbbk^*)$. Any H-stable vector in a finite-dimensional G-module is a sum of highest weight vectors. The restriction of any dominant weight to T' has a non-trivial kernel and the stabilizer of such a vector contains H as a proper subgroup.

Proof of Theorem 4. Let $(V,\ v)$ be the couple from Lemma 2. Denote by H' the stabilizer G_v of the vector v. By (1)–(3) and since H_1/H is isomorphic to $\Bbbk^*$, H' is an overgroup of H with $(H')^0 = H^0$. By (3), the closure of Gv in V is a cone, so by (4) the property (AF) does not hold for G/H'. Lemma 1 completes the proof.

4. Proof of Theorem 3

Let H be a reductive subgroup of G. If there exists a nonspherical extension of H by a one-dimensional torus, then (AF) fails for G/H by Theorem 4. To prove the converse, we begin with the following:

Lemma 3. ([K1, 7.3.1]) *Let X be an irreducible G-variety, and v a G-invariant valuation of $\Bbbk(X)/\Bbbk$ with residue field $\Bbbk(v)$. Then $\Bbbk(v)^B$ is the residue field of the restriction of v to $\Bbbk(X)^B$.*

Proof. For completeness, we give the proof in the case where X is affine (the only case we need below). It suffices to prove that any B-invariant element of $\Bbbk(v)$ is the residue class of a B-invariant rational function on X.

For any $f_1, f_2 \in \Bbbk(X)$, we shall write $f_1 \equiv f_2$ if $v(f_1) = v(f_2) < v(f_1 - f_2)$. Such "congruences" are G-stable and may be multiplied term by term, as numerical congruences.

Assume $f = p/q$, $p, q \in \Bbbk[X]$, $v(f) = 0$, and the residue class of f belongs to $\Bbbk(v)^B$. Then $v(p) = v(q) = d$, and $bf \equiv f$, $\forall b \in B$, i.e., $bp \cdot q \equiv p \cdot bq$.

Let M be a complementary G-submodule to $\{h \in \Bbbk[X] \mid v(h) > d\}$ in $\{h \in \Bbbk[X] \mid v(h) \geq d\}$, and p_0, q_0 be the projections of p, q on M. Then $bp_0 \cdot q \equiv bp \cdot q \equiv p \cdot bq \equiv p \cdot bq_0$, $\forall b \in B$. By the Lie–Kolchin theorem, we may choose finitely many $b_i \in B$, $\lambda_i \in \Bbbk$ so that $q_1 = \sum \lambda_i b_i q_0$ is a B-eigenfunction in M of some weight μ. Put $p_1 = \sum \lambda_i b_i p_0$. Then $p_1 \cdot q \equiv p \cdot q_1$, whence $p_1/q_1 \equiv f \equiv bf \equiv bp_1/\mu(b)q_1$, $\forall b \in B$. It follows that

$bp_1 \equiv \mu(b)p_1$, hence $bp_1 = \mu(b)p_1$, because $p_1 \in M$. Thus p_1, q_1 are B-eigenfunctions of the same weight, and $f_1 = p_1/q_1 \in \Bbbk(X)^B$ has the same residue class in $\Bbbk(v)$ as f.

Definition 2. ([K2, §7]) Let X be a normal G-variety. A discrete $\mathbb{Q}$-valued G-invariant valuation of $\Bbbk(X)$ is called *central* if it vanishes on $\Bbbk(X)^B \setminus \{0\}$. A *source* of X is a nonempty G-stable subvariety $Y \subseteq X$ which is the center of a central valuation of $\Bbbk(X)$.

For affine X, central valuations are described in a simple way. Consider the isotypic decomposition $\Bbbk[X] = \bigoplus_{\mu \in \Xi(X)_+} \Bbbk[X]_\mu$, where the *rank semigroup* $\Xi(X)_+ \subseteq \Xi(G)_+$ is the set of all dominant weights μ such that $\Bbbk[X]_\mu \neq 0$. For any $\lambda, \mu \in \Xi(X)_+$, we have

$$(*) \qquad \Bbbk[X]_\lambda \cdot \Bbbk[X]_\mu \subseteq \Bbbk[X]_{\lambda+\mu} \oplus \bigoplus_{\alpha \in \mathcal{T}_{\lambda,\mu}(X)} \Bbbk[X]_{\lambda+\mu-\alpha},$$

where $\mathcal{T}_{\lambda,\mu}(X)$ is a finite set of positive integral linear combinations of positive roots, and the inclusion fails for all proper subsets of $\mathcal{T}_{\lambda,\mu}(X)$.

Let $\Xi(X)$ be a sublattice spanned by $\Xi(X)_+$ in the weight lattice of G, and $\Xi(X)_\mathbb{Q} = \Xi(X) \otimes \mathbb{Q}$. Define the "cone of tails" $\mathcal{T}(X)$ to be the convex cone in $\Xi(X)_\mathbb{Q}$ spanned by the union of all $\mathcal{T}_{\lambda,\mu}(X)$.

A central valuation v is constant on each $\Bbbk[X]_\mu$ and defines a linear function $\nu \in \mathrm{Hom}(\Xi(X), \mathbb{Q}) = \Xi(X)^*_\mathbb{Q}$ so that $\langle \nu, \mu \rangle = v(f)$, $f \in \Bbbk[X]_\mu \setminus \{0\}$. By definition of a valuation, we must have $\langle \nu, \alpha \rangle \leq 0$ for $\forall \alpha \in \mathcal{T}_{\lambda,\mu}(X)$, $\lambda, \mu \in \Xi(X)_+$. Conversely, each linear function $\nu \in \mathrm{Hom}(\Xi(X), \mathbb{Q})$ which is nonpositive on $\mathcal{T}(X)$ defines a central valuation v of $\Bbbk(X)$ by the formula $v(f) = \min\{\langle \nu, \mu \rangle \mid f_\mu \neq 0\}$, where f_μ is the projection of $f \in \Bbbk[X]$ on $\Bbbk[X]_\mu$. The valuation v has a center on X iff ν is nonnegative on $\Xi(X)_+$, and the respective source $Y \subseteq X$ is determined by a G-stable ideal $I(Y) = \bigoplus_{\langle \nu, \mu \rangle > 0} \Bbbk[X]_\mu \lhd \Bbbk[X]$. Central valuations of $\Bbbk(X)$, identified with respective linear functions on $\Xi(X)_\mathbb{Q}$, form a solid convex cone $\mathcal{Z}(X) \subseteq \Xi(X)^*_\mathbb{Q}$, namely the dual cone to $-\mathcal{T}(X)$. Knop proved [K1, 9.2], [K2, 7.4] that $\mathcal{Z}(X)$ is a fundamental domain for a finite group $W_X \subset \mathrm{Aut}\,\Xi(X)$ (the *little Weyl group* of X) acting on $\Xi(X)^*_\mathbb{Q}$ as a crystallographic reflection group.

The following lemma is an easy consequence of results of Knop [K2].

Lemma 4. *If X is a normal affine G-variety containing a proper source, then there exists a one-dimensional torus $S \subseteq \mathrm{Aut}_G(X)$ such that $\Bbbk(X)^B \subseteq \Bbbk(X)^S$. (Here $\mathrm{Aut}_G(X)$ denotes the group of G-equivariant automorphisms of X.)*

Proof. If X is as above, then Knop has shown that the algebra $\Bbbk[X]$ admits a non-trivial G-invariant grading, whose homogeneous components are sums of isotypic components of the G-module $\Bbbk[X]$, see [K2, 7.9] and its proof. This grading is constructed as follows. Under the above assumptions, there is a central valuation v of $\Bbbk(X)$ such that the respective linear function ν on $\Xi(X)_\mathbb{Q}$ lies in $\mathcal{Z}(X) \cap -\mathcal{Z}(X)$, hence ν vanishes on $\mathcal{T}(X)$. In view of $(*)$, this ν defines a grading of $\Bbbk[X]$ such that isotypic components $\Bbbk[X]_\mu$ are homogeneous of degree $\langle \nu, \mu \rangle$.

Let $S \subseteq \mathrm{Aut}_G(X)$ be the one-dimensional torus corresponding to this grading. Take any $f \in \Bbbk(X)^B$, $f = p/q$, $p, q \in \Bbbk[X]$. By the Lie–Kolchin theorem, we may choose finitely many $b_i \in B$, $\lambda_i \in \Bbbk$ so that $q_0 = \sum \lambda_i b_i q$ is a B-eigenfunction of some weight $\mu \in \Xi(X)_+$. Then $p_0 = \sum \lambda_i b_i p$ is a B-eigenfunction of the same weight, and $f = p_0/q_0$. Since $p_0, q_0 \in \Bbbk[X]_\mu$, the torus S acts on them by the same weight $\langle \nu, \mu \rangle$, thence $f \in \Bbbk(X)^S$. This shows the inclusion $\Bbbk(X)^B \subseteq \Bbbk(X)^S$.

Proof of Theorem 3. It remains to prove that (AF) holds for G/H whenever any extension of H by a one-dimensional torus is spherical. As H is reductive, $W(H)$ is reductive, too. If there exists no one-parameter extension of H at all, then $W(H)$ is finite and G/H is affinely closed by Theorem 2. Otherwise $c(G/H) \leq 1$. As the spherical case is clear, we may suppose $c(G/H) = 1$.

Let X be an affine embedding of G/H. To prove that X has finitely many G-orbits, we may assume that X is normal. If X contains a proper source, then a one-dimensional torus $S \subseteq \mathrm{Aut}_G(X) \subseteq \mathrm{Aut}_G(G/H) = W(H)$ provided by Lemma 4 yields a nonspherical extension of H. Indeed, if H_1 is the preimage of S in $N_G(H)$, then $\Bbbk(G/H_1)^B = \Bbbk(G/H)^{B\times S} = \Bbbk(X)^{B\times S} = \Bbbk(X)^B \neq \Bbbk$, since $c(X) = 1$. This implies $c(G/H_1) = 1$, a contradiction.

If X contains no proper source, then any proper G-stable subvariety $Y \subset X$ is the center of a noncentral G-invariant valuation v. There is an inclusion of residue fields $\Bbbk(Y) \subseteq \Bbbk(v) \implies \Bbbk(Y)^B \subseteq \Bbbk(v)^B$. By Lemma 3, $\Bbbk(v)^B$ is the residue field of the restriction of v to $\Bbbk(G/H)^B$, which is the field of rational functions in one variable. As v is noncentral, $\Bbbk(Y)^B = \Bbbk(v)^B = \Bbbk$, thence Y is spherical. It follows that X has finitely many orbits. (Otherwise, a one-parameter family of G-orbits provides a nonspherical G-subvariety.)

Proof of Corollary 2. The reductive group $W(H)$ acts on $\Bbbk(G/H)^B$, which is the field of rational functions on a projective line. If the kernel of this action has positive dimension, then it contains a one-dimensional torus extending H to a nonspherical subgroup.

Otherwise, either $W(H)$ is finite or $\mathrm{rk}\, W(H) = 1$ and each subtorus of $W(H)$ has a dense orbit on the projective line. The corollary follows.

Remark 3. In the proof of Theorem 3, we have used reductivity of H only in the following assertion: *If $W(H)$ contains no subtori, then it is finite, and G/H is affinely closed.* In fact, we need this assertion only if $c(G/H) > 1$. Theorem 3 holds for quasi-affine G/H of complexity ≤ 1.

One might hope that the situation described in the above assertion never occurs for nonreductive H, i.e., $W(H)$ always contains a subtorus. Unfortunately, $W(H)^0$ may be a non-trivial unipotent group, as the following example (suggested by a referee) shows.

Example 4. Let e be a regular nilpotent in the Lie algebra $\mathfrak{sl}(3)$, $G = SL(3) \times SL(3)$, and H the two-dimensional unipotent subgroup with the Lie algebra generated by (e, e^2) and (e^2, e). Then the Lie algebra of the normalizer of H is the linear span of $(e, 0)$, $(e^2, 0)$, $(0, e)$ and $(0, e^2)$. Hence the group $W(H)^0$ is two-dimensional and unipotent. (Another example was suggested by E. A. Tevelev.)

We are not able to characterize quasi-affine, but not affine, homogeneous spaces that have the property (AF). In this context we would like to formulate the following

Conjecture. If $H \subseteq G$ is observable, but not reductive, then $W(H)$ is infinite.

5. Very symmetric embeddings

The group of G-equivariant automorphisms of a homogeneous space G/H is isomorphic to $W(H)$. (The action $W(H) : G/H$ is induced by the action $N_G(H) : G/H$ by right multiplication.) Let $G/H \hookrightarrow X$ be an affine embedding. The group $\mathrm{Aut}_G X$ of G-equivariant automorphisms of X is a subgroup of $W(H)$.

Definition 3. An embedding $G/H \hookrightarrow X$ is said to be *very symmetric* if $W(H)^0 \subseteq \mathrm{Aut}_G X$.

Any spherical affine variety is very symmetric. In fact, for a spherical homogeneous space G/H, any isotypic component $\Bbbk[G/H]_\mu$ of the G-algebra $\Bbbk[G/H]$ is an irreducible G-module (see [Se] or [VK, Th.2]), and $W(H)$ acts on $\Bbbk[G/H]_\mu$ by scalar multiplications. This shows that any G-invariant subalgebra in $\Bbbk[G/H]$ is also $W(H)$-invariant.

In the case of affine $SL(2)/\{e\}$-embeddings, only the embedding $X = SL(2)$ is very symmetric; in all other cases the group $\mathrm{Aut}_{SL(2)} X$ is isomorphic to a Borel subgroup in $SL(2)$, see [Kr, III.4.8, Satz 1]. More generally, if X is an affine embedding of the homogeneous space $G/\{e\}$, then X is very symmetric if and only if the action $G : X$ can be extended to an action of the group $G \times G$ with an open orbit isomorphic to $(G \times G)/H$, where H is the diagonal in $G \times G$. Hence X can be considered as an affine $(G \times G)/H$-embedding. Theorem 2 implies that if G is a semisimple group, then $X = (G \times G)/H$, for other proofs see [Wa] and [V2, Prop. 1].

If G is a reductive group, then the set of all very symmetric embeddings of the homogeneous space $G/\{e\}$ is exactly the set of all affine algebraic monoids with G as the group of units [V2]. The classification of reductive algebraic monoids is obtained in [V2] and [Rit].

Put $\widehat{G} = G \times W(H)^0$, $N = \gamma^{-1}(W(H)^0)$, and $\widehat{H} = \{(n, nH) \mid n \in N\}$ (the "diagonal" embedding of N). Any very symmetric affine embedding of G/H may be considered as an embedding of $\widehat{G}/\widehat{H}$.

Proposition 1. *Under assumptions of Theorem 4, if $\lambda(\Bbbk^*)$ is central in $W(H)^0$, then there exists a very symmetric affine embedding $G/H \hookrightarrow X$ with infinitely many G-orbits.*

Proof. We follow the proof of Theorem 4. Put $\widehat{H}_1 = \widehat{H} \cdot \lambda(\Bbbk^*)$; then $\widehat{H}_1 \cap G = H_1$. We modify the proof of Lemma 2 (b) to obtain a $\widehat{G}$-module V and an $\widehat{H}_1$-eigenvector $v \in V$ such that $\widehat{G}_{\langle v \rangle} = \widehat{H}_1$, $\widehat{G}_v$ is a finite extension of $\widehat{H}$, and $\overline{\widehat{G}\langle v \rangle} \subseteq \mathbb{P}(V)$ contains infinitely many G- (not $\widehat{G}$-) orbits. Arguing as in the proof of Theorem 4, we see that the closure X of $Gv = \widehat{G}v \subseteq V$ is $\widehat{G}$-stable and has infinitely many G-orbits. (Observe that $\widehat{G}$ may be not reductive, but Lemma 1, required in the proof, does not use the reductivity assumption.)

To construct such a couple (V, v), it suffices, in the notation of Lemma 2, to construct a $\widehat{G}$-module V' and a vector $v' \in V'$ such that $G\langle v' \rangle = \widehat{G}\langle v' \rangle \cong \widehat{G}/\widehat{H}_1$ and $\overline{G\langle v' \rangle}$ has infinitely many G-orbits. Then we proceed as in Lemma 2 (a), replacing G by $\widehat{G}$. (Note that the reductivity of G is not essential in Lemma 2 (a).) It remains to construct a couple (V', v'). For this purpose, we refine Akhiezer's construction [Akh].

By assumption, $c(G/H_1) > 0$, hence there exists a character $\xi : H_1 \to \Bbbk^*$ such that for the associated line bundle L_ξ on G/H_1, the multiplicity of a certain simple G-module V_μ in $H^0(G/H_1, L_\xi)$ is greater than one [VK, Th. 1].

The group $W(H)^0$ acts on $H^0(G/H_1, L_\xi)$ and on the isotypic component $E = H^0(G/H_1, L_\xi)_\mu$ by G-module automorphisms.

Take a $\widehat{G}$-module M and a vector $m \in M$ such that $\widehat{G}_{\langle m \rangle} = \widehat{H}_1$. Let Y be the closure of $\widehat{G}\langle m \rangle = G\langle m \rangle$ in $\mathbb{P}(M)$. The natural rational map $f : Y \dashrightarrow \mathbb{P}(E^*)$ is $\widehat{G}$-equivariant.

Consider a decomposition $E = E_0 \oplus \ldots \oplus E_k$ into irreducible G-submodules and fix isomorphisms $\psi_i : V_\mu \to E_i$. Choose a basis $\{\varepsilon_0, \ldots, \varepsilon_m\}$ of T-eigenvectors with

weights $\mu_0 = \mu, \mu_1, \ldots, \mu_m$ in V_μ, and put $\varepsilon_j^{(i)} = \psi_i(\varepsilon_j)$. In projective coordinates, $f(gH_1) = [\varepsilon_0^{(0)}(gH_1) : \ldots : \varepsilon_0^{(k)}(gH_1) : \ldots : \varepsilon_m^{(0)}(gH_1) : \ldots : \varepsilon_m^{(k)}(gH_1)]$.

The closure Z of the graph of f in $Y \times \mathbb{P}(E^*)$ is $\widehat{G}$-stable. We claim that Z contains infinitely many G-orbits. To prove it, take a strictly dominant one-parameter subgroup $\delta : \Bbbk^* \to T$. If all $\varepsilon_0^{(i)}(gH_1) \neq 0$, then $f(\delta(t)gH_1) = [\ldots : t^{-\langle \mu_j, \delta\rangle}\varepsilon_j^{(i)}(gH_1) : \ldots] = [\ldots : t^{\langle \mu_0 - \mu_j, \delta\rangle}\varepsilon_j^{(i)}(gH_1) : \ldots] \longrightarrow [\varepsilon_0^{(0)}(gH_1) : \ldots : \varepsilon_0^{(k)}(gH_1) : \ldots : 0 : \ldots : 0]$ as $t \to 0$, because $\mu_0 - \mu_j$ is a positive linear combination of positive roots for all $j > 0$.

We may identify E^* with $V_\mu^* \otimes \Bbbk^{k+1}$ and consider the Segre embedding $\mathbb{P}(V_\mu^*) \times \mathbb{P}^k \hookrightarrow \mathbb{P}(E^*)$. Then $\lim_{t\to 0} \delta(t) f(gH_1) = (\langle \varepsilon_0^* \rangle, p) \in \mathbb{P}(V_\mu^*) \times \mathbb{P}^k$, where $\{\varepsilon_j^*\}$ is the dual basis to $\{\varepsilon_j\}$, and $p = [\varepsilon_0^{(0)}(gH_1) : \ldots : \varepsilon_0^{(k)}(gH_1)] \in \mathbb{P}^k$.

As the sections $\varepsilon_0^{(0)}, \ldots, \varepsilon_0^{(k)}$ are linearly independent on G/H_1, $\overline{f(Y)}$ intersects infinitely many closed disjoint G-stable subvarieties $\mathbb{P}(V_\mu^*) \times \{p\} \hookrightarrow \mathbb{P}(E^*)$, $p \in \mathbb{P}^k$. This proves the claim, because Z projects G-equivariantly onto $\overline{f(Y)}$.

Finally, a $\widehat{G}$-module $V' = M \otimes E^*$ and a vector $v' = m \otimes e$ such that $f(\langle m \rangle) = \langle e \rangle$ are the desired, because $\overline{G\langle v' \rangle} \cong Z$. The proof is complete.

Now we are interested in the following problem: when does any very symmetric affine embedding of a homogeneous space G/H have finitely many G-orbits? The example of $SL(3)/\{e\}$-embeddings shows that the latter property is not equivalent to (AF).

Proposition 2. *Let H be a reductive subgroup in a reductive group G. Every very symmetric affine embedding of G/H has finitely many G-orbits iff either* (AF) *holds or $W(H)^0$ is semisimple. In the second case, there is only one very symmetric affine embedding, namely $X = G/H$.*

Proof. The Lie algebra of $N_{\widehat{G}}(\widehat{H})$ equals $\widehat{\mathfrak{h}} + \widehat{\mathfrak{z}}$, where $\widehat{\mathfrak{z}}$ is the centralizer of $\widehat{H}$ in $\widehat{\mathfrak{g}}$. We have $\widehat{\mathfrak{z}} = \mathfrak{z}(N) \oplus \mathfrak{z}(W(H)^0)$, and $\mathfrak{z}(N) \cong \mathfrak{z}(H) \oplus \mathfrak{z}(W(H)^0)$.

If $W(H)^0$ is semisimple, then $\widehat{\mathfrak{z}} \subseteq \mathfrak{h} \subseteq \widehat{\mathfrak{h}}$, and $N_{\widehat{G}}(\widehat{H})$ is finite. Theorem 2 implies the assertion for this case.

Now suppose that $W(H)^0$ is not semisimple. If there exists a nonspherical extension of H by a one-dimensional torus $S \subseteq Z(N)$, then by Proposition 1, there exists a very symmetric affine embedding of G/H with infinitely many G-orbits.

Finally, suppose that any extension of H by a one-dimensional torus in $Z(N)$ is spherical. Then $c(G/H) \leq 1$. As the spherical case is clear, we may assume that $c(G/H) = 1$.

The connected kernel W_0 of the action $W(H) : \Bbbk(G/H)^B$ acts on isotypic components of $\Bbbk[G/H]$ by scalar multiplications, whence W_0 is diagonalizable and central in $W(H)$. By assumption, $W_0 = \{e\}$. Hence $W(H)^0$ is a one-dimensional torus acting on $\Bbbk(G/H)^B$ with finite kernel. By Corollary 2, (AF) holds for G/H. The proof is complete.

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