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COMPLEXITY OF HOMOGENEOUS SPACES AND GROWTH OF MULTIPLICITIES

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Abstract. The complexity of a homogeneous space G/H under a reductive group G is by definition the codimension of general orbits in G/H of a Borel subgroup $B \subseteq G$. We give a representation-theoretic interpretation of this number as the exponent of growth for multiplicities of simple G-modules in the spaces of sections of homogeneous line bundles on G/H. For this, we show that these multiplicities are bounded from above by the dimensions of certain Demazure modules. This estimate for multiplicities is uniform, i.e., it depends not on G/H, but only on its complexity.

1. Introduction

Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of characteristic 0, and $H \subseteq G$ a closed subgroup. Consider the homogeneous space G/H. Choose a Borel subgroup $B \subseteq G$. By lower semicontinuity, general B-orbits in G/H have maximal dimension. The minimal (=typical) codimension $c = c(G/H)$ of B-orbits is called the *complexity* of G/H. (Clearly, it does not depend on the choice of B since all Borel subgroups are conjugate.) By the Rosenlicht theorem [VP, 2.3], $c(G/H)$ equals the transcendence degree of $\Bbbk(G/H)^B$ over $\Bbbk$.

This numerical invariant plays an important rôle in the geometry of G/H. For instance, the class of homogeneous spaces of complexity zero, called *spherical* spaces, is particularly nice [Kn1], [Bri2], [Vin]. It includes many classical spaces, all symmetric spaces, etc. Also the equivariant embedding theory of G/H depends crucially on its complexity, see [LV], [Tim].

In this note, we describe $c(G/H)$ in terms of representation theory related to G/H.

Let $\Lambda = \Lambda(B)$ be the weight lattice of B, and $\Lambda_+ \subseteq \Lambda$ the set of dominant weights. By $V(\lambda)$ denote the simple G-module of highest weight $\lambda \in \Lambda_+$. For any rational G-module M, let $\operatorname{mult}_\lambda M$ denote the multiplicity of $V(\lambda)$ in M.

It turns out that $c(G/H)$ characterizes the growth of multiplicities in spaces of global sections of G-line bundles over G/H. Here is our main result:

Theorem 1. *The complexity $c(G/H)$ is the minimal nonnegative integer c such that $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}) = O(|\lambda|^c)$ over all $\lambda \in \Lambda_+$ and all G-line bundles $\mathcal{L} \to G/H$, where*

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$|\cdot|$ *is any fixed norm on the vector space spanned by* Λ. *Moreover, this estimate for multiplicities is uniform over all* $H \subseteq G$ *such that* $c(G/H) = c$. *If* G/H *is quasiaffine, then it suffices to consider only* $\operatorname{mult}_\lambda \Bbbk[G/H]$.

A weaker version of Theorem 1 under some restrictive conditions (multiplicities in $\Bbbk[G/H]$ for quasiaffine G/H provided that $\Bbbk[G/H]$ is finitely generated; no uniform estimate) appeared in [AP]. The relation between complexity and growth of multiplicities is known for quite a time, see partial results in [Pa1, 1.1], [Pa2, 2.4], [Bri2, 1.3].

We prove Theorem 1 in Section 2. The idea is to embed the "space of multiplicity" in the dual of a certain Demazure submodule in $V(\lambda)$, associated with an element of length c in the Weyl group (Lemma 2).

In Section 3, we justify the term "complexity" by providing a much more precise information on multiplicities on homogeneous spaces of complexity ≤ 1. Actually, the spherical case is well known [VK] and is included in the text only for convenience of the reader. In the case of complexity 1, a formula similar to ours for multiplicities in $\Bbbk[G/H]$ provided that it is a finitely generated unique factorization domain, and G/H is quasiaffine, was obtained in [Pa1, 1.2], see also [Pa2, 2.4.19].

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Notation

- The character lattice of an algebraic group H is denoted by $\Lambda(H)$ and is written additively.
- By M^H we denote the set of H-fixed points in the set M acted on by a group H. If M is a vector space, and H acts linearly, then $M_\chi = M_\chi^{(H)}$ is the H-eigenspace of eigenweight $\chi \in \Lambda(H)$.
- Throughout the paper, G is a connected reductive group, and $B \subseteq G$ a fixed Borel subgroup. The semigroup Λ_+ of dominant weights is considered relative to B. We fix an opposite Borel subgroup B^-, a maximal torus $T = B \cap B^-$, and consider the Weyl group W of G relative to T.
- By λ^* denote the highest weight of the dual G-module to the simple G-module $V(\lambda)$ of highest weight $\lambda \in \Lambda_+$.

2. Upper bound for multiplicities

We begin with basic facts about line bundles on homogeneous spaces. Every line bundle $\mathcal{L} \to G/H$ admits a G-linearization, i.e., a fiberwise linear G-action compatible with the projection onto the base if we replace G by a finite cover [Po], [KKLV, §2]. (Alternatively, one may replace $\mathcal{L}$ by its sufficiently large tensor power.) Every G-line bundle is isomorphic to a homogeneous bundle $\mathcal{L}(\chi) = \mathcal{L}_{G/H}(\chi) = G \times^H \Bbbk_\chi$, where H acts on the fiber $\Bbbk_\chi \simeq \Bbbk$ via a character $\chi \in \Lambda(H)$. The bundle $\mathcal{L}(\chi)$ is trivial (regardless of the G-linearization) iff χ is the restriction of a character of G.

The space of global sections $H^0(G/H, \mathcal{L}(\chi)) \simeq \Bbbk[G]^{(H)}_{-\chi}$ is a rational G-module, where H (resp. G) acts on G by right (resp. left) translations. By the Frobenius reciprocity [Jan, I.3.3–3.4], we have $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim V(\lambda^*)^{(H)}_{-\chi}$.

In particular, $H^0(G/B, \mathcal{L}(-\lambda)) = V(\lambda^*)$ whenever $\lambda \in \Lambda_+$, and 0, otherwise (the Borel–Weil theorem).

Observe that for any rational G-module M we have $\operatorname{mult}_\lambda M = \dim M^{(B)}_\lambda$, $\forall \lambda \in \Lambda_+$. In particular,

$$\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim \Bbbk[G]^{(B\times H)}_{(\lambda,-\chi)}$$

The nonzero spaces $\Bbbk[G]^{(B\times H)}_{(\lambda,-\chi)}$ represent complete linear systems of B-stable divisors on G/H, i.e., linear systems of pairwise rationally equivalent B-stable divisors which cannot be enlarged by adding new B-stable effective divisors. The respective biweights (λ, χ) form a subsemigroup $\Sigma = \Sigma(G/H) \subseteq \Lambda(B\times H)$. Two biweights $(\lambda, \chi), (\lambda', \chi') \in \Sigma$ determine the same linear system on G/H iff (λ, χ) differs from (λ', χ') by a twist of G-linearization, i.e., by $(\varepsilon|_B, -\varepsilon|_H)$, $\varepsilon \in \Lambda(G)$.

We need a useful result, essentially due to Brion. Replacing H by a conjugate, we may assume that $\dim B(eH)$ is maximal among all B-orbits, i.e., $\operatorname{codim} B(eH) = c = c(G/H)$.

Lemma 1 (cf. [Bri1, 2.1]). *There exists a sequence of minimal parabolic subgroups $P_1, \ldots, P_c \supset B$ such that $\overline{P_c \cdots P_1(eH)} = G/H$. The decomposition $w = s_1 \cdots s_c$, where $s_i \in W$ is the simple reflection corresponding to P_i, is reduced, and $D_w = \overline{BwB} = P_1 \cdots P_c$ is a "Schubert subvariety" in G of dimension $c + \dim B$.*

Proof. If $c > 0$, then $\overline{B(eH)}$ is not G-stable, whence it is not stabilized by some minimal parabolic $P_1 \supset B$. Since $P_1/B \simeq \mathbb{P}^1$, the natural map $P_1 \times^B B(eH) \to G/H$ is generically finite, and $\operatorname{codim} P_1(eH) = c - 1$. Continuing in the same way, we construct a sequence of minimal parabolics $P_1, \ldots, P_c \supset B$ such that $\overline{P_c \cdots P_1(eH)} = G/H$, i.e., $P_c \cdots P_1 H$ is dense in G. The map $P_c \cdots P_1 \times^B B(eH) \to G/H$ is generically finite, hence $\dim P_c \cdots P_1 = c + \dim B$, which yields all the remaining assertions. □

Remark 1. The "Schubert subvarieties" D_w, $w \in W$, form a monoid w.r.t. the multiplication of sets in G, called the *Richardson–Springer monoid.* It is generated by the D_s, $s \in W$ a simple reflection, and is naturally identified with W, the defining relations being $s^2 = s$ and the braid relations for W. The action of the Richardson–Springer monoid on the set of B-stable subvarieties (appearing implicitly in Lemma 1) is studied in [Kn2].

Let $v_\lambda \in V(\lambda)$ be a highest weight vector. The B-submodule $V_w(\lambda) \subseteq V(\lambda)$ generated by wv_λ, $w \in W$, is called a *Demazure module.* In the notation of Lemma 1, we have $V_w(\lambda) = \langle D_w v_\lambda \rangle \simeq H^0(S_w, \mathcal{L}_{G/B}(-\lambda))^*$, where $S_w = D_w/B$ is a Schubert subvariety in G/B of dimension c. Indeed, the restriction map $V(\lambda^*) = H^0(G/B, \mathcal{L}(-\lambda)) \to H^0(S_w, \mathcal{L}(-\lambda))$ is surjective [Jan, II.14.15,e)], and $D_w v_\lambda$ is the affine cone over the image of S_w under the map $G/B \to \mathbb{P}(V(\lambda))$.

Lemma 2. *In the notation of Lemma 1,* $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) \le \dim V_w(\lambda)$.

Proof. We show that the pairing between $V(\lambda^*)$ and $V(\lambda)$ provides an embedding $V(\lambda^*)^{(H)}_{-\chi} \hookrightarrow V_w(\lambda)^*$. Otherwise, if $v^* \in V(\lambda^*)^{(H)}_{-\chi}$ vanishes on $V_w(\lambda)$, then it vanishes on $D_w v_\lambda$, i.e., $\langle D_w^{-1} v^* \rangle = \langle G v^* \rangle = V(\lambda^*)$ vanishes at v_λ, a contradiction. ☐

Remark 2. A similar idea was used in [Pa2, 2.4.18] to obtain an upper bound for multiplicities in coordinate algebras of homogeneous spaces of complexity 1.

Remark 3. The assertion of Lemma 2 can be refined and viewed in a more geometric context as follows. Consider the natural B-equivariant proper map $\varphi : D_w^{-1}/B \cap H \simeq D_w^{-1} \times^B B(eH) \to G/H$, which is generically finite by construction. For $\mathcal{L} = \mathcal{L}_{G/H}(\chi)$ we have $\varphi^*\mathcal{L} = \mathcal{L}_{D_w^{-1}/B\cap H}(\chi|_{B\cap H})$, and $\varphi^* : H^0(G/H, \mathcal{L}) \hookrightarrow H^0(D_w^{-1}/B \cap H, \varphi^*\mathcal{L})$ gives rise to

$$\begin{aligned} H^0(G/H, \mathcal{L})^{(B)}_\lambda \hookrightarrow H^0(D_w^{-1}/B \cap H, \varphi^*\mathcal{L})^{(B)}_\lambda &= \Bbbk[D_w^{-1}]^{(B\times B\cap H)}_{(\lambda,-\chi)} \\ \simeq \Bbbk[D_w]^{(B\cap H\times B)}_{(-\chi,\lambda)} = H^0(S_w, \mathcal{L}(-\lambda))^{(B\cap H)}_{-\chi} &= V_w(\lambda)^{*(B\cap H)}_{-\chi} \end{aligned}$$

I am indebted to M. Brion for this remark.

Lemma 2 applies to obtaining upper bounds for multiplicities in branching to reductive subgroups, cf. [AP, Thm. 2].

Corollary. *Let $L \subseteq G$ be a connected reductive subgroup.*

(1) Choose $w \in W$ as in Lemma 1 for H equal to a Borel subgroup of L. Then $\operatorname{mult}_\mu \operatorname{res}^G_L V(\lambda) \leq \dim V_w(\lambda^*)$ *for any two dominant weights λ, μ of G, L, respectively.*

(2) Similarly, if $w \in W$ corresponds to H equal to a maximal unipotent subgroup of L, then $\operatorname{length} \operatorname{res}^G_L V(\lambda) \leq \dim V_w(\lambda^*)$. *(Here* length *is the number of simple factors in an L-module.)*

Proof. Just note that $\operatorname{mult}_\mu \operatorname{res}^G_L V(\lambda) = \dim V(\lambda)^{(H)}_\mu = \operatorname{mult}_{\lambda^*} H^0(G/H, \mathcal{L}(-\mu))$ in the first case, and $\operatorname{length} \operatorname{res}^G_L V(\lambda) = \dim V(\lambda)^H = \operatorname{mult}_{\lambda^*} \Bbbk[G/H]$ in the second case, and then apply Lemma 2. ☐

Example 1. Let $P \supseteq B^-$ be a parabolic subgroup with the Levi decomposition $P = L \leftthreetimes P_{\mathrm{u}}$, $L \supseteq T$. By w_L denote the longest element in the Weyl group of L, and consider the decomposition $w_G = w_L w^L$. Let H be the unipotent radical of $B^- \cap L$. Then we may take $w = w^L$ and obtain $\operatorname{length} \operatorname{res}^G_L V(\lambda) \leq \dim V_{w^L}(\lambda^*)$.

Proof of Theorem 1. The first way is to use the character formula for Demazure modules [Jan, II.14.18, b)]:

$$\operatorname{ch}_T V_w(\lambda) = \frac{1 - e^{-\alpha_1} s_1}{1 - e^{-\alpha_1}} \cdots \frac{1 - e^{-\alpha_c} s_c}{1 - e^{-\alpha_c}}\, e^\lambda$$

where e^μ is the monomial in the group algebra $\mathbb{Z}[\Lambda]$ corresponding to $\mu \in \Lambda$, α_i are the simple roots defining P_i, and $s_i \in W$ are the respective simple reflections acting on $\mathbb{Z}[\Lambda]$ in a natural way. It is easy to deduce that $\dim V_w(\lambda) = O(|\lambda|^c)$, and Lemma 2 yields the desired estimate in Theorem 1. Indeed, suppose that $\mu \in \operatorname{conv} W\lambda$. One easily computes

$$\frac{1-e^{-\alpha_i}s_i}{1-e^{-\alpha_i}}\,e^{\mu}=\begin{cases}e^{\mu}(1+e^{-\alpha_i}+\dots+e^{-\langle\mu,\alpha_i^\vee\rangle\alpha_i}), & \langle\mu,\alpha_i^\vee\rangle\geq 0,\\ 0, & \langle\mu,\alpha_i^\vee\rangle=-1,\\ -e^{\mu}(e^{\alpha_i}+e^{2\alpha_i}+\dots+e^{|1+\langle\mu,\alpha_i^\vee\rangle|\alpha_i}), & \langle\mu,\alpha_i^\vee\rangle\leq -2,\end{cases}$$

$\forall i=1,\dots,c$, $\forall\mu\in\Lambda$, where $\alpha_i^\vee$ is the respective simple coroot. The r.h.s. is the sum of $|1+\langle\mu,\alpha_i^\vee\rangle|\leq 1+\max\langle W\lambda,\alpha_i^\vee\rangle$ exponentials (with a sign) of weights from the segment $[\mu,s_i\mu]\subseteq\operatorname{conv}W\lambda$. Starting with $\mu=\lambda$ and expanding the r.h.s. of the character formula step by step, we see that $\operatorname{ch}_T V_w(\lambda)$ is a (possibly alternating) sum of $\leq(1+\max\langle W\lambda,\alpha_1^\vee\rangle)\cdots(1+\max\langle W\lambda,\alpha_c^\vee\rangle)=O(|\lambda|^c)$ exponentials, which proves the claim.

Alternatively, observing that S_w is a projective variety of dimension c, one may deduce that $\dim H^0(S_w,\mathcal{L}(-\lambda))$ grows no faster than $|\lambda|^c$ as follows.

Without loss of generality we may assume G to be semisimple and simply connected. Let $\omega_1,\dots,\omega_l$ be the fundamental weights of G. Put $X=\mathbb{P}_{S_w}(\mathcal{L}(\omega_1)\oplus\dots\oplus\mathcal{L}(\omega_l))$, a a projective space bundle over S_w with fiber $\mathbb{P}^{l-1}$. Let $\mathcal{O}_X(1)$ be the antitautological line bundle over X, and $\pi:X\to S_w$ the projection map. Then

$$\begin{aligned}\pi_*\mathcal{O}_X(k)&=\bigoplus_{k_1+\dots+k_l=k,\ k_j\geq 0}\mathcal{L}(-k_1\omega_1-\dots-k_l\omega_l)\\ R^i\pi_*\mathcal{O}_X(k)&=0,\qquad\forall i>0\end{aligned}$$

Now $R_w=\bigoplus_{k\geq 0}H^0(X,\mathcal{O}(k))=\bigoplus_{\lambda\in\Lambda_+}H^0(S_w,\mathcal{L}(-\lambda))$ is a Λ-graded algebra, and the quotient field of R_w is a monogenic transcendental extension of $\Bbbk(X)$, hence it has transcendence degree $n=c+l$. Furthermore, R_w is finitely generated. Indeed, R_w is a quotient of the G-algebra $R=\bigoplus_{\lambda\in\Lambda_+}H^0(G/B,\mathcal{L}(-\lambda))$. The multihomogeneous components of R are the simple G-modules $V(\lambda^*)$, hence R is generated by its components with $\lambda=\omega_1,\dots,\omega_l$. (The spectrum of R_w is the so called *multicone* over the Schubert variety S_w, studied in [KR]. For instance, this multicone has rational singularities.) Finally, we apply the following general property of finitely generated multigraded algebras (see e.g. [AP, Thm. 3]):

Lemma 3. *Suppose $A=\bigoplus_{\lambda\in\Lambda}A_\lambda$ is a finitely generated algebra graded by a lattice Λ so that $A_0=\Bbbk$ and Λ is spanned by those λ with $A_\lambda\neq 0$. Then $\dim A_\lambda=O(|\lambda|^{d-r})$, where $|\cdot|$ is any norm on $\Lambda\otimes\mathbb{R}$, d is the Krull dimension of A, and $r=\operatorname{rk}\Lambda$.*

Lemma 3 implies that $\dim H^0(S_w,\mathcal{L}(-\lambda))=O(|\lambda|^{n-l})=O(|\lambda|^c)$.

This estimate is uniform in H, because there are finitely many choices for w. It remains to show that the exponent c cannot be made smaller.

Let $f_1,\dots,f_c$ be a transcendence base of $\Bbbk(G/H)^B$. There exist a G-line bundle $\mathcal{L}$ and B-eigenvectors $\sigma_0,\dots,\sigma_c\in H^0(G/H,\mathcal{L})$ of the same weight λ such that $f_i=\sigma_i/\sigma_0$, $\forall i=1,\dots,c$. (Indeed, $\mathcal{L}$ and σ_0 may be determined by a sufficiently big B-stable effective divisor majorizing the poles of all f_i.)

Consider the graded algebra $R = \bigoplus_{n\geq 0} R_n$, $R_n = H^0(G/H, \mathcal{L}^{\otimes n})^{(B)}_{n\lambda}$. Clearly, $\sigma_0, \dots, \sigma_c$ are algebraically independent in R, whence

$$\operatorname{mult}_{n\lambda} H^0(G/H, \mathcal{L}^{\otimes n}) = \dim R_n \geq \binom{n+c}{c} \sim n^c$$

This proves our claim.

Finally, if G/H is quasiaffine, then there even exist $\sigma_0, \dots, \sigma_c \in \Bbbk[G/H]$ with the same properties. □

Remark 4. For a given H, the estimate of the multiplicity by $\dim V_w(\lambda)$ may be not sharp. However, it is natural to ask whether it is a sharp uniform estimate over all homogeneous spaces with given complexity. More precisely, we formulate the following

Question. Given an element $w \in W$ of length c, does there exist a subgroup $H \subseteq G$ such that $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim V_w(\lambda)$ for sufficiently general $(\lambda, \chi) \in \Sigma(G/H)$?

Example 2. In the notation of Example 1, put $H = P_{\mathrm{u}}$. Then always $\chi = 0$, so that $H^0(G/H, \mathcal{L}(\chi)) = \Bbbk[G/H]$. We may take $w = w_L$. Then $V(\lambda^*)^H$ is a simple L-module of lowest weight $-\lambda$, and $V_w(\lambda)$ is the dual L-module of highest weight λ. It follows that $\operatorname{mult}_\lambda \Bbbk[G/H] = \dim V_w(\lambda)$.

3. Case of small complexity

Homogeneous spaces of complexity ≤ 1 are distinguished among all homogeneous spaces by their nice behaviour. For instance, they have the well-developed equivariant embedding theory [LV, 8–9], [Tim, 2–5]. There are also more explicit formulæ for multiplicities in this case.

Theorem 2. *In the above notation,*

(1) If $c(G/H) = 0$, then $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = 1$, $\forall(\lambda, \chi) \in \Sigma$.
(2) If $c(G/H) = 1$, then there exists a pair $(\lambda_0, \chi_0) \in \Sigma$, unique up to a shift by $(\varepsilon|_B, -\varepsilon|_H)$, $\varepsilon \in \Lambda(G)$, such that $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = n+1$, where n is the maximal integer such that $(\lambda, \chi) - n(\lambda_0, \chi_0) \in \Sigma(G/H)$.

Proof. The assertion is well known in the case $c = 0$, and we prove it just to keep the exposition self-contained. Assuming the contrary yields two non-proportional B-eigenvectors $\sigma_0, \sigma_1 \in H^0(G/H, \mathcal{L}(\chi))$ of the same weight λ. Hence $f = \sigma_1/\sigma_0 \in \Bbbk(G/H)^B$, $f \neq \mathrm{const}$, a contradiction.

In the case $c = 1$, we have $\Bbbk(G/H)^B \simeq \Bbbk(\mathbb{P}^1)$ by the Lüroth theorem. Consider the respective rational map $\pi : G/H \dashrightarrow \mathbb{P}^1$, whose general fibers are (the closures of) general B-orbits. By a standard argument, π is given by two B-eigenvectors $\sigma_0, \sigma_1 \in H^0(G/H, \mathcal{L}(\chi_0))$ of the same weight λ_0 for a certain $(\lambda_0, \chi_0) \in \Sigma$. Moreover, σ_0, σ_1 are algebraically independent, and each $f \in \Bbbk(G/H)^B$ can be represented as a homogeneous rational fraction in σ_0, σ_1 of degree 0.

Now put $(\mu, \tau) = (\lambda, \chi) - n(\lambda_0, \chi_0)$, fix $\sigma_\mu \in H^0(G/H, \mathcal{L}(\tau))^{(B)}_\mu$, and take any $\sigma_\lambda \in H^0(G/H, \mathcal{L}(\chi))^{(B)}_\lambda$. Then $f = \sigma_\lambda/\sigma_0^n\sigma_\mu \in \Bbbk(G/H)^B$, whence $f = F_1/F_0$ for some m-forms F_0, F_1 in σ_0, σ_1. We may assume the fraction to be reduced and decompose

$F_1 = L_1 \cdots L_m$, $F_0 = M_1 \cdots M_m$, as products of linear forms, with all L_i distinct from all M_j. Then $\sigma_\lambda M_1 \cdots M_m = \sigma_\mu \sigma_0^n L_1 \cdots L_m$.

Being fibers of π, the divisors of σ_0, L_i, M_j on G/H either coincide or have no common components. The divisor of σ_μ does not majorize any one of M_j: otherwise σ_μ/M_j is a regular section of $\mathcal{L}(\tau-\chi_0)$, whence $(\mu-\lambda_0, \tau-\chi_0) \in \Sigma$, a contradiction with the maximality of n. Thus each $\operatorname{div} M_j$ must have a common component with one of $\operatorname{div}\sigma_0, \operatorname{div} L_1, \dots, \operatorname{div} L_m$. It follows that $M_1 = \cdots = M_m = \sigma_0$ (up to proportionality), $m \le n$, and $\sigma_\lambda/\sigma_\mu$ is an n-form in σ_0, σ_1. Therefore $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi))$ equals the dimension of the space of n-forms in two variables, which is $n+1$. $\square$

Example 3. Let $G = \mathrm{SL}_3$, $H = T$ (the diagonal torus), B be the upper-triangular subgroup. The space G/H can be regarded as the space of ordered triangles in $\mathbb{P}^2$, i.e., the set of all ordered triples $p = (p_1, p_2, p_3) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ such that p_1, p_2, p_3 do not lie on one line. Let $\ell_i \subset \mathbb{P}^2$ denote the line joining p_j and p_k, where (i,j,k) is a cyclic permutation of $(1,2,3)$. By p_0 denote the B-fixed point in $\mathbb{P}^2$, and by ℓ_0 the B-stable line.

There are the following B-stable prime divisors on G/H:

$$D_i = \{p \mid p_i \in \ell_0\} = \operatorname{div} g_{3i}, \qquad \lambda_i = \omega_2,\ \chi_i = -\varepsilon_i$$

$$D_i' = \{p \mid p_0 \in \ell_i\} = \operatorname{div}\Delta_i, \quad \Delta_i = \begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{vmatrix}, \qquad \lambda_i' = \omega_1,\ \chi_i' = \varepsilon_i$$

$$D_t = \overline{B \cdot p(t)} = \operatorname{div}(g_{32}\Delta_2 + t g_{33}\Delta_3), \qquad \lambda_0 = \omega_1 + \omega_2,\ \chi_0 = 0$$

where $i = 1,2,3$, $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the vertices of the triangle $p(t)$ are:

$$p_1(t) = (0:0:1), \quad p_2(t) = (0:1:1), \quad p_3(t) = (1:t:1).$$

Here g_{ij} are matrix entries of $g \in G$, and H-semiinvariant polynomials in g_{ij} are regarded as sections of G-line bundles on G/H. We also indicate their biweights $(\lambda, \chi) \in \Sigma$, denoting by ω_i the fundamental weights, and by ε_i the diagonal matrix entries of H. Observe that $g_{31}\Delta_1 + g_{32}\Delta_2 + g_{33}\Delta_3 = 0$.

As generic B-orbits $B \cdot p(t) \subset G/H$ form a one-parameter family, we have $c(\mathrm{SL}_3/T) = 1$. The semigroup Σ is generated by $(\omega_1, \varepsilon_i)$, $(\omega_2, -\varepsilon_i)$, $i = 1,2,3$. Now it is an easy combinatorial exercise to deduce that $(\lambda, \chi) \in \Sigma$ iff $\lambda = k_1\omega_1 + k_2\omega_2$, $\chi = l_1\varepsilon_1 + l_2\varepsilon_2 + l_3\varepsilon_3$, $k_1, k_2 \ge 0$, $k_1 - k_2 \equiv l_1 + l_2 + l_3 \pmod 3$, and

$$n = \frac{k_1 + k_2}{2} - \frac{1}{6}\sum_{i=1}^{3} |k_1 - k_2 + 2l_i - l_j - l_k| \ge 0$$

Indeed, $(\lambda, \chi) \in \Sigma$ iff there exist nonnegative integers l_{ij}, $i = 1,2,3$, $j = 1,2$, such that $k_j = l_{1j} + l_{2j} + l_{3j}$, and the l_i's differ from $l_{i1} - l_{i2}$ by one and the same number. It follows that $k_j \ge 0$ and $k_1 - k_2 \equiv l_1 + l_2 + l_3 \pmod 3$. Replacing l_i by $l_i + (k_1 - k_2 - l_1 - l_2 - l_3)/3 = (k_1 - k_2 + 2l_i - l_j - l_k)/3$, we may assume that $l_i = l_{i1} - l_{i2}$. Then $n = \sum_{i=1}^3 (l_{i1} + l_{i2} - |l_i|)/2 = \sum_{i=1}^3 \min\{|l_{i1}|, |l_{i2}|\} \ge 0$.

Conversely, suppose that (λ, χ) satisfies the above conditions. Subtracting $n(\omega_1 + \omega_2, 0)$ from (λ, χ), we may assume that $k_1 - k_2 = l_1 + l_2 + l_3$, $k_1 + k_2 = |l_1| + |l_2| + |l_3|$. Now there is a unique choice for l_{ij}'s: $l_{i1} = l_i$, $l_{i2} = 0$, or $l_{i1} = 0$, $l_{i2} = |l_i|$, depending on whether $l_i \ge 0$ or $l_i \le 0$, which yields $(\lambda, \chi) \in \Sigma$.

The above reasoning also shows that n is the maximal integer such that $(\lambda, \chi) - n(\omega_1 + \omega_2, 0) \in \Sigma$. By Theorem 2(2), $\operatorname{mult}_\lambda H^0(\mathrm{SL}_3/T, \mathcal{L}(\chi)) = n + 1$.

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