Reductive group actions of complexity 1

D. A. Timashev

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Introduction

The problem of classifying algebraic varieties is central in algebraic geometry. In the theory of algebraic transformation groups it is natural to formulate its equivariant analogue: to describe varieties supplied with the action of an algebraic group G, up to G-isomorphism. This problem divides in two parts: "birational" and "biregular". Birational classification of G-varieties (with a given field of G-invariant functions) may be obtained in terms of Galois cohomology [PV]. We are interested in the biregular part of the classification problem: to describe all G-actions in a given birational class. More precisely, let K be a fixed function field with G acting on it birationally (i. e., K is the function field on some G-variety). We want to describe all G-varieties with function field G-isomorphic to K. We call such varieties *G-models* of the field K.

The classification of G-models in terms of certain combinatorial data is an important problem in the theory of algebraic transformation groups, since it provides a method to construct G-varieties, which is more natural than, say, representation by equations. Moreover, it often happens that many geometric properties of a G-variety are expressed naturally in terms of these combinatorial data, i. e., algebraic geometry is translated into the combinatorial language. The theory of toric varieties is a bright example. Toric varieties are normal varieties with the action of an algebraic torus having the open orbit. Smooth toric varieties appeared in the paper of Demazure [Dem] dealing with algebraic group actions on rational varieties. The general theory of toric varieties was developed in [KKMS], see also [Dan], [O]. The main result is the description of toric varieties and geometric objects on them (invertible sheaves, divisors, differential forms, ...) in terms of certain objects of combinatorial convex geometry (polyhedral cones, fans, piecewise linear functions, ...). This gives a way to translate such geometric questions as the respective position of orbits, smoothness, etc. into the combinatorial language.

In 1983 Luna and Vust introduced a general method to solve the problem of classifying G-models in case, where K has no non-trivial G-invariants, see [LV]. In this case, there is a distinguished G-model, namely the homogeneous space G/H, and our problem may be considered as description of all possible equivariant (partial) completions of G/H. The most complete results were obtained for reductive G. For some reasons, it is natural to restrict all considerations to normal varieties (the general case is reduced to this special case by equivariant normalization). The crucial notion here is the *complexity* of an action $c(K) = \operatorname{tr.deg} K^B$ (where $B \subseteq G$ is a Borel subgroup). Complexity is a birational invariant of an action, it is equal to the codimension of a B-orbit in general position on (any) G-model of K. As noted in [LV], one may hope to obtain a final answer in the problem of classifying G-models only in case, where $c(K) \leqslant 1$.

The case $c(K) = 0$ is considered in [LV, §8], see also [Kn1] and §1 of this paper. Varieties of complexity 0 are called spherical. The theory of such varieties is well developed and is strikingly close to the theory of toric varieties (which are a special case), see [Kn1] and the bibliography there.

In this paper we consider the case $c(K) = 1$. We obtain a classification of normal G-varieties with a given function field K of complexity 1 in terms of certain combinatorial objects constructed from invariants of K. The results remind those for spherical or toric case, but of course, they look more complicated. As in the toric (spherical) theory, a number of geometric questions is translated into the language of these combinatorial data: respective position and adherence of G-orbits, description of divisors etc.

The paper is organized a follows. In §1, we generalize main results of the Luna–Vust theory to the case of arbitrary function field K (with no restriction on triviality of G-invariants). Let us review the key points.

A G-variety X with the function field K is completely determined by the set of local rings $\mathcal{O}_{X,Y}$ of G-stable subvarieties $Y \subseteq X$. It is natural to think of them as of germs of the G-model X in neighborhoods of G-stable subvarieties Y (*G-germs*). Therefore our task is to realize how to (a) describe G-germs in terms of invariants of K, not of X; (b) point out which sets of G-germs do actually correspond to some G-model.

The first problem can be solved as follows: a G-germ of $Y \subseteq X$ is completely determined by the set of B-stable divisors containing Y. In their turn, B-stable divisors on G-models of K may be described in terms depending only on K, not on X. The conditions for determining "admissible" sets of B-stable divisors (corresponding actually to some G-germs) are given in

[LV]. However these conditions are too general, and an effective classification of G-germs cannot be derived from them in general case.

To solve the second problem, we use the local technique of studying G-models. Every G-model can be covered by a finite number of "simple" G-stable open subsets that are G-spans of B-stable open affine subsets (B-*charts*). A B-chart is completely determined by the set of B-stable divisors intersecting it. Thus the similar problem arises: to recognize "admissible" sets of B-stable divisors and after this, to determine subsets corresponding to G-germs (intersecting a given B-chart). Another problem is to glue "simple" G-models so as to satisfy the separation axiom.

In §2, we describe the "environment" for B-stable divisors on a variety of complexity 1. It turns out to be a collection of linear semispaces indexed by points of a smooth projective curve and glued together along their common boundary hyperplane.

In §3, we give the description of B-charts and G-germs (or admissible sets of B-stable divisors) in terms of convex geometry of these semispaces, similarly to the toric and spherical case, and derive the description of G-models. The main results are formulated in theorems 3.1–3.3.

In §4, examples are considered. Most of them describe completely the situation, where the completions of homogeneous spaces G/H are considered, G being a group of semisimple rank 1, H its finite subgroup. The respective G-varieties are "local models" for arbitrary reductive group actions of complexity 1 with dense orbit [Bri]. Some of these cases were considered earlier: $G = SL_2$, $H = \{e\}$, — in [LV, §9]; $G = SL_2$, H an arbitrary finite subgroup, — in [M-J]. As a by-product of our considerations, we obtain a uniform description for polynomial invariants of finite subgroups of SL_2 in its standard two-dimensional representation (invariants of binary polyhedral groups, cf. [PV, 0.13]). In case, where G is a torus, the classification of G-models with complexity 1 admitting a "good" quotient (in a sense) was obtained in [KKMS]. We consider this example in the spirit of this paper and generalize it to the case, where G-action may have no quotient (see 4.1).

The main results of this paper are published in [Tim1]–[Tim2].

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Notation and conventions. All schemes and (pre)varieties under consideration are locally of finite type over an algebraically closed field k of characteristic 0.

$\mathbb{A}^n$ is the affine space, $\mathbb{P}^n$ is the projective space of dimension n.

$k[X]$ is the algebra of regular functions, $k(X)$ is the field of rational functions on a (pre)variety X.

G is a connected (linear) algebraic group with tangent algebra $\mathfrak{g}$. From 1.2 and below, G is supposed to be reductive and of simply connected type.

$B \subseteq G$ is a Borel subgroup, $U \subseteq B$ is the maximal unipotent subgroup.

K is a finitely generated field extension of k with the birational G-action.

$\mathfrak{X}(H)$ is the character group of an algebraic group H.

M^H is the set of stable elements under the action of a group H on a set M.

$M_\chi^{(H)}$ is the set of (nonzero) eigenvectors (H-semiinvariants) with the weight χ in a linear representation $H : M$.

$k[S]$ is a k-algebra generated by a set S of its elements (e. g., the semigroup algebra of a semigroup S).

$A^\times$ is the group of units in an algebra A.

QA is the quotient field for an algebra A.

§1. The Luna–Vust theory

The Luna–Vust theory [LV] deals with the description of G-varieties in a given G-birational class (with the function field K). In this section we expose its main results with some improvements and generalizations due to F. Knop and the author. The theory is based on the following ideas.

1.1. "Rational point of view". All models of the given function field K may be glued together into one large (non-noetherian, non-separated) scheme (prevariety) $\mathbb{X} = \mathbb{X}(K)$. Points of $\mathbb{X}$ are just local rings that are localizations of finitely generated k-algebras with quotient field K. Basic open sets $\operatorname{Spec} A \subseteq \mathbb{X}$ (all localizations of a given algebra A) define the *Zariski topology* on $\mathbb{X}$. The structure sheaf $\mathcal{O}_{\mathbb{X}}$ is defined in a natural way. Local rings $\mathcal{O}_{\mathbb{X},Y}$ of points $Y \in \mathbb{X}$ in the sense of this sheaf are just those that appear in the definition of points of $\mathbb{X}$.

The scheme $\mathbb{X}$ may be considered as a prevariety if we consider only closed points of $\mathbb{X}$ (i. e., the residue field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ of the respective local ring $\mathcal{O}_x = \mathcal{O}_{\mathbb{X},x}$ equals k). Non-closed schematic points are identified with closed subvarieties $Y \subseteq \mathbb{X}$.

We distinguish in $\mathbb{X}$ the subsets $\mathbb{X}^{\mathrm{reg}}$, $\mathbb{X}^{\mathrm{norm}}$, ... of smooth, normal, ... points.

From this point of view, a model of K, i. e., a variety X with $k(X) = K$ is nothing else than a noetherian separated open subset $X \subseteq \mathbb{X}$.

The group G acts on $\mathbb{X}$ in a natural way (through its action on local rings) but this is not the action in the category of schemes or prevarieties. However there exists the largest open subset $\mathbb{X}_G \subseteq \mathbb{X}$ such that G acts on it as an algebraic group. Local rings of points in $\mathbb{X}_G$ are just those $\mathcal{O}_{\mathbb{X},x}$ that are mapped to $\mathcal{O}_{G\times\mathbb{X},e\times x}$ by the map $K \to k(G)\cdot K$ induced by the action $G : K$ (here $k(G)\cdot K = Q\big(k(G)\otimes_k K\big)$ is a free composite of fields). It is equivalent (when $\operatorname{char} k = 0$) to $\mathfrak{g}$-invariance of $\mathcal{O}_x$ [LV, §1], [PV, 1.1].

Proposition 1.1 ([LV, 1.5]). *Assume X is an open subset in $\mathbb{X}_G$. Then GX is noetherian (separated) iff X is noetherian (separated).*

Remark 1.1. In [LV] only the case $K^G = k$ is considered (in other words, all G-varieties with the function field K contain an open G-orbit) but most of the results generalize immediately.

Denote by ${}_G\mathbb{X}$ the set of G-stable schematic points of $\mathbb{X}_G$ (i. e., G-stable subvarieties $Y \subseteq \mathbb{X}_G$ or $(\mathfrak{g}, G)$-stable local rings). The similar notation ${}_GX$ is used for an arbitrary open subset $X \subseteq \mathbb{X}_G$. The Zariski topology is induced on ${}_G\mathbb{X}$, ${}_GX$ being its open subsets. If X is G-stable, then X and ${}_GX$ determine each other. If moreover X is a variety (i. e., is noetherian and separated), then elements of ${}_GX$ are interpreted naturally as germs of a G-variety X in a neighborhood of its G-stable subvariety (*G-germs*). Thus a *G-germ* is an element of ${}_G\mathbb{X}$, or a G-stable subvariety of $\mathbb{X}_G$, or a $(\mathfrak{g}, G)$-stable local ring. A G-variety X such that a given G-germ is contained in ${}_GX$ (i. e., intersects X in a G-stable subvariety Y) is called a *geometric realization* or a *model* of the G-germ. Every G-germ admits a geometric realization [LV, 2.1].

1.2. B-charts. From now on, G is supposed to be reductive and of simply connected type (i. e., G is a direct product of a torus and a simply connected semisimple group). Furthermore, all G-models are supposed to be normal (i. e., lying in $\mathbb{X}_G^{\mathrm{norm}}$).

The following theorem of Sumihiro is the basic tool for local study of G-varieties.

Theorem 1.1 ([Su], [KKLV]). *Assume X is a normal G-variety. Then any point $x \in X$ has a G-stable neighborhood $X_0 \subseteq X$ admitting an equivariant locally closed embedding in a projective space.*

Thus if we are interested in local properties of a G-model X in a neighborhood of its point x or a subvariety Y, then we may assume X quasiprojective (replacing X by X_0).

Lemma 1.1 ([Kn2, 2.10. Satz], [Kn1, Th. 1.3]). *Assume $Y \subseteq X$ is a closed G-subvariety in a normal G-variety. Then there exists an affine B-stable open subset $\mathring{X} \subseteq X$ such that $\mathring{Y} = \mathring{X} \cap Y \neq \varnothing$.*

Proof. We may assume X quasiprojective. It suffices to find a B-semiinvariant homogeneous polynomial F (in homogeneous coordinates) that equals 0 on $\overline{X} \setminus X$ but not on Y (here $\overline{X}$ is the projective closure of X). Then we put $\mathring{X} = \{\, x \in \overline{X} \mid F(x) \neq 0 \,\}$. The existence of F is an easy consequence of complete reducibility of G-modules and of the highest weight theory. □

We call such an $\mathring{X}$ a *B-chart* (of the G-germ $Y \subseteq X$). In general, a *B-chart* is any affine B-stable open subset of a G-variety X. Therefore any G-germ has a B-chart $\mathring{X} \subseteq \mathbb{X}_G^{\mathrm{norm}}$ in any geometric realization. Conversely, any affine B-stable open subset $\mathring{X} \subseteq \mathring{\mathbb{X}}_G^{\mathrm{norm}}$ is a B-chart in a suitable G-model (e. g., in $X = G\mathring{X}$). Sets ${}_G\mathring{X}$ form a base of topology on ${}_G\mathbb{X}$.

1.3. B-semiinvariants, G-valuations, B-divisors. The representation theory of G plays an important role in the Luna–Vust theory. So it is not surprising that the main results are formulated in terms of B-semiinvariant and invariant objects.

Let X be a G-variety with $k(X) = K$. We introduce the following notations.

$K_B \subseteq K$ is a subalgebra of rational functions with B-stable divisor of poles on X. The existence of a B-chart implies $QK_B = K$.

$\Gamma = \Gamma(X) = \Gamma(K)$ is a group of weights of all rational B-eigenfunctions. There is an exact sequence

$$(*) \qquad 1 \longrightarrow \left(K^B\right)^\times \longrightarrow K^{(B)} \longrightarrow \Gamma \longrightarrow 0$$

$c(X) = c(K) = \operatorname{tr.deg} K^B$ is the *complexity* of the variety X or of the field K; it is equal to the codimension of a B-orbit in general position on X.

$r(X) = r(K) = \operatorname{rk}\Gamma$ is the *rank* of the variety X or of the field K. The sum $c(X) + r(X)$ equals the codimension of a U-orbit in general position on X.

$\mathcal{V} = \mathcal{V}(X) = \mathcal{V}(K)$ is the set of G-stable discrete $\mathbb{Q}$-valued valuations of K/k whose restrictions on K^B are geometric (i. e., correspond to some prime divisors on a suitable model of K^B). The last requirement is equivalent to the valuation of K being geometric itself [Kn2, §4]. Elements of $\mathcal{V}$ are called *G-valuations.*

$\mathcal{D} = \mathcal{D}(X) = \mathcal{D}(K)$ is the set of not G-stable prime divisors on X. The valuation corresponding to a divisor $D \in \mathcal{D}$ is denoted by v_D. B- but not G-stable prime divisors, i. e., elements of $\mathcal{D}^B$ are called *B-divisors.*

Remark 1.2. The sets K_B, $\mathcal{D}$ and $\mathcal{D}^B$ do not depend on the choice of a G-model X [Kn2, 3.4. Definition].

The following technical result is crucial for reducing various questions to B-semiinvariants.

Lemma 1.2 ([Kn2, 3.5]). *For any G-valuation $v \in \mathcal{V}$ and any rational function $f \in K_B$ there exists a rational B-eigenfunction $\overline{f} \in K^{(B)}$ such that:*

$$\begin{cases} v(\overline{f}) = v(f) \\ w(\overline{f}) \geqslant w(f) & (\forall w \in \mathcal{V}) \\ v_D(\overline{f}) \geqslant v_D(f) & (\forall D \in \mathcal{D}^B) \end{cases}$$

Corollary 1.1 ([Kn2, 3.6]). *G-valuations are determined uniquely by their restriction on $K^{(B)}$.*

1.4. Admissibility conditions for B-charts. Let $\mathring{X}$ be a B-chart. Then $A = k[\mathring{X}]$ is an integrally closed finitely generated algebra, in particular, it is a Krull ring (see [Bou, Chap. VII, §1]). Therefore

$$A = \bigcap \mathcal{O}_{\mathring{X},D} \text{ (in all prime divisors } D \subset \mathring{X}) \;=\; \bigcap_{w \in \mathcal{W}} \mathcal{O}_w \cap \bigcap_{D \in \widetilde{\mathcal{R}}} \mathcal{O}_{v_D},$$

where $\mathcal{O}_v$ is the valuation ring of v, $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{R} \subseteq \mathcal{D}^B$, $\widetilde{\mathcal{R}} = \mathcal{R} \sqcup (\mathcal{D} \setminus \mathcal{D}^B)$. We may assume that the value group of $\forall w \in \mathcal{W}$ is exactly $\mathbb{Z} \subset \mathbb{Q}$. If we take another B-chart, then we change $\mathcal{W} \sqcup \mathcal{R}$ in a finite number of elements. Hence all possible $\mathcal{W} \sqcup \mathcal{R}$ lie in a certain distinguished class $\mathfrak{V}$ of equivalent subsets of $\mathcal{V} \sqcup \mathcal{D}^B$ w. r. t. the equivalence relation "differ in a finite number of elements".

Conversely, if $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{R} \subseteq \mathcal{D}^B$, and $\mathcal{W} \sqcup \mathcal{R}$ belongs to $\mathfrak{V}$, then

$$A = A(\mathcal{R}, \mathcal{W}) = \bigcap_{w \in \mathcal{W}} \mathcal{O}_w \cap \bigcap_{D \in \widetilde{\mathcal{R}}} \mathcal{O}_{v_D}$$

is a Krull ring. Indeed, for $\forall f \in K$ almost all valuations from $\mathcal{W} \sqcup \mathcal{R}$ (except a finite number) vanish on f, since it is true for a subset corresponding to a B-chart, hence for any subset in the class $\mathfrak{V}$.

Remark 1.3. Here and below, we identify prime divisors and respective valuations including the notation. Thus for $\mathcal{V}_0 \subseteq \mathcal{W} \sqcup \mathcal{R}$, we write $\mathcal{V}_0(f) \geqslant 0$ iff $\forall v \in \mathcal{V}_0$ ($v = v_D$ for $D \in \mathcal{V}_0 \cap \mathcal{R}$) : $v(f) \geqslant 0$, etc.

Remark 1.4. $K_B = A(\varnothing, \varnothing)$

All valuations from $\widetilde{\mathcal{R}}$ are essential for A (see [LV, 8.5. Lemme 3] for the case $K^G = k$; the proof for general case is based on similar ideas using the results from [Kn2, §3]).

If (a) $QA = K$ and (b) A is finitely generated, then $\mathring{X} = \operatorname{Spec} A \subset \mathbb{X}_G^{\mathrm{norm}}$ is a B-chart, since A (and all its localizations) is $\mathfrak{g}$-stable. The following theorem gives a reformulation of (a), (b) in terms of B-semiinvariants.

Theorem 1.2. (1) *$QA = K$ iff*

(C) $\forall \mathcal{V}_0 \subseteq \mathcal{W} \sqcup \mathcal{R}$, *$\mathcal{V}_0$ finite,*

$$\exists f \in K^{(B)} \ : \ (\mathcal{W} \sqcup \mathcal{R})(f) \geqslant 0,\, \mathcal{V}_0(f) > 0$$

(2) *All valuations from $\mathcal{W}$ are essential for A iff*

(W) $$\forall w \in \mathcal{W}\ \exists f \in K^{(B)} \ : \ (\mathcal{W} \sqcup \mathcal{R} \setminus \{w\})(f) \geqslant 0,\, w(f) < 0$$

(3) *A is finitely generated iff*

(F) $$A^U = k[A^{(B)}] \text{ is finitely generated}$$

Proof. (1) We may assume $\mathcal{V}_0 = \{v\}$. Assume $QA = K$; then $\exists f \in A \subseteq K_B \ : \ v(f) > 0$. Applying lemma 1.2, we replace f by an element of $A^{(B)}$ and obtain (C). Conversely, assume (C) is true and $f \in K_B$; then we take $\mathcal{V}_0 = \{\, v \in \mathcal{W} \sqcup \mathcal{R} \mid v(f) \neq 0 \,\}$ and fall into A multiplying f by the respective eigenfunction from $A^{(B)}$ (killing the poles). Hence $K_B \subseteq QA$, and this yields $K = QA$.

(2) Assume $w \in \mathcal{W}$ is essential for A; then $\exists f \in K \ : \ (\mathcal{W} \sqcup \widetilde{\mathcal{R}} \setminus \{w\})(f) \geqslant 0$, $w(f) < 0$, and we obtain (W) applying lemma 1.2 once more. The converse statement is obvious.

(3) Let X be a smooth G-model of K. Take an effective divisor on X with the support $(\mathcal{D}^B \setminus \mathcal{R})$ and consider the corresponding section $\eta \in H^0(X, L)^{(B)}$ of the line bundle L (admitting a G-linearization by [KKLV]). Consider an algebra $R = \bigoplus_{n \geqslant 0} H^0(X, L^{\otimes n})$. Then $A = \bigcup R(\eta^n)\eta^{-n} \subseteq QR$, where $R(\eta^n) = \{\, \sigma \in H^0(X, L^{\otimes n}) \mid \sigma\eta^{-n} \in A \,\}$. Since every G-valuation on K can be extended to a G-stable valuation on QR [Kn2, 3.2], we have that $R(\eta^n) = \{\, \sigma \mid \forall w \in \mathcal{W} \ : \ w(\sigma) \geqslant nw(\eta) \,\}$ is G-stable.

Consider a U-equivariant homomorphism $\varphi : R[t] \to QR,\ t \mapsto \eta^{-1}$.

Lemma. *Assume $\widehat{A} = \bigoplus_{n \geqslant 0} \widehat{A}_n t^n \subseteq R[t]$ is a graded G-stable subalgebra containing ηt, so that $\varphi(\widehat{A}) \subseteq A$. Then $\varphi(\widehat{A}) = A \iff \varphi(\widehat{A}^U) = A^U$.*

Proof of the lemma. Assume $\varphi(\widehat{A}) = A$. If $\sum \sigma_n t^n \in \widehat{A}$, and $\varphi(\sum \sigma_n t^n) = \sum \sigma_n \eta^{-n} \in A^U$, then we have for n_0 sufficiently large $\sum \sigma_n \eta^{-n} = \sum(\sigma_n \eta^{n_0-n}) \cdot \eta^{-n_0} = \varphi(\sum \sigma_n t^n (\eta t)^{n_0-n})$ and $\sum \sigma_n t^n (\eta t)^{n_0-n} \in \widehat{A}^U$.

Conversely, assume $\varphi(\widehat{A}) \neq A$. The U-module $A/\varphi(\widehat{A})$ contains a nonzero U-invariant, which is the image of $f = \frac{\sigma}{\eta^n} \in A$. We may assume (for n sufficiently large) that σ determines a nonzero element of $\left(R(\eta^n)/\widehat{A}_n\right)^U$. Without changing this element, we can change σ so as to obtain $\sigma \in R(\eta^n)^U$, hence $f \in A^U \neq \varphi(\widehat{A})^U$. So we are done. □

Now if A is finitely generated, then it is easy to construct a finitely generated graded G-algebra $\widehat{A}$ such that $\varphi(\widehat{A}) = A$. Hence $\widehat{A}^U$ and $A^U = \varphi(\widehat{A}^U)$ as well are finitely generated. Conversely, if A^U is finitely generated, then we construct a finitely generated graded G-algebra $\widehat{A}$ such that $\varphi(\widehat{A}^U) = A^U$. This yields that $\varphi(\widehat{A}) = A$ is finitely generated. □

Remark 1.5. Conditions (W), (F) were considered in [LV, 8.5–8.6] in case $K^G = k$ though the proof of (3) was insufficient. The condition (C) appeared in [Kn1] in the spherical case (the condition in [LV] is more complicated).

Thus a pair $\mathcal{W}$, $\mathcal{R}$ (from the class $\mathfrak{V}$) is admissible for some B-chart (i. e., corresponds to all its B-stable divisors) iff (C), (W), (F) hold.

1.5. ***G*-germs.** As we have seen in 1.2, every G-germ intersects some B-chart. A question arises: how to describe a G-germ in terms of the combinatorial data determining a B-chart (see 1.4)?

Consider an arbitrary geometric realization $Y \subseteq X$ of a given G-germ. There exists a valuation $v \in \mathcal{V}$ having the center Y (i. e., $\mathcal{O}_v$ dominates $\mathcal{O}_{X,Y}$)—e. g., the valuation corresponding to a component of the exceptional divisor in the normalized blow-up of X along Y. The collection of all such v (independent on the choice of a geometric realization) is denoted by $\mathcal{S}_Y$ and called the *support* of the G-germ.

If the G-germ intersects a B-chart $\mathring{X}$, then $\mathring{Y} = Y \cap \mathring{X}$ is the center of any $v \in \mathcal{S}_Y$, i. e., $v|_{k[\mathring{X}]} \geqslant 0$, and the ideal $\mathbb{I}(\mathring{Y}) \lhd k[\mathring{X}]$ is given by $v > 0$. Conversely, if a G-valuation $v \in \mathcal{V}$ is nonnegative on $A = k[\mathring{X}]$, then it determines a G-germ intersecting $\mathring{X}$.

We use the notation from 1.4. Let $\mathcal{V}_Y \subseteq \mathcal{W}$, $\mathcal{D}_Y^B \subseteq \mathcal{R}$ be the subsets corresponding to all B-stable divisors on $\mathring{X}$ containing $\mathring{Y}$.

Proposition 1.2 ([Kn2, 3.8. Satz]). *The pair of sets $\mathcal{V}_Y$, $\mathcal{D}_Y^B$ determines the G-germ uniquely.*

The proof is based on lemma 1.2 and on the formula

$$\mathcal{O}_Y^{(B)} = \{ f \in K^{(B)} \mid (\mathcal{V}_Y \sqcup \mathcal{D}_Y^B)(f) \geqslant 0 \}.$$

Now we reformulate the conditions describing G-germs in a given B-chart in terms of B-semiinvariants.

Theorem 1.3. (1) *$v \in \mathcal{V}$ has a center on $\mathring{X}$ iff*

$$\text{(V)} \qquad \forall f \in K^{(B)} \ : \ (\mathcal{W} \sqcup \mathcal{R})(f) \geqslant 0 \implies v(f) \geqslant 0$$

(2) *Assume $v \in \mathcal{S}_Y$. A valuation $w \in \mathcal{W}$ belongs to $\mathcal{V}_Y$ iff*

$$\text{(V}'_w\text{)} \qquad \forall f \in K^{(B)} \ : \ (\mathcal{W} \sqcup \mathcal{R})(f) \geqslant 0,\, v(f) = 0 \implies w(f) = 0$$

Similarly, $D \in \mathcal{R}$ belongs to $\mathcal{D}_Y^B$ iff

$$\text{(D}'_D\text{)} \qquad \forall f \in K^{(B)} \ : \ (\mathcal{W} \sqcup \mathcal{R})(f) \geqslant 0,\, v(f) = 0 \implies v_D(f) = 0$$

Proof. (1) v has a center iff $v|_A \geqslant 0$. This clearly implies (V). On the other hand, if $f \in A$, $v(f) < 0$, then we replace f by an element of $A^{(B)}$ applying lemma 1.2 and see that (V) is false.

(2) Assume $w \in \mathcal{W}$ (or $D \in \mathcal{R}$) belongs to $\mathcal{V}_Y$ (or $\mathcal{D}_Y^B$); then every function $f \in A$ not vanishing on $\mathring{Y}$ (i. e., $v(f) = 0$) does not vanish on the respective B-stable divisor of $\mathring{X}$ as well (i. e., $w(f) = 0$ or $v_D(f) = 0$). For $f \in A^{(B)}$, we obtain (V$'_w$) (or (D$'_D$)).

Conversely, assume $w \notin \mathcal{V}_Y$ (or $D \notin \mathcal{D}_Y^B$); then there exists $f \in A$ vanishing on the respective B-stable divisor of $\mathring{X}$ (i. e., $w(f) > 0$ or $v_D(f) > 0$) but not on $\mathring{Y}$ (i. e., $v(f) = 0$). Applying lemma 1.2, we replace f by an element of $A^{(B)}$ and see that (V$'_w$) (or (D$'_D$)) is false. ☐

Remark 1.6. The theorem was proved in [LV, 8.8] in case $K^G = k$.

We notice in addition that one G-germ may have a lot of different B-charts. E. g., we may consider a principal open subset $\mathring{X}_f = \{ x \mid f(x) \neq 0 \}$ in $\mathring{X}$, where $f \in A^{(B)}$, $v(f) = 0$ (not to cut $\mathring{Y}$ off), i. e., pass from A to its localization A_f. This corresponds to removing from $(\mathcal{W} \setminus \mathcal{V}_Y) \sqcup (\mathcal{R} \setminus \mathcal{D}_Y^B)$ a finite set $\mathcal{W}_0 \sqcup \mathcal{R}_0$ of those valuations that are positive on f. By (V$'$) and (D$'$), this set may contain any finite number of elements from $(\mathcal{W} \setminus \mathcal{V}_Y) \sqcup (\mathcal{R} \setminus \mathcal{D}_Y^B)$. In particular, if $(\mathcal{W} \setminus \mathcal{V}_Y) \sqcup (\mathcal{R} \setminus \mathcal{D}_Y^B)$ is finite, then there exists a *minimal* B-chart with $\mathcal{W} = \mathcal{V}_Y$, $\mathcal{R} = \mathcal{D}_Y^B$.

1.6. Résumé: the description of G-models. Now we sum up. As noted in 1.1, a G-model is completely determined by the set ${}_GX$ of its G-germs. We call a set of G-germs *separated* if the supports of its elements are disjoint.

Proposition 1.3 ([LV, 6.1–6.2]). *An open subset $X \subseteq \mathbb{X}_G$ is noetherian (separated) iff ${}_GX$ is noetherian (separated).*

Therefore to determine a G-model is the same thing as to determine the set of its G-germs, i. e., of pairs $(\mathcal{V}_Y, \mathcal{D}_Y^B)$. Any separated set of pairs $(\mathcal{V}_Y, \mathcal{D}_Y^B)$ of subsets in $\mathcal{V}$ and $\mathcal{D}^B$ obtained through the conditions (V), (V′), (D′) from a finite collection of B-charts, i. e., of pairs $(\mathcal{W}_\alpha, \mathcal{R}_\alpha)$ satisfying (C), (W), (F), suits for this purpose. The supports of these germs must be disjoint since the G-model must be separated. Note that we may assume the collection of B-charts to be finite, since the G-model must be noetherian (i. e., it can be covered by a finite number of B-charts and their translates). Of course, this collection of charts is not unique.

Some geometric properties of a G-model are expressed in terms of this description. We point out two of them.

A G-model is affine iff it can be determined by a single G-stable B-chart, where G-stability means that all B-divisors intersect this B-chart, i. e., $\mathcal{R} = \mathcal{D}^B$ (in general, elements of $\mathcal{D}^B \setminus \mathcal{R}$ are just B-divisors in the complement of a B-chart in its G-span).

Further, a set of G-germs is called *complete* if it is separated and supports of its elements cover all $\mathcal{V}$. The valuative criterion of completeness implies

Proposition 1.4 ([LV, 6.4]). *A G-model $X \subseteq \mathbb{X}_G$ is a complete variety iff ${}_GX$ is complete.*

1.7. An example: spherical varieties. Varieties of complexity 0 are called *spherical*. For a theory of them see [LV, 8.10], [Kn1]. Let us obtain their combinatorial description in the framework of the general theory.

By corollary 1.1 and by $K^B = k$, G-valuations are identified with linear functionals on the lattice Γ, i. e., $\mathcal{V}$ lives in a space $\mathcal{Q} = \mathrm{Hom}(\Gamma, \mathbb{Q})$ (and forms there a convex solid cone [Kn1, §5], [Kn2, §5]). The set $\mathcal{D}^B$ is finite (since a spherical variety contains a dense B-orbit) and maps to $\mathbb{Q}$ (not injectively) by means of the restriction on $K^{(B)}$. B-semiinvariants are interpreted as linear functionals on $\mathbb{Q}$.

The class $\mathfrak{V}$ consists of finite sets.

(C) means that elements of $\mathcal{W} \sqcup \mathcal{R}$ generate a strictly convex cone in $\mathcal{Q}$, and no element from $\mathcal{R}$ maps to 0.

(W) means that elements of $\mathcal{W}$ are exactly generators of those edges of $\mathcal{C}$ that contain no images of elements from $\mathcal{R}$.

(F) holds automatically: $k[A^{(B)}]$ is the semigroup algebra of a semigroup $\Gamma \cap \mathcal{C}^\vee$ of integral points in a finitely generated dual cone $\mathcal{C}^\vee = \{ \lambda \in \mathcal{Q}^* \mid \langle \mathcal{W} \sqcup \mathcal{R}, \lambda \rangle \geqslant 0 \}$.

(V) means that $v \in \mathcal{C}$.

(V$'$) and (D$'$) select (elements of $\mathcal{W} \sqcup \mathcal{R}$ lying in) the face $\mathcal{C}_Y \subseteq \mathcal{C}$ containing v in its interior. The support $\mathcal{S}_Y$ of a germ equals $\mathcal{V} \cap \operatorname{int} \mathcal{C}_Y$.

Every G-germ has a minimal B-chart with $\mathcal{C} = \mathcal{C}_Y$. It suffices to consider only such charts.

The separation condition means that interiors of cones corresponding to different germs (or charts) are disjoint inside $\mathcal{V}$ (i. e., $\mathcal{V} \cap \operatorname{int} \mathcal{C}_1 \cap \operatorname{int} \mathcal{C}_2 = \varnothing$).

A *colored cone* is a pair $(\mathcal{C}, \mathcal{R})$, where $\mathcal{C}$ is a finitely generated strictly convex cone, $\mathcal{R} \subseteq \mathcal{D}^B$ is mapped to $\mathcal{C} \setminus \{0\}$ and generates $\mathcal{C}$ together with a finite number of elements from $\mathcal{V}$, and $\mathcal{V} \cap \operatorname{int} \mathcal{C} \neq \varnothing$. Elements of $\mathcal{R}$ are called its *colors*. A *face* of the colored cone is a colored cone $(\mathcal{C}', \mathcal{R}')$, where $\mathcal{C}'$ is a face of $\mathcal{C}$, $\mathcal{V} \cap \operatorname{int} \mathcal{C}' \neq \varnothing$, and $\mathcal{R}'$ consists of all elements from $\mathcal{R}$ mapped to $\mathcal{C}'$.

Thus a G-model is given by a finite set of colored cones closed under passing to a face such that different cones intersect by faces inside $\mathcal{V}$. Such a collection of colored cones is called a *colored fan*.

In case, where $G = B$ is a torus, we obtain the most well-known example of spherical varieties—toric varieties. They are determined by fans of finitely generated strictly convex cones (the word "colored" is needless here, since there are no colors). For toric varieties see [KKMS], [Dan], [O]. Their importance is based on the fact that many geometric notions (the respective position of orbits, line bundles etc.) are described in the language of the geometry of cones and fans. The analogies exist in the theory of spherical varieties, too—see [Kn1] and the bibliography there. It is desirable to develop the same approach in more general situation.

§2. G-valuations and B-divisors

From now on, we assume $c(K) = 1$. Our goal is to give a combinatorial-geometric description of varieties with complexity 1 (with the given function field K) in the framework of the general Luna–Vust theory, similar to the spherical case (see 1.7). In this section we give an appropriate interpretation for the main ingredients of such description—G-valuations, B-divisors, B-semiinvariants—and describe the methods for their computation used in examples.

2.1. "Environment". The exact sequence $(*)$ from 1.3 splits, i. e., there exists an embedding $e : \Gamma \to K^{(B)},\ \lambda \mapsto e_\lambda$ (a B-eigenfunction of the weight λ). A G-valuation of K/k is determined uniquely by its restriction on $K^{(B)}$, i. e., by the pair consisting of the linear functional on $e(\Gamma) \cong \Gamma$ and of the restriction of the valuation on K^B. The last field is the function field on a smooth projective curve C. To determine its valuation is the same as to give a point $x \in C$ and the number $\delta = v(t_x)$, where t_x is a uniformising element at x. If $\delta = 0$, then the restriction of v on K^B is trivial (such valuations are called *central*), and the point $x \in C$ may be chosen arbitrary.

Thus $\mathcal{V}$ lives in the union of semispaces $\mathcal{Q}_{x,+} \cong \mathbb{Q}_+ \times \mathcal{Q},\ x \in C$, glued together along their common boundary hyperplane $\mathcal{Q} = \operatorname{Hom}(\Gamma, \mathbb{Q})$. Such an aggregate $\mathcal{Q}_{C,+} = \bigcup_{x\in C} \mathcal{Q}_{x,+}$ will be called a *hyperspace* over C.

Thus an element $v \in \mathcal{V}_x = \mathcal{V} \cap \mathcal{Q}_{x,+}$ is given by a pair (δ, ℓ), where $\delta \in \mathbb{Q}_+$ is a *jump* in the point x and $\ell \in \mathcal{Q}$ is a *central component.* The set $\mathcal{V}_x$ is a convex solid cone [Kn2, §5], its face $\mathcal{Z} = \mathcal{V} \cap \mathcal{Q}$ is called the *cone of central valuations.*

Also we have a map $\mathcal{D}^B \to \mathcal{Q}_{C,+}$ (the restriction on $K^{(B)}$), not necessarily injective.

Remark 2.1. A splitting homomorphism $\Gamma \to K^{(B)}$ is defined not uniquely. A pass from one homomorphism to another may be considered as a "change of coordinates" in the hyperspace, which is controlled easily. Indeed, assume e and e' are two splitting homomorphisms; then the jump does not change when we pass from e to e': $\delta' = \delta$; and the central component of $v \in \mathcal{V}_x$ is shifted: $\ell' = \ell + \delta\ell_x$, where $\langle \ell_x, \lambda\rangle = v_x\left(\frac{e'_\lambda}{e_\lambda}\right)$. In addition, $\sum_{x\in C} \ell_x = 0$ (since $\deg \operatorname{div}\left(\frac{e'_\lambda}{e_\lambda}\right) = 0$). Conversely, any such balanced collection of integral shifts $\ell_x \in \Gamma^* = \operatorname{Hom}(\Gamma, \mathbb{Z}) \subset \mathcal{Q},\ \sum \ell_x = 0$, where $\sum_{x\in C}\langle \ell_x, \lambda\rangle \cdot x \subset C$ is a principal divisor for $\forall\lambda$, defines a pass from one splitting to another: for $\left(\frac{e'_\lambda}{e_\lambda}\right)$, we should take rational functions on C having zeros (or poles) of order $\langle \ell_x, \lambda\rangle$ in $x \in C$. For $C = \mathbb{P}^1$, it suffices to have $\sum \ell_x = 0$.

One could give a "coordinateless" definition of a hyperspace, but it is more cumbersome.

Any element of $K^{(B)}$ looks like $f = f_0 e_\lambda,\ f_0 \in K^B,\ \lambda \in \Gamma$. We associate with it a function φ on $\mathcal{Q}_{C,+}$ whose restriction on every $\mathcal{Q}_{x,+}$ is a linear functional $\varphi_x = \varphi|_{\mathcal{Q}_{x,+}}$: $\langle q, \varphi_x\rangle = \delta v_x(f_0) + \langle \ell, \lambda\rangle$, where $q = (\delta, \ell) \in \mathcal{Q}_{x,+}$. In this setting, $\langle q_x, \varphi_x\rangle = 0$ for almost all $x \in C$, where $q_x = (1, 0) \in \mathcal{Q}_{x,+}$. We call such functions *linear functionals on the hyperspace.* A B-eigenfunction is determined by its linear functional uniquely (up to a scalar multiple). Conversely, any linear functional φ on the hyperspace that is integral on Γ^* and on all q_x is determined by an element $s \cdot e_\lambda \in L_{(C)} \otimes k[\Gamma]$, where

$\lambda \in \Gamma$, and $L_{(C)}$ is the space of rational sections of an invertible sheaf L on C. Namely, $\forall q = (\delta, \ell) \in \mathcal{Q}_{x,+} \ : \ \langle q, \varphi_x \rangle = \delta v_x(s) + \langle \ell, \lambda \rangle$. The functional φ determines an element of $K^{(B)}$ (i. e., $L = \mathcal{O}_C$ is the structure sheaf) iff $\sum_{x \in C} \langle q_x, \varphi_x \rangle \cdot x \subset C$ is a principal divisor (in particular, $\sum \langle q_x, \varphi_x \rangle = 0$; for $C = \mathbb{P}^1$, this is enough). The set $\operatorname{Ker} \varphi = \bigcup_{x \in C} \operatorname{Ker} \varphi_x$ is called the *kernel* of the linear functional.

2.2. Quasihomogeneous and one-parametric cases. Consider an arbitrary G-model X of K. There is a rational map $X \dashrightarrow C$ (rational quotient w. r. t. the B-action) separating B-orbits of general position. Thus B-orbits of general position determine a one-parametric family of B-stable divisors on X parametrized by points from an open subset $\mathring{C} \subseteq C$. Decreasing $\mathring{C}$ if necessary, we obtain that divisors from this family do not occur inside $\operatorname{div} e_\lambda$, $\lambda \in \Gamma$, and that they are preimages of points $x \in \mathring{C}$ with multiplicity 1. The respective valuations are represented in the hyperspace by points $q_x \in \mathcal{Q}_{x,+}$; the collection of such points belongs to the class $\mathfrak{V}$.

For varieties of complexity 1, there are 2 alternatives:

(a) *Quasihomogeneous case*: $K^G = k$. In this case, any G-model X of K contains a dense orbit G/H—a minimal model.

(b) *One-parametric case*: $K^G = K^B$. In this case, a G-orbit of general position has codimension 1 and contains a dense B-orbit.

First we consider the quasihomogeneous case. The variety G/H is unirational, hence $C = \mathbb{P}^1$ by Lüroth theorem. There is a 1–1 correspondence between B-divisors on G/H and B-eigensections (determined up to multiplying by an invertible function) of homogeneous line bundles $L_\chi = G *_H k_\chi$, where k_χ is the one-dimensional representation of H, corresponding to a character $\chi \in \mathfrak{X}(H)$. Since $H^0(G/H, L_\chi) \cong k[G]^{(H)}_{-\chi}$, elements of $D^B(G/H)$ correspond bijectively to indecomposable eigenfunctions from $k[G]^{(B \times H)}$ determined up to multiplying by functions proportional to elements of $\mathfrak{X}(G)$ (here B acts on functions by left and H by right translates of an argument). Elements of $k[G]^{(B \times H)}$ may be considered as equations for preimages of B-divisors in the group G, which is a factorial variety, being of simply connected type by assumption [KKLV].

Consider the map $G/H \dashrightarrow \mathbb{P}^1$ of rational factorization w. r. t. the B-action. It is determined by a one-dimensional linear system of B-stable divisors, i. e., there exists a line bundle on G/H and a two-dimensional subspace $k[G]^{(B \times H)}_{(\lambda_0, \chi_0)}$ of its B-eigensections that defines this linear system and consists (except a finite number of lines) of indecomposable semiinvariants corresponding to B-divisors of general position. Elements of $k[G]^{(B\times H)}_{(\lambda_0,\chi_0)}$ play the role of "homogeneous coordinates" on $\mathbb{P}^1$. Indecomposable elements of $k[G]^{(B\times H)}_{(\lambda_0,\chi_0)}$ and the respective B-divisors are called *regular*.

Besides, there exists a finite set of one-dimensional subspaces $k[G]^{(B\times H)}_{(\lambda,\chi)}$ consisting of indecomposable semiinvariants that correspond to other B-divisors. If an element of $k[G]^{(B\times H)}_{(\lambda,\chi)}$ divides some element of $k[G]^{(B\times H)}_{(\lambda_0,\chi_0)}$, i. e., the respective B-divisor occurs inside the respective element of the linear system (maybe with multiplicity), then we call such a semiinvariant and B-divisor *subregular*. In the converse case, such elements and divisors are called *central*.

Regular B-divisors have $\delta = 1$ and almost always $\ell = 0$.

Subregular B-divisors have δ equal to the multiplicity of the respective $(B\times H)$-semiinvariant factor in the decomposition of the element from $k[G]^{(B\times H)}_{(\lambda_0,\chi_0)}$.

Central B-divisors have $\delta = 0$.

Now we consider the one-parametric case.

Lemma 2.1. *In the one-parametric case, all B-divisors are central.*

Proof. Assume D is a non-central B-divisor; then $\exists f \in K^B = K^G \ : \ v_D(f) > 0$, hence D is G-stable. A contradiction. □

The following statement concerns both quasihomogeneous and one-parametric case.

Lemma 2.2 ([Kn2, 6.2. Lemma]). *For $\forall x \in C$, only a finite number of B-divisors maps to $\mathcal{Q}_{x,+}$. In particular, there are finitely many central B-divisors.*

2.3. Methods for computation. The description of B-divisors in the quasihomogeneous case was given in 2.2. The problem of finding indecomposable $(B\times H)$-semiinvariants belongs essentially to the representation theory (the study of the regular representation).

In order to compute G-valuations in the quasihomogeneous case, we use the method of germs of formal curves [LV, §4]. First consider G-valuations associated with divisors having a dense G-orbit (i. e., the residue field $k(v) = \mathcal{O}_v/\mathfrak{m}_v$ has no invariants: $k(v)^G = k$). All of them, up to calibration, are obtained in the following way. Consider the group $G\big(k((t))\big)$ of points of G over the field of Loran series. Let $x(t) \in G\big(k((t))\big)$. For $f \in K = k(G)^H$ we define a valuation $v_{x(t)}(f) = v_t\big(f(gx(t))\big)$, where g is the generic point of G (over the field $k(G)$) and $v_t(\cdot)$ is the order of a formal series. The right-hand side of the equality may be interpreted as $\min_{g\in G} v_t\bigl(f(gx(t))\bigr)$ as well. The valuation $v_{x(t)}$ corresponds to a divisor with a dense G-orbit and does not change if we multiply $x(t)$ by an element of $G\bigl(k[[t]]\bigr)$ from the left and by an element of $H\bigl(k((t))\bigr)$ from the right. Using the "Iwasawa decomposition" [Sp]

$$G\bigl(k((t))\bigr) = G\bigl(k[[t]]\bigr) \cdot \mathfrak{Y}(T) \cdot U\bigl(k((t))\bigr),$$

where $\mathfrak{Y}(T)$ is the set of one-parametric subgroups of a maximal torus $T \subseteq B$, we may assume $x(t) \in \mathfrak{Y}(T) \cdot U\bigl(k((t))\bigr)$.

G-valuations associated with divisors without dense G-orbit are obtained as "limits" of G-valuations of the previous type, see [LV, 4.11] and §3.

Now let us suppose that a finite group A of birational G-automorphisms acts on K, e. g., $K = k(G/H)$, H is a normal subgroup of finite index in $\overline{H} \subseteq G$, and $A = \overline{H}/H$ acts on G/H by right translations. Suppose we know the structure of the hyperspace $\mathcal{Q}_{C,+}$ supplied with the subset $\mathcal{V} = \bigcup \mathcal{V}_x$ of G-valuations and with the map $\mathcal{D}^B \to \mathcal{Q}_{C,+}$, for the field K. We expose an algorithm of producing the respective description for $\overline{K} = K^A$. All respective objects for $\overline{K}$ will be denoted by the same letters as for K but overlined (e. g., $\overline{\mathcal{V}} = \mathcal{V}(\overline{K})$).

First we pass from the lattice Γ to $\overline{\Gamma}$ (a sublattice of finite index). Under this transformation, $\overline{\mathcal{Q}} = \mathrm{Hom}(\overline{\Gamma}, \mathbb{Q}) \cong \mathcal{Q}$ will not change and $\mathrm{Hom}(\overline{\Gamma}, \mathbb{Z}) \subset \overline{\mathcal{Q}}$ will increase: there will be more integral points.

Then we perform a series of balanced integral (in the new sense) shifts in each of $\mathcal{Q}_{x,+}$ (see remark 2.1) so as to pass from e_λ, $\lambda \in \overline{\Gamma}$, to $\overline{e}_\lambda \in K^{(B)\times A} = \overline{K}^{(B)}$.

Now we consider the action $A : C$. The group A acts also on the hyperspace, mapping each $\mathcal{Q}_{x,+}$ to another isomorphically; on the set $\mathcal{V}$, permuting $\mathcal{V}_x$; and on the set $\mathcal{D}^B$—in accordance with permutations of $x \in C$. The pass $C \to \overline{C}$ is just the factorization w. r. t. the A-action.

Every G-valuation of $\overline{K}$ extends to a G-valuation of K [Kn2, 3.2]. The preimage of any B-divisor from $\overline{\mathcal{D}}^B$ consists of one or more elements from $\mathcal{D}^B$ (maybe with multiplicities). Hence the hyperspace $\overline{\mathcal{Q}}_{\overline{C},+}$ is obtained from $\mathcal{Q}_{C,+}$ by the "factorization" w. r. t. the A-action. In other words, different subspaces $\mathcal{Q}_{x,+}$ for different x having the same image $\overline{x}$ are identified; the same holds for $\mathcal{V}_x$ and $\mathcal{D}^B$. But the following refinement is needed. Let A_* be the kernel of the action $A : C$ (which coincides with the stabilizer of a point in general position). Then we have to increase the jump for elements in $\overline{\mathcal{Q}}_{\overline{x},+}$ in $(A_x : A_*)$ times w. r. t. $\mathcal{Q}_{x,+}$, where x is (any) preimage of $\overline{x}$.

In the one-parametric case, if the action has trivial birational type, i. e., $K = k(C \times G/H)$, $c(G/H) = 0$, then $K^{(B)} = k(C)^\times \cdot k(G/H)^{(B)}$, $\mathcal{Q} =$

$\mathrm{Hom}(\Gamma(G/H),\mathbb{Q})$, $\mathcal{D}^B = \mathcal{D}^B(G/H)$, and $\forall x \in C \;:\; \mathcal{V}_x = \mathbb{Q}_+ \times \mathcal{V}(G/H)$. Indeed, B-divisors on $C \times G/H$ are just $C \times D$, where $D \in \mathcal{D}^B(G/H)$ (a finite set). Further, any G-valuation of K restricted on $k(G/H)$ gives an element of $\mathcal{V}(G/H)$, i. e., $\mathcal{V}_x \subseteq \mathbb{Q}_+ \times \mathcal{V}(G/H)$. On the other hand, every G-valuation of $k(G/H)$ extends to a central G-valuation of K, and every divisor $\{x\} \times G/H \subset C \times G/H$ determines a G-valuation of K represented by the point $q_x \in \mathcal{Q}_{x,+}$. Since $\mathcal{V}_x$ is a convex cone, we have $\mathcal{V}_x \supseteq \mathbb{Q}_+ q_x + \mathcal{V}(G/H) = \mathbb{Q}_+ \times \mathcal{V}(G/H)$, i. e., an equality holds.

Now assume the action has non-trivial birational type, and the stabilizer of general position exists; then there exists a G-equivariant Galois extension $K \subseteq L$ such that the action $G : L$ has trivial birational type (see [PV, §2]). So we may apply the algorithm of factorizing by a finite group for finding $\mathcal{Q}_{C,+}$, $\mathcal{V}$ and $\mathcal{D}^B$. There is a conjecture that every one-parameter action of complexity 1 has stabilizer of general position (i. e., there are no continuous families of pairwise non-conjugate spherical subgroups).

§3. B-charts and G-germs

In this section we give the description of B-charts, G-germs and G-models in terms of combinatorial geometry of certain objects in the hyperspace similar to colored cones and fans for spherical case. We preserve the notation introduced above.

3.1. B-charts and hypercones. Assume $\mathcal{W} \sqcup \mathcal{R} \in \mathfrak{V}$. We want to establish under what conditions does the pair $(\mathcal{W},\mathcal{R})$ determine some B-chart $\mathring{X} = \operatorname{Spec} A,\ A = A(\mathcal{W},\mathcal{R})$ (see 1.4), i. e., what restrictions do the conditions (C), (W), (F) impose on $(\mathcal{W},\mathcal{R})$.

First note that eigenfunctions $f \in K^{(B)}$ figuring in (C), (W), (V), (V′), (D′) may be replaced by f^n, $n \in \mathbb{N}$, without any loss (and conversely, one may pass from f^n to f). Thus the respective linear functionals are interesting to us up to a positive multiple. Actually, we are interested only in "positive and negative semispaces" for a given linear functional φ, i. e., in subsets of the hyperspace, where $\varphi \geqslant 0$ and $\varphi \leqslant 0$. So it suffices that a multiple of φ is given by an element of $K^{(B)}$, i. e., the divisor $\sum N\langle\varphi_x, q_x\rangle \cdot x$ is principal for some $N > 0$. Such linear functionals will be called *admissible*.

There are two types of functionals on the hyperspace. The first type corresponds to rational sections $s \in L_{(C)}$ (in particular, to B-stable rational functions $f \in K^B$). The respective linear functional φ vanishes on the central hyperplane $\mathcal{Q}$ and on almost all $\mathcal{Q}_{x,+}$. Besides, φ takes positive and negative values on a finite collection of semispaces $\mathcal{Q}_{x,+}$. Conversely, for any given collections of "positive and negative" semispaces $\mathcal{Q}_{x,+}$, we may construct respective linear functionals. To determine such a functional uniquely, up to proportionality, it remains to give a collection of weights for the semispaces, i. e., a collection of numbers $d_x = \langle \varphi_x, q_x \rangle$ determined up to proportionality.

The functionals of the second type do not vanish on $\mathcal{Q}$. They are determined, up to proportionality, by the set of hyperplanes $\operatorname{Ker} \varphi_x$ in each $\mathcal{Q}_{x,+}$, which have common intersection with $\mathcal{Q}$ and contain q_x for almost all x. The admissibility condition may be reformulated geometrically as the condition on balanced inclinations of these hyperplanes from the vertical line: for $\forall \ell \in \mathcal{Q} \setminus \operatorname{Ker} \varphi$, if $p_x = (1, d_x \ell) \in (q_x + \mathbb{Q}\ell) \cap \operatorname{Ker} \varphi_x$, then the divisor $\sum N d_x x$ is principal for some $N > 0$ (in particular, $\sum d_x = 0$; for $C = \mathbb{P}^1$, this is enough). Any such collection of hyperplanes having common intersection with $\mathcal{Q}$ (with the "balancing condition for inclinations from the vertical line") determines an (admissible) linear functional φ of the 2-nd type uniquely, up to proportionality. Besides, we have to point out "positive and negative subspaces" w. r. t. φ in each $\mathcal{Q}_{x,+}$.

Elements of $\mathcal{W} \sqcup \mathcal{R}$ are represented by points in the hyperspace, $q_x \in \mathcal{Q}_{x,+}$ being among them for almost all x. In the quasihomogeneous case, these q_x represent elements from $\mathcal{R}$ (regular B-divisors), and in the one-parametric case they represent elements from $\mathcal{W}$.

Consider a set $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ of those elements of the hyperspace, where all admissible functionals nonnegative on $\mathcal{W} \sqcup \mathcal{R}$ are nonnegative as well. This is a "bidual cone" to the set $\mathcal{W} \sqcup \mathcal{R}$. (If we considered a finite-dimensional linear space instead of the hyperspace and a finite set $\mathcal{W} \sqcup \mathcal{R}$, as in the spherical case, then we would obtain just the cone generated by $\mathcal{W} \sqcup \mathcal{R}$, see 1.7.) The sets $\mathcal{C}_x = \mathcal{C} \cap \mathcal{Q}_{x,+}$ and $\mathcal{K} = \mathcal{C} \cap \mathcal{Q}$ are cones in respective (semi)spaces, and our nearest goal is to describe their structure.

Lemma 3.1. *Assume φ is a linear functional on $\mathcal{Q}_{C,+}$ such that $\sum \langle \varphi_x, q_x \rangle < 0$. Then there exists an admissible functional $\psi \geqslant \varphi$ on $\mathcal{Q}_{C,+}$ coinciding with φ on any given finite collection of semispaces $\mathcal{Q}_{x,+}$ such that all differences $\langle \psi_x, q_x \rangle - \langle \varphi_x, q_x \rangle$ are less than any given value. (I. e., φ may be increased to an admissible functional by means of a "small variation".)*

Proof. $\sum \langle \varphi_x, q_x \rangle < 0 \implies -\sum N \langle \varphi_x, q_x \rangle x$ is a very ample divisor for N sufficiently large (divisible by sufficiently many prime numbers). Moving slightly the respective hyperplane section in the embedding of C in a projective space, we find an equivalent very ample divisor without multiple components that does not pass through any given finite subset of C. It looks like $\sum n_x \cdot x$, where $n_x = 0, 1$. Then the divisor $\sum (\langle N\varphi_x, q_x \rangle + n_x) x$ is principal, and we obtain a linear functional on the hyperspace induced by an element of $K^{(B)}$ if we increase the value of $N\varphi$ in q_x by n_x. Then we divide this functional by N and obtain a linear functional whose values in q_x differ from $\langle\varphi_x, q_x\rangle$ by n_x/N, i. e., less than any given value. $\square$

Consider in any $\mathcal{Q}_{x,+}$ an affine hyperplane $\mathcal{Q}_{x,1}$ determined by the condition $\delta = 1$. Let us identify $\mathcal{Q}_{x,1}$ with $\mathcal{Q}$ via the translation by q_x. A non-central element $q \in \mathcal{Q}_{x,+}$ is determined by the point of intersection $\mathbb{Q}_+ q \cap \mathcal{Q}_{x,1}$ and by its jump δ. G-valuations are interesting for us up to proportionality, hence we may determine a non-central G-valuation $v \in \mathcal{V}_x$ by the point $\mathbb{Q}_+ v \cap \mathcal{Q}_{x,1}$, i. e., assume that v has jump 1. It is also convenient to determine non-central B-divisors (more precisely, their images in $\mathcal{Q}_{x,+}$) by points of $\mathcal{Q}_{x,1}$, indicating their jump.

Thus non-central elements of $\mathcal{W} \sqcup \mathcal{R}$ generate a collection of polytopes $q_x + \mathcal{P}_x \subseteq \mathcal{Q}_{x,1}$ (convex hulls of respective points in $\mathcal{Q}_{x,1}$) so that almost all $\mathcal{P}_x = \{0\}$. Consider a polytope $\mathcal{P} = \sum_x \mathcal{P}_x$. If some of $\mathcal{Q}_{x,+}$ contains no non-central elements of $\mathcal{W} \sqcup \mathcal{R}$, then formally $\mathcal{P}_x = \varnothing$ and $\mathcal{P} = \varnothing$. Note that if an admissible functional φ is nonnegative on $\mathcal{W} \sqcup \mathcal{R}$, then $\forall x: \ \langle\varphi_x, q_x + \mathcal{P}_x\rangle \geqslant 0 \implies \sum_x \langle\varphi_x, q_x + \mathcal{P}_x\rangle = \sum\langle\varphi_x, q_x\rangle + \sum\langle\varphi, \mathcal{P}_x\rangle = \langle\varphi, \mathcal{P}\rangle \geqslant 0$.

Lemma 3.2. *Assume φ_x is a linear functional on $\mathcal{Q}_{x,+}$ (x is given) that is positive on $\mathcal{P}$ and nonnegative on elements of $\mathcal{W} \sqcup \mathcal{R}$ getting into $\mathcal{Q}_{x,+}$. Then there exists a functional $\psi_x \leqslant \varphi_x$ on $\mathcal{Q}_{x,+}$ that extends to an admissible functional ψ on $\mathcal{Q}_{C,+}$ nonnegative on $\mathcal{W} \sqcup \mathcal{R}$.*

Proof. The functional ψ_x is constructed as follows: without change of $\varphi_x|_{\mathcal{Q}}$, we turn the hyperplane $\operatorname{Ker}\varphi_x$ so that it becomes supporting for $q_x + \mathcal{P}_x$. Then we extend $\psi_x|_{\mathcal{Q}} = \varphi_x|_{\mathcal{Q}}$ to similar supporting functionals for $q_y + \mathcal{P}_y$ for $\forall y$ (if $\mathcal{P}_y = \varnothing$, then we extend this functional to $\mathcal{Q}_{y,+}$ meanwhile in an arbitrary way). If $\forall y: \ \mathcal{P}_y \neq \varnothing$, then $\forall y: \ \langle\psi_y, q_y + \mathcal{P}_y\rangle \geqslant 0$ (and the equality is achieved) $\implies \sum_y \langle\psi_y, q_y\rangle + \langle\psi, \mathcal{P}\rangle \geqslant 0$ (the equality is achieved) $\implies \sum_y \langle\psi_y, q_y\rangle < 0$. If otherwise $\exists y: \ \mathcal{P}_y = \varnothing$, then we decrease $\langle\psi_y, q_y\rangle$ and obtain $\sum_y \langle\psi_y, q_y\rangle < 0$. By lemma 3.1, we can increase the functional ψ to an admissible one via "small variation" (moving some hyperplanes $\operatorname{Ker}\psi_y$ apart from supporting ones) without changing $\psi_x = \psi|_{\mathcal{Q}_{x,+}}$. $\square$

Consequently, if $\mathcal{P} \not\ni 0$ (in particular, if $\mathcal{P} = \varnothing$), then the cones $\mathcal{C}_x$ are generated by $\mathcal{P}$ and by elements of $\mathcal{W} \sqcup \mathcal{R}$ getting into $\mathcal{Q}_{x,+}$.

The condition (C) means that all cones $\mathcal{C}_x$ are strictly convex (i. e., their common central face $\mathcal{K}$ is strictly convex), $\mathcal{P} \not\ni 0$, and no one element of $\mathcal{R}$ is mapped to 0. Indeed, (C) holds iff there exists an admissible functional φ that is nonnegative on $\mathcal{W} \sqcup \mathcal{R}$ and positive on elements of $\mathcal{W} \sqcup \mathcal{R}$ getting into $\mathcal{Q}_{x,+}$, for any given finite set of x, hence positive on $\mathcal{P}$ as well. Thus the conditions listed above (strict convexity, etc.) are necessary. They are sufficient, too. Indeed, let us take a linear functional on $\mathcal{Q}$ positive on $\mathcal{K} \setminus \{0\}$ and extend it to supporting functionals φ_x on $\mathcal{Q}_{x,+}$, as in lemma 3.2, so that $\sum_x \langle \varphi_x, q_x \rangle < 0$. Then we increase $\langle \varphi_x, q_x \rangle$ slightly in the given set of x (so that still $\sum_x \langle \varphi_x, q_x \rangle < 0$ holds) and construct a required functional, applying lemma 3.1.

Note that we may forget about $\mathcal{W}$ while checking (C) and consider only the pair $(\mathcal{C}, \mathcal{R})$, whereas $q_x + \mathcal{P}_x$ is defined as the convex hull of the vertex set of the polyhedral domain $\mathcal{K}_x = \mathcal{C}_x \cap \mathcal{Q}_{x,1}$. Thus we come to the following definition.

Definition 3.1. A *colored hypercone* is a pair $(\mathcal{C}, \mathcal{R})$, where $\mathcal{R} \subseteq \mathcal{D}^B$, and $\mathcal{C} = \bigcup_{x \in C} \mathcal{C}_x$ is a union of finitely generated strictly convex cones $\mathcal{C}_x = \mathcal{C} \cap \mathcal{Q}_{x,+}$, so that the following conditions hold:

(a) $\mathcal{P} = \sum \mathcal{P}_x \not\ni 0$, where $q_x + \mathcal{P}_x$ is the convex hull of the vertex set of the polyhedral domain $\mathcal{K}_x = \mathcal{C}_x \cap \mathcal{Q}_{x,1}$;

(b) all $\mathcal{C}_x$ are generated by $\mathcal{P}$ together with a finite number of elements from $\mathcal{V}$ and elements of $\mathcal{R}$ mapped to $\mathcal{Q}_{x,+}$ (more precisely, their images);

(c) almost all $\mathcal{C}_x$ (except a finite number) are generated by $\mathcal{K} = \mathcal{C} \cap \mathcal{Q}$ and q_x;

(d) no one element of $\mathcal{R}$ is mapped to 0.

Elements of $\mathcal{R}$ are called *colors* of the hypercone.

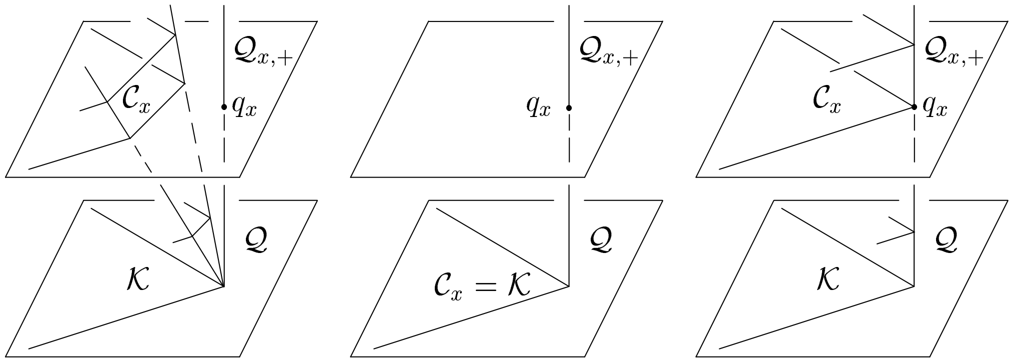
There are two types of hypercones:

I. $\exists x \in C \ : \ \mathcal{C}_x \subseteq \mathcal{Q} \iff \exists x \in C \ : \ \mathcal{P}_x = \varnothing$

II. $\forall x \in C \ : \ \mathcal{C}_x \nsubseteq \mathcal{Q} \iff \forall x \in C \ : \ \mathcal{P}_x \neq \varnothing$

We shall distinguish "true" faces of $\mathcal{K}$ (not intersecting $\mathcal{P}$) and "pseudo-faces" (intersecting $\mathcal{P}$). For a hypercone of type I, $\mathcal{P} = \varnothing$ (since $\exists x : \ \mathcal{P}_x = \varnothing$), and all faces are true.

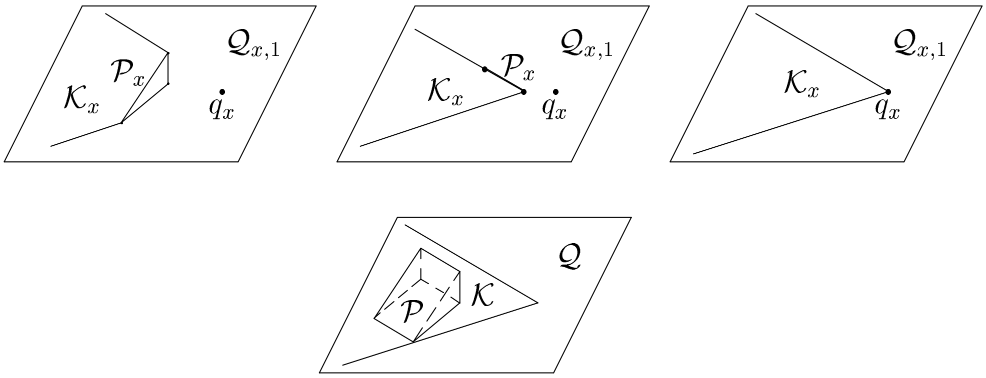
The main elements of definition 3.1 are illustrated at the following picture:



Type I

$\delta = 1$

$\delta = 0$



Type II

The condition (W) means that elements of $\mathcal{W}$ (getting into $\mathcal{Q}_{x,+}$) are just generators of those edges of $\mathcal{C}_x$ that contain no (images of) elements from $\mathcal{R}$ and $\mathcal{P}$. Indeed, (W) means that for $\forall w \in W$ there exists an admissible functional φ that is negative on w and non-negative on $(\mathcal{W} \setminus \{w\}) \sqcup \mathcal{R}$, hence w lies on an edge of some $\mathcal{C}_x$, and no elements of $\mathcal{R}$ and $\mathcal{P}$ get into this edge ($w \in \mathcal{K} \implies \langle \varphi, \mathcal{P} \rangle \geqslant 0$). Conversely, for any such edge $\mathbb{Q}_+ w,\ w \in \mathcal{W}$, we may construct a functional φ_x on $\mathcal{Q}_{x,+}$ that is negative on w and positive on $\mathcal{P}$ and on elements of $(\mathcal{W} \setminus \{w\}) \sqcup \mathcal{R}$ getting into $\mathcal{C}_x$. Then we apply lemma 3.2 to φ_x and to the set $(\mathcal{W} \setminus \{w\}) \sqcup \mathcal{R}$.

Thus the set $\mathcal{W}$ is recovered from the hypercone $(\mathcal{C}, \mathcal{R})$.

Definition 3.2. A *hyperface* of the colored hypercone is a colored hypercone $(\mathcal{C}', \mathcal{R}')$ such that $\mathcal{C}' = \mathcal{C} \cap \operatorname{Ker} \varphi$, where φ is a linear functional nonnegative on $\mathcal{C}$, and $\mathcal{R}'$ is the set of all elements from $\mathcal{R}$ mapped to $\operatorname{Ker} \varphi$, so that either

I. $(\mathcal{C}', \mathcal{R}')$ is of type I, and $\mathcal{C}' \cap \mathcal{Q}$ is a true face of $\mathcal{K}$, or

II. $(\mathcal{C}', \mathcal{R}')$ is of type II, and $\mathcal{C}' \cap \mathcal{Q}$ is a pseudoface of $\mathcal{K}$.

The functional φ is called *supporting* for the hyperface $(\mathcal{C}', \mathcal{R}')$. It can always be changed so as to satisfy the condition $\sum_x \langle \varphi_x, q_x \rangle = 0$ (which is fulfilled automatically for the hyperface of type II). A hyperface is called *admissible* if its supporting functional is admissible. For $C = \mathbb{P}^1$, every hyperface is admissible.

Proposition 3.1. *The following conditions are equivalent:*

(F)

(1) *Weights of elements from $A^{(B)}$ form a finitely generated semigroup.*

(2) *All supporting functionals for hyperfaces of type II of the hypercone $(\mathcal{C}, \mathcal{R})$ are admissible. Such hypercones will be called* admissible.

Proof. (F) $\Longrightarrow$ (1) It is obvious.

(1) $\Longrightarrow$ (2) Every linear functional on $\mathcal{Q}$, nonnegative on $\mathcal{K}$ and positive on $\mathcal{P}$, i. e., lying in the dual cone $\mathcal{K}^\vee$ with some faces removed, may be extended by lemma 3.2 to an admissible functional on the hyperspace, nonnegative on $\mathcal{C}$. A multiple of such a functional is induced by an element from $A^{(B)}$. Functionals lying on the remaining faces of $\mathcal{K}^\vee$ extend uniquely to supporting functionals for hyperfaces of type II. Hence for the weight semigroup of $A^{(B)}$ being finitely generated, it is necessary that all these supporting functionals are admissible, too.

(2) $\Longrightarrow$ (1) Let $S = \Gamma \cap \mathcal{K}^\vee$ be the (finitely generated) semigroup of integral points in the dual cone, and Σ the weight semigroup of $A^{(B)}$. Then $S \supseteq \Sigma \supseteq N \cdot S$ for N sufficiently large, i. e., the algebra $k[\Sigma]$ is integral over its finitely generated subalgebra $k[N \cdot S]$, and the extension of quotient fields $Qk[\Sigma] \supseteq Qk[N \cdot S]$ is finite. Consequently, $k[\Sigma]$ is finitely generated [Bou, Chap. V, §3, n°2], hence Σ is finitely generated as well.

(2) $\Longrightarrow$ (F) Assume $f \in A^{(B)}$ and consider the respective functional φ. Consider a decomposition $\varphi = \varphi_0 + \varphi_+$, where φ_+ is nonnegative on the whole hyperspace and φ_0 extends $\varphi|_{\mathcal{Q}}$ and has the following properties: $\varphi_0|_{\mathcal{K}_x} \geqslant 0$, and the equality is achieved at some point whenever $\mathcal{C}_x \neq \mathcal{K}$; $\langle \varphi_0, q_x \rangle < 0$ whenever $\mathcal{C}_x = \mathcal{K}$; and either $\sum \langle \varphi_0, q_x \rangle < 0$, or $\varphi_0 = \varphi$ is a supporting functional for a hyperface of type II. If φ is of type 1, then, obviously, such a decomposition exists. Assume φ is of type 2. It is easy to construct φ_0, rotating the hyperplanes $\operatorname{Ker} \varphi_x$ until they touch $\mathcal{K}_x$. Consider a collection of faces $\mathcal{F}_x \subseteq \mathcal{K}_x$ determined by $\varphi_0 = 0$. There are finitely many such collections for all possible φ_0. Consider a set of all functionals ψ nonnegative on $\mathcal{C}$ and vanishing on $\mathcal{F}_x$ (φ_0 is among them). Their restrictions on $\mathcal{Q}$ form a finitely generated cone $\left[\bigcup_{x \in C} (\mathcal{K}_x - \mathcal{F}_x)\right]^\vee$.

Let $\lambda_1, \dots, \lambda_m$ be the generators of all such cones, and $\psi_1, \dots, \psi_m$ the respective linear functionals on the hyperspace. If the hypercone $(\mathcal{C}, \mathcal{R})$ is of type I, then we decrease $\langle \psi_i, q_y \rangle$ in points $y \in C$ such that $\mathcal{C}_y = \mathcal{K}$ and obtain $\sum \langle \psi_i, q_x \rangle < 0$. Moreover, for every such y, we add to $\{\psi_i\}$ a functional nonpositive on $\mathcal{Q}_{y,+}$ and vanishing on other $\mathcal{Q}_{x,+}$. If the hypercone is of type II, then we already have $\sum \langle \psi_i, q_x \rangle \leqslant 0$, and ψ_i is admissible whenever the equality holds (by condition (2)).

We may assume that ψ_i are integral on Γ^* and on all q_x (replacing them by multiples), hence ψ_i are induced by $s_i e_{\lambda_i}$, where $s_i \in \left(L_i^{-1}\right)_{(C)}$, $\deg L_i \geqslant 0$, and $\deg L_i = 0 \implies L_i = \mathcal{O}_C$ (because in this case ψ_i is admissible).

Replacing φ by its multiple, we may assume that φ_0 and φ_+ are integral (on Γ^* and on all q_x), and that $\varphi_0 = \sum k_i \psi_i$, $k_i \geqslant 0$. Then φ_0 is induced by a monomial $(s_1 e_{\lambda_1})^{k_1} \dots (s_m e_{\lambda_m})^{k_m} \in \left(L_1^{-k_1} \dots L_m^{-k_m}\right)_{(C)} \otimes k[\Gamma]$, and φ_+ is induced by an element from $H^0(C, L_1^{k_1} \dots L_m^{k_m})$. Therefore some power of f is a product of $(s_1 e_{\lambda_1})^{k_1} \dots (s_m e_{\lambda_m})^{k_m}$ and an element from $H^0(C, L_1^{k_1} \dots L_m^{k_m})$. Conversely, any such product gives an element of $A^{(B)}$.

Lemma. *The algebra* $\bigoplus_{k_1, \dots, k_m \geqslant 0} H^0(C, L_1^{k_1} \dots L_m^{k_m})$ *is finitely generated.*

Proof of the lemma. Assume that $\deg L_i > 0$ $(= 0)$ whenever $i \leqslant l$ ($> l$, respectively). Then $\bigoplus H^0(C, L_1^{k_1} \dots L_m^{k_m}) \cong \left[\bigoplus H^0(C, L_1^{k_1} \dots L_l^{k_l})\right] \otimes k[\mathbb{A}^{m-l}]$, i. e., we may assume that all $\deg L_i > 0$. Then all line bundles L_i are ample, hence a vector bundle $E = L_1 \oplus \dots \oplus L_m$ is ample [Ha1, Prop. (2.2)] (see [Ha1, §2, p. 65] for the definition of an ample vector bundle). Therefore a line bundle $L = \mathcal{O}_{\mathbb{P}_C(E)}(1)$ is ample, where $\mathbb{P}_C(E)$ is the projectivization of E [Ha1, Prop. (3.2)]. It follows that the algebra $\bigoplus_{k_1, \dots, k_m \geqslant 0} H^0(C, L_1^{k_1} \dots L_m^{k_m}) \cong \bigoplus_{k \geqslant 0} H^0(C, S^k E) \cong \bigoplus_{k \geqslant 0} H^0(\mathbb{P}_C(E), L^k)$ ([Ha2, II.7.11(a)]) is finitely generated. $\square$

An algebra $\widehat{A} = \bigoplus_{k_1, \dots, k_m \geqslant 0} H^0(C, L_1^{k_1} \dots L_m^{k_m})[s_1 e_{\lambda_1}, \dots, s_m e_{\lambda_m}]$ is supplied with the action of a torus $(k^\times)^m$: $t = (t_1, \dots, t_m) \in (k^\times)^m$ multiplies $H^0(C, L_1^{k_1} \dots L_m^{k_m})$ by $t_1^{k_1} \dots t_m^{k_m}$, and $s_i e_{\lambda_i}$ by t_i^{-1}. $\widehat{A}$ is finitely generated, hence by Hilbert theorem on invariants, an algebra $\widehat{A}_0 = (\widehat{A})^{(k^\times)^m}$ is finitely generated, too.

The algebra A^U is integral over $\widehat{A}_0$. Let us show that the extension of quotient fields is finite.

First we have $Q\widehat{A}_0 \supseteq K^B$. Indeed, assume $f \in K^B$ and consider the respective functional ψ of the 1-st type. We can find an element $p \in \widehat{A}_0^{(B)}$

inducing a functional $\underset{\frown}{\varphi} = \varphi_0 + \varphi_+$ such that the functional $\varphi_+ + \psi$ is non-negative. Then $fp \in \widehat{A}_0$.

Secondly $\forall \lambda \in \Gamma\ \exists k > 0\ :\ e_{k\lambda} \in Q\widehat{A}_0$. Indeed, there exists an element $f_0 e_{k\lambda} \in Q\widehat{A}_0,\ f_0 \in K^B$, and it remains to divide it by f_0.

Consequently the field extension $QA^U \supseteq Q\widehat{A}_0$ is finite, hence A^U is finitely generated [Bou, Chap. V, §3, n°2]. ☐

Remark 3.1. Every hypercone of type I is admissible. For $C = \mathbb{P}^1$, every colored hypercone is admissible.

There are two types of B-charts:

I. The function algebra on $\mathring{X}$ has non-trivial B-invariants: $A^B \neq k$. This means that for some x no elements from $\mathcal{W} \sqcup \mathcal{R}$ get into $\mathcal{Q}_{x,+} \setminus \mathcal{Q}$.

II. $A^B = k$, i. e., all $\mathcal{Q}_{x,+}$ contain elements from $\mathcal{W} \sqcup \mathcal{R}$ with nonzero jump.

Summing up, we obtain

Theorem 3.1. *B-charts of type I (type II) are determined by admissible colored hypercones of type I (type II). Localizing in a B-eigenfunction is the same thing as passing to an admissible hyperface.*

Proof. As we have already seen, the pair $(\mathcal{C}, \mathcal{R})$ is an admissible colored hypercone, where $\mathcal{C}$ is a "bidual" set to the pair $(\mathcal{W}, \mathcal{R})$ satisfying (C), (W), and (F). Moreover, the set $\mathcal{W}$ is recovered uniquely. On the other hand, any admissible colored hypercone $(\mathcal{C}, \mathcal{R})$ is a "bidual" set to the pair $(\mathcal{W}, \mathcal{R})$, where $\mathcal{W}$ is the set of generators for all edges of $\mathcal{C}_x$ that contain no colors and elements from $\mathcal{P}$. Clearly, the pair $(\mathcal{W}, \mathcal{R})$ satisfies (C), (W), and (F). The assertion on localization is obvious (see 1.5). ☐

3.2. G-germs and faces. Assume $(\mathcal{C}, \mathcal{R})$ is a colored hypercone, $v \in \mathcal{V}$.

The condition (V) means that $v \in \mathcal{C}$ (by construction of $\mathcal{C}$ as a "bidual" set).

Definition 3.3. A *colored cone* is a pair $(\mathcal{C}', \mathcal{R}')$, where $\mathcal{R}' \subseteq \mathcal{D}^B$ is a finite set whose elements do not map to 0, and $\mathcal{C}'$ is a strictly convex cone in some $\mathcal{Q}_{x,+}$ generated by finitely many elements from $\mathcal{V}$ and by $\mathcal{R}'$.

A *face* of a colored hypercone $(\mathcal{C}, \mathcal{R})$ is either

I. a colored cone $(\mathcal{C}', \mathcal{R}')$ such that $\mathcal{C}'$ is a face of some $\mathcal{C}_x$ intersecting $\mathcal{K}$ in a true face, and $\mathcal{R}'$ is the set of all elements from $\mathcal{R}$ mapped to $\mathcal{C}'$, or

II. a hyperface $(\mathcal{C}', \mathcal{R}')$ of type II.

The *interior* of a colored cone (colored hypercone of type II) $(\mathcal{C}',\mathcal{R}')$ is just the set $\operatorname{int}\mathcal{C}'$ $(=\bigcup \operatorname{int}\mathcal{C}'_x \cup \operatorname{int}\mathcal{K}'$, respectively).

Lemma 3.3. *The face $(\mathcal{C}',\mathcal{R}')$ of an admissible colored hypercone $(\mathcal{C},\mathcal{R})$ such that $v\in\operatorname{int}\mathcal{C}'$ is the intersection of admissible hyperfaces containing v.*

Proof. If φ is an admissible linear functional such that $\langle\varphi,\mathcal{C}\rangle\geqslant 0$, $\langle\varphi,v\rangle=0$, then $\langle\varphi,\mathcal{C}'\rangle=0$ (for the face of type II, $\langle\varphi,\mathcal{K}'\rangle=0$, hence $\forall x:\ \langle\varphi,\mathcal{C}'_x\rangle=0$). Conversely, for $q\in\mathcal{C}\setminus\mathcal{C}'$ it is easy to construct a functional φ such that $\langle\varphi,\mathcal{C}\rangle\geqslant 0$, $\langle\varphi,\mathcal{C}'\rangle=0$, $\langle\varphi,q\rangle>0$. For the face of type II, φ is already admissible (since the hypercone is admissible). For the face of type I, we may assume $\sum_x\langle\varphi_x,q_x\rangle<0$ and increase φ to an admissible functional by lemma 3.1 preserving the conditions $\langle\varphi,\mathcal{C}'\rangle=0$, $\langle\varphi,q\rangle>0$. □

Consequently, the conditions (V′) and (D′) select (elements of $\mathcal{W}\sqcup\mathcal{R}$ lying in) the face $(\mathcal{C}_Y,\mathcal{D}_Y^B)$ containing v in its interior.

The support $\mathcal{S}_Y$ of the corresponding G-germ equals $\mathcal{V}\cap\operatorname{int}\mathcal{C}_Y$.

Summing up, we have proved

Theorem 3.2. *G-germs (intersecting a given B-chart) are determined by colored cones and admissible hypercones of type II (being faces of the given admissible hypercone) whose interiors intersect $\mathcal{V}$. The adherence of G-germs (i. e., inclusion of respective closed subvarieties in any G-model) corresponds to the inverse inclusion of colored (hyper)cones as faces in each other.*

A G-germ admitting a chart of type I is called a *G-germ of type I*; G-germs admitting only B-charts of type II are called *G-germs of type II*. G-germs of type I (type II) are given by colored cones (by admissible hypercones of type II, respectively). Germs of type I are characterized by finiteness of $\mathcal{V}_Y\sqcup\mathcal{D}_Y^B$, and germs of type II are characterized by the existence of a minimal B-chart.

Corollary 3.1. *In the quasihomogeneous case, $q_x\notin\mathcal{V}_x$ for almost all x.*

Proof. There exists a G-valuation w corresponding to a divisor with open G-orbit (we may construct it using the method of germs of formal curves if we take $x(t)\in G\big(k((t))\big)\setminus G\big(k[[t]]\big)H\big(k((t))\big)$). It is easy to construct its geometric realization consisting of two G-orbits (a dense one and an orbit of codimension 1). Take a B-chart of type I in it intersecting an orbit of codimension 1. For this chart we have $\mathcal{W}=\{w\}$ and $\mathbb{Q}_+w$ is the only face of the respective hypercone whose interior intersects $\mathcal{V}$. At the same time, any element of $\mathcal{R}$ mapping to $q_x\in\mathcal{V}_x$ determines a face $\mathbb{Q}_+q_x$. □

Corollary 3.2. *Non-central G-valuations correspond to divisors with an open G-orbit; central ones correspond to divisors with infinitely many G-orbits, with the exception of proper faces $\mathcal{Z}' \subset \mathcal{Z}$ such that $(\mathcal{Z}' + \mathbb{Q}_+ q_x) \cap \mathcal{V} = \mathcal{Z}'$ for almost all x, in the quasihomogeneous case.*

Proof. Assume w is a non-central G-valuation. There exists a hypercone $(\mathcal{C}, \mathcal{R})$ of type I with $\mathcal{C}_x = \mathbb{Q}_+ w$ (for some x). $\mathcal{C}_x$ is not contained in any other face of $(\mathcal{C}, \mathcal{R})$, i. e., the respective divisor has no proper G-stable subvarieties intersecting the B-chart associated with $(\mathcal{C}, \mathcal{R})$, hence contains an open G-orbit.

Similarly, if $w \in \mathcal{Z}$, then any hypercone having a face $\mathbb{Q}_+ w$ has infinitely many faces containing w and q_x (for almost all x) whose interiors intersect $\mathcal{V}$. The exceptions are possible only in the quasihomogeneous case ($q_x \notin \mathcal{V}_x$) when w gets into some proper face $\mathcal{Z}' \subset \mathcal{Z}$ so that $(\mathcal{Z}' + \mathbb{Q}_+ q_x) \cap \mathcal{V} = \mathcal{Z}'$. $\square$

Corollary 3.3. *In the quasihomogeneous case, a germ of type I admits a geometric realization with finitely many orbits iff $\mathcal{C}_Y \cap \mathcal{Z}$ is contained in one of the faces of $\mathcal{Z}$ mentioned in corollary 3.2 (e. g., it equals $\{0\}$).*

3.3. The description of G-models. The procedure of constructing G-models from G-germs is described in 1.6. It remains to translate in into the language of (hyper)cones.

Theorem 3.3. *A normal G-model is given by a set of colored cones and admissible hypercones of type II obtained from a finite collection of admissible colored hypercones as the set of all their faces whose interiors intersect $\mathcal{V}$, under the condition that interiors of all these faces are disjoint inside $\mathcal{V}$.*

By analogy with the spherical case, we may call such a set of cones and admissible hypercones of type II a *colored fan.* A G-model is affine iff its fan consists of all possible faces of a *single* colored hypercone having all elements of $\mathcal{D}^B$ for colors. A G-model is complete iff its fan covers all $\mathcal{V}$.

§4. Examples

In the last section we consider several examples of describing varieties with complexity 1. The first example describes the situation, where G is an algebraic torus. Other examples concern the quasihomogeneous case, i. e., the field K equals $k(G/H)$ with the natural G-action. In each of these examples, we indicate:

—indecomposable eigenfunctions from $k[G]^{(B\times H)}$ (more precisely, their biweights $(\lambda, \chi) \in \mathfrak{X}(B \times H)$), i. e., the finite set of subregular and central semiinvariants and the two-dimensional subspace $k[G]^{(B\times H)}_{(\lambda_0,\chi_0)}$ filled with regular semiinvariants. Denote by F an (arbitrary) fixed regular semiinvariant from $k[G]^{(B\times H)}_{(\lambda_0,\chi_0)}$, and call the respective point on $\mathbb{P}^1$ *distinguished.*

—the weight group Γ and the generators of $e(\Gamma)$.

—the hyperspace $\mathcal{Q}_{\mathbb{P}^1,+}$ together with the cones $\mathcal{V}_x \subseteq \mathcal{Q}_{x,+}$ of G-valuations (we shade them at the pictures) and with the images of B-divisors (denoted by white circles $\circ$ at the pictures). Besides, from 4.8 and below, we keep the conventions of 3.1, i. e., we describe the hyperspace by the set of hyperplanes $\mathcal{Q}_{x,1}$ together with the central hyperplane $\mathcal{Q}$, non-central G-valuations by polyhedral domains $\mathcal{V}_x \cap \mathcal{Q}_{x,1} \subseteq \mathcal{Q}_{x,1}$, and non-central B-divisors by points of $\mathcal{Q}_{x,1}$, indicating their jump. G-valuations (and their images in the hyperspace) are computed via combining the method of germs of formal curves and factorization by a finite group of automorphisms (see 2.3).

Moreover, in examples 4.2–4.7 we classify all kinds of G-germs and describe the respective colored (hyper)cones. In the general case, we content ourselves with the clarity of the situation (see §3).

4.1. $G = B = T$ **is a torus.** This class of examples belongs to the one-parametric case. We may assume (after factorizing by the kernel of the action) that the stabilizer of general position is trivial. Since a torus has no non-trivial Galois cohomology (Hilbert's theorem 90, see [Se], [PV]), the birational type of the action is trivial, i. e., (any) T-model is birationally isomorphic to $T\times C$. By 2.3, $\mathcal{Q} = \operatorname{Hom}(\mathfrak{X}(T), \mathbb{Q})$, $\mathcal{V} = \mathcal{Q}_{C,+}$ (since $\mathcal{V}(T) = \mathcal{Q}$), $\mathcal{D}^B = \mathcal{D}^B(T) = \varnothing$. Normal T-models are given by sets of cones and admissible hypercones of type II (being faces of a finite collection of admissible hypercones) with disjoint interiors (the word "colored" is needless here, since there are no colors). A T-chart is of type I (type II) iff $\mathring{X}/\!\!/T \subset C$ is an open subset ($\mathring{X}/\!\!/T$ is a point). If all germs of a T-model are of type I, then quotient morphisms of its T-charts may be glued together into a regular map $X \to C$ separating T-orbits of general position. Such T-models were classified in [KKMS, Chap. IV].

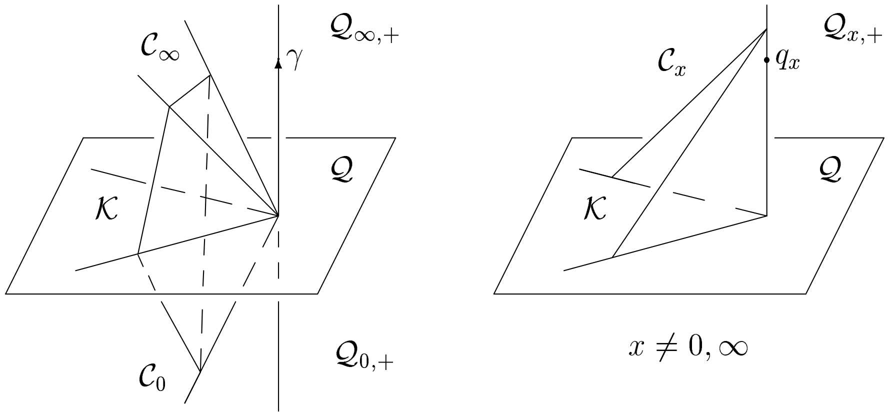
The example, where $T \subset \widetilde{T}$ is a codimension 1 subtorus in a torus $\widetilde{T}$ acting on a toric variety X, belongs to this class. By 1.7, the toric variety X is given by a fan in a space $\widetilde{\mathcal{Q}} = \operatorname{Hom}(\mathfrak{X}(\widetilde{T}), \mathbb{Q})$. Now we construct a fan in the hyperspace, describing the action of the subtorus T. Since a toric variety is covered by affine $\widetilde{T}$-charts, we may assume that X is affine and its fan in $\widetilde{\mathcal{Q}}$ is given by a single cone $\widetilde{\mathcal{C}}$ (and its faces).

Consider a decomposition $\widetilde{T} = T \times k^\times$ and the respective decompositions $\mathfrak{X}(\widetilde{T}) = \mathfrak{X}(T) \oplus \mathbb{Z}\chi$, $\widetilde{\mathcal{Q}} = \mathcal{Q} \oplus \mathbb{Z}\gamma$, where $\forall t \in T,\, x \in k^\times \ : \ \chi(t \times x) = x$, $\langle \gamma, \chi\rangle = 1$. Since X contains an open subset isomorphic to $\widetilde{T}$, we see that

$C = \mathbb{P}^1$. We have $K^{(\widetilde{T})} = e(\mathfrak{X}(\widetilde{T})) \cdot k^\times$, where a rational function e_μ of the weight μ on X is associated with every $\mu \in \mathfrak{X}(\widetilde{T})$; under the restriction on $\widetilde{T}$, $e_\mu(\tilde{t}) = \mu(\tilde{t})^{-1}$. It follows that $K^{(T)} = e(\mathfrak{X}(T)) \cdot k(e_{-\chi})^\times$ $(e_{-\chi}(t \times x) = x)$.

There are two types of T-stable divisors on X. First there are closures of T-orbits on $\widetilde{T}$ determined by equations $e_{-\chi}(\tilde{t}) = x \in k^\times$; they are represented by $q_x \in \mathcal{Q}_{x,+}$, $x \in \mathbb{P}^1 \setminus \{0, \infty\}$. Secondly there are $\widetilde{T}$-stable divisors in the complement to $\widetilde{T}$. The respective valuations (restricted on $K^{(\widetilde{T})}$) are represented in $\widetilde{\mathcal{Q}}$ by generators of edges of $\widetilde{\mathcal{C}}$ via the formula: $v(e_\mu) = \langle v, \mu \rangle = \langle v, \lambda \rangle + n\langle v, \chi \rangle$, where $\mu = \lambda + n\chi$, $\lambda \in \mathfrak{X}(T)$. Since $e_{-\chi}$ $(e_{-\chi}^{-1})$ is a uniformising element at $0 \in \mathbb{P}^1$ (at $\infty \in \mathbb{P}^1$), we may identify the subspaces $\mathcal{Q}_{0,+}$ and $\mathcal{Q}_{\infty,+}$ with those given by partitioning $\widetilde{\mathcal{Q}}$ by $\mathcal{Q}$ that contain $-\gamma$ or γ, respectively. In this setting, ℓ is the projection of v on $\mathcal{Q}$, and $\delta = |\langle v, \chi \rangle|$.

A hypercone $\mathcal{C}$ describing a T-chart X is determined by admissible linear functionals on $\mathcal{Q}_{\mathbb{P}^1,+}$, nonnegative in points representing T-stable divisors. Moving such a functional, we may assume that it vanishes in q_x, $x \neq 0, \infty$. Such a functional is induced by an element from $K^{(\widetilde{T})}$ and defines a linear functional on $\widetilde{\mathcal{Q}}$. Consequently, $\mathcal{C}_0 = \widetilde{\mathcal{C}} \cap \mathcal{Q}_{0,+}$, $\mathcal{C}_\infty = \widetilde{\mathcal{C}} \cap \mathcal{Q}_{\infty,+}$, $\mathcal{K} = \widetilde{\mathcal{C}} \cap \mathcal{Q}$, $\mathcal{C}_x = \mathcal{K} + \mathbb{Q}_+ q_x$ whenever $x \neq 0, \infty$,—see the picture:



In the general case, where a toric $\widetilde{T}$-variety X is pasted from several $\widetilde{T}$-charts, we may construct the respective hypercone for each chart. These hypercones define a fan in the hyperspace, describing the action $T : X$ (see 3.3), in particular, the respective position of T-orbits etc.

4.2. $G = SL_2$, $H = \{e\}$**.** This example is examined in detail in [LV, §9]. Elements of G are matrices

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad g_{11}g_{22} - g_{21}g_{12} = 1,$$

B consists of upper-triangular matrices ($g_{21} = 0$). The group $\mathfrak{X}(B)$ is generated by the fundamental weight ω, $\omega(g) = g_{11}$ for $g \in B$.

There are no subregular and central semiinvariants. Regular semiinvariants fill the two-dimensional subspace $k[G]^{(B)}_\omega$, generated by $f_0(g) = g_{21}$, $f_\infty(g) = g_{22}$.

The group Γ equals $\mathfrak{X}(B) = \mathbb{Z}\omega$, $e_\omega = F \in k[G]^{(B)}_\omega$.

Any G-valuation corresponding to a divisor with dense G-orbit is proportional to $v_{x(t)}$, where $x(t) = \begin{pmatrix} t^p & u(t) \\ 0 & t^{-p} \end{pmatrix} \in G\bigl(k((t))\bigr)$, $p \in \mathbb{Z}$, $u(t) \in k((t))$, $v_t(u(t)) = q$, and we may assume $q \leqslant -p$ (see 2.3). Consider a number

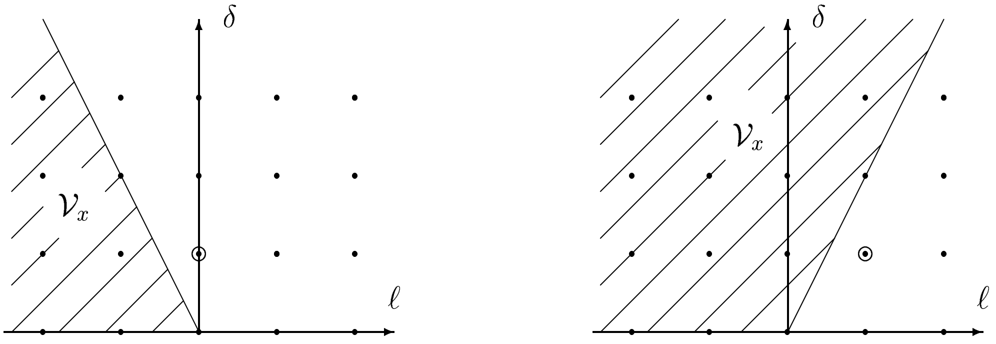
$$d = v_{x(t)}(\alpha f_0 + \beta f_\infty) = v_t\bigl((\alpha t^p + \beta u(t))g_{21} + \beta t^{-p} g_{22}\bigr).$$

The following cases are possible:

$p \leqslant q \implies \begin{cases} d = p, & v_t(\alpha t^p + \beta u(t)) = p \\ d \in (p, -p], & v_t(\alpha t^p + \beta u(t)) > p \end{cases} \implies \delta \in (0, -2p]$ (the jump in $x \in \mathbb{P}^1$ corresponding to $\alpha f_0 + \beta f_\infty$ such that $v_t(\alpha t^p + \beta u(t)) > p$), $\langle \ell, \omega\rangle = v_{x(t)}(e_\omega) = p$ (or $p+\delta$ if the jump occurs in the distinguished point).

$p > q \implies \begin{cases} d = q, & \beta \neq 0 \\ d = p, & \beta = 0 \end{cases} \implies \delta = p - q$ (the jump in $x \in \mathbb{P}^1$ corresponding to f_0), $\langle \ell, \omega\rangle = q$ (or $q + \delta$).

Identifying the functional ℓ with the number $\langle \ell, \omega\rangle$, we see that the set of G-valuations having the jump in x and corresponding to divisors with dense G-orbit is determined by inequalities $\delta > 0$, $2\ell + \delta \leqslant 0$ (or $2\ell - \delta \leqslant 0$ if x is distinguished), hence the cone $\mathcal{V}_x$ is determined by inequality $2\ell + \delta \leqslant 0$ (or $2\ell - \delta \leqslant 0$). Summing up, we obtain the following picture for the hyperspace:

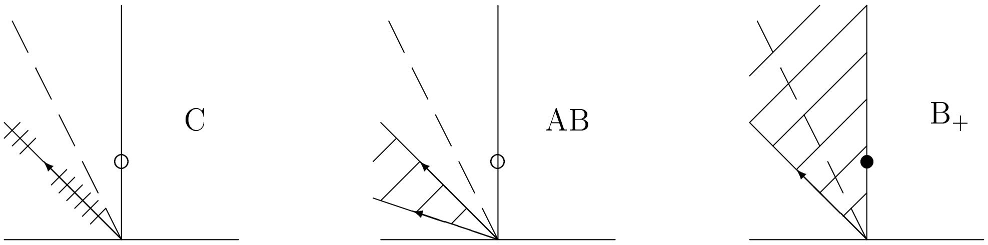


point of general position   distinguished point

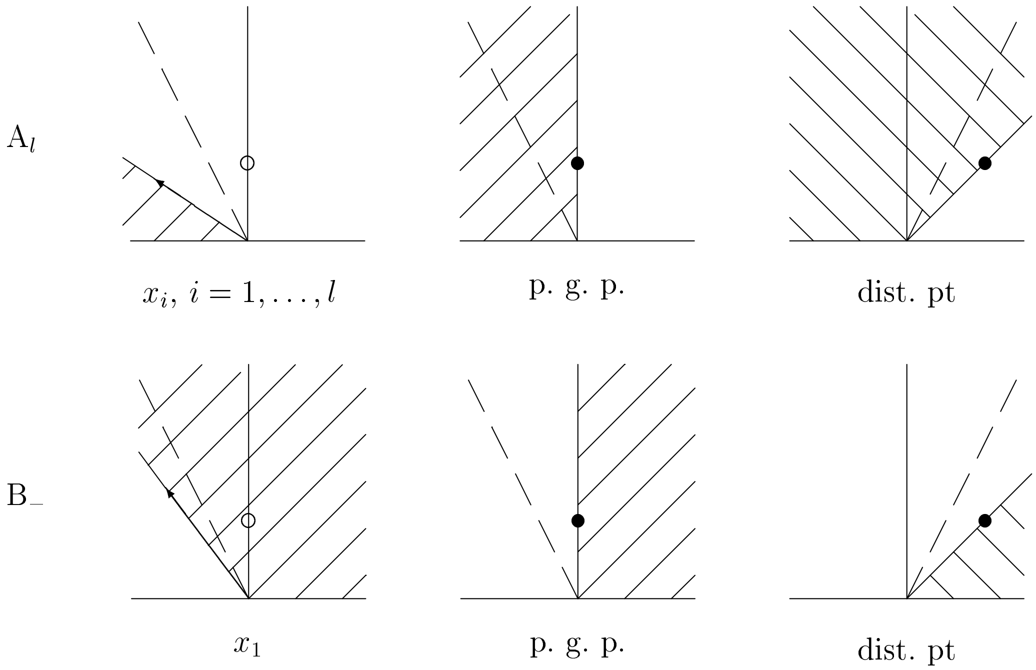
(elements of $\Gamma^* \times \mathbb{Z}_+ \subset \mathcal{Q}_{x,+}$ are marked by points).

Using the results of §3, it is easy to obtain a complete description for all possible kinds of G-germs in this case (the notation for various kinds of G-germs is taken from [LV, §9]).

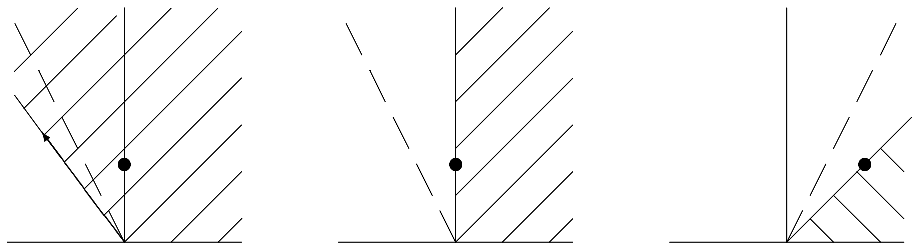
G-germs of type I are given by colored cones (shaded at the pictures; their colors are marked by black circles •; the point $x \in \mathbb{P}^1$ such that the colored cone is contained in $\mathcal{Q}_{x,+}$ may be assumed non-distinguished, due to a “coordinate change”—see remark 2.1):



G-germs of type II are given by colored hypercones of type II (shaded at the pictures; their colors are marked by black circles):



B_0



x_1 p. g. p. dist. pt

Besides, the total inclination of edges of $\mathcal{C}_x \subseteq \mathcal{Q}_{x,+}$ from the vertical line must have the same direction as the ray $\mathcal{K}$, i. e., $\sum \ell_x \in \mathcal{K} \setminus \{0\}$, where ℓ_x is the projection on $\mathcal{Q}$ of the intersection point of the respective edge in $\mathcal{C}_x$ with $\mathcal{Q}_{x,1}$. This means that: $\ell_{x_1} < -1$ for the type A_1; ℓ_{x_1} or $\ell_{x_2} < -\frac{1}{2}$ for the type A_2; $\ell_{x_1} > -1$ for the types B_- and B_0.

B-charts intersecting G-germs of type B_0 are G-stable (since they intersect all B-divisors) and they exhaust all affine G-models of K (affine SL_2-embeddings, see [Po]).)

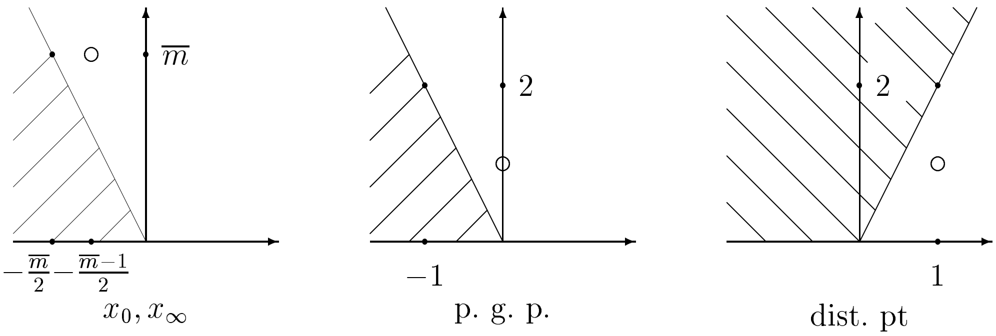
4.3. $G = SL_2$, $H = Z_m$ is a cyclic group. We use the notation of the preceding subsection. H is embedded in G as a subgroup of diagonal matrices generated by an element $h = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, where ε is a primitive m-th root of 1. $\mathfrak{X}(H)$ is isomorphic to Z_m and consists of characters $h \mapsto \varepsilon^k$ denoted simply by ε^k (multiplicative notation as against additive notation for elements of $\mathfrak{X}(B)$). Put $\overline{m} = \begin{cases} m, & m \text{ odd} \\ \frac{m}{2}, & m \text{ even} \end{cases}$.

Subregular semiinvariants are f_0, f_∞, with respective biweights (ω, ε), $(\omega, \varepsilon^{-1})$.

Regular semiinvariants fill the two-dimensional subspace $k[G]^{(B\times H)}_{(\overline{m}\omega, \varepsilon^{\overline{m}})} \ni f_0^{\overline{m}}, f_\infty^{\overline{m}}$.

$$\Gamma = \begin{cases} \mathbb{Z}\omega, & m \text{ is odd, } e_\omega = \dfrac{F}{(f_0 f_\infty)^{\frac{m-1}{2}}} \\ \mathbb{Z}\cdot 2\omega, & m \text{ is even, } e_{2\omega} = \dfrac{F^2}{(f_0 f_\infty)^{\frac{m}{2}-1}} \end{cases}.$$

The hyperspace looks like:

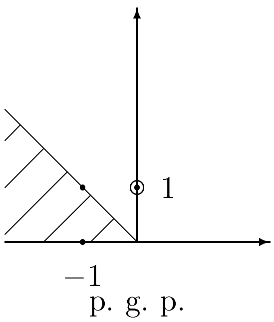


so that Γ^* is identified with $\mathbb{Z}$ whenever $2 \nmid m$, and with $\frac{1}{2}\mathbb{Z}$ whenever $2 \mid m$.

The colored (hyper)cones determining the germs are similar to 4.2 (and denoted similarly). There are the following restrictions: for the type A_1, $\ell_{x_1} < -\frac{1}{2} - \frac{1}{2\overline{m}}$ (if $x_1 = x_0$ or x_∞), or $\ell_{x_1} < -\frac{1}{\overline{m}}$ (otherwise); for the type A_2, ℓ_{x_1} or $\ell_{x_2} < -\frac{1}{2}$ (if $x_1 = x_0$, $x_2 = x_\infty$), or $\ell_{x_1} + \ell_{x_2} < -\frac{1}{2} - \frac{1}{2\overline{m}}$ (if one of $x_i = x_0$ or x_∞), or $\ell_{x_1} + \ell_{x_2} < -\frac{1}{\overline{m}}$ (if $x_i \neq x_0, x_\infty$); for the types B_- and B_0, $\ell_{x_1} > -\frac{1}{2} - \frac{1}{2\overline{m}}$ and $x_1 = x_0$ or x_∞ (except the case $\overline{m} = 1$, then x_1 is arbitrary).

In the next four examples, $G = SL_2$, H is a binary group of a regular polyhedron. When $k = \mathbb{C}$, $\mathbb{P}^1$ may be identified with the Riemann sphere S^2, $SU_2 \subset SL_2$ acting on it via the epimorphism $SU_2 \to SO_3$. Then H is the preimage of a symmetry group of a regular polyhedron inscribed in S^2. Its vertices, edge midpoints, and face centers (more precisely, their projections on S^2) form three orbits for $H : \mathbb{P}^1$ with non-trivial stabilizers. The respective points of $\mathbb{P}^1/H$ are denoted by x_{v}, x_{e}, x_{f}, and the respective products of linear forms on $\mathbb{P}^1$ (i. e., of elements from $k[G]^{(B)}_{\omega}$) corresponding to points of each orbit are denoted by f_{v}, f_{e}, f_{f}.

The semispace $\mathcal{Q}_{x,+}$ in a point $x \neq x_{\mathrm{v}}, x_{\mathrm{e}}, x_{\mathrm{f}}$ (a point of general position) has the same structure as for the subgroup $H = Z_m$, m is even. In other words, it looks like:



so that Γ^* is identified with $\mathbb{Z}$. We shall omit this picture below and depict only $\mathcal{Q}_{x,+}$ for $x = x_{\mathrm{v}}, x_{\mathrm{e}}, x_{\mathrm{f}}$.

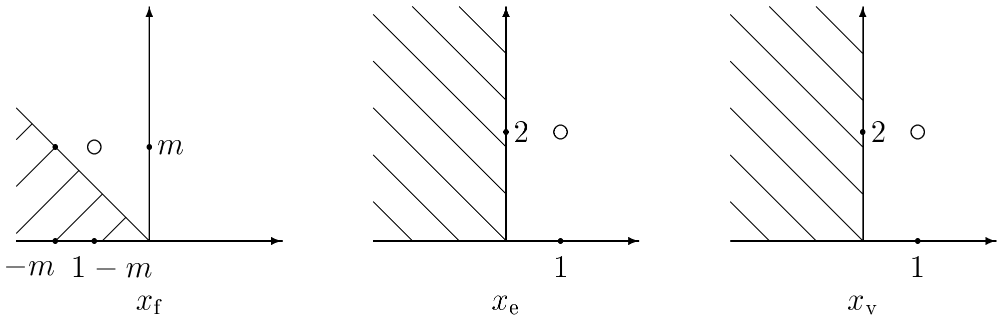
4.4. $G = SL_2$**,** $H = \widetilde{D}_m$ **is a binary dihedral group.** The group H is generated by elements $h = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, where ε is a primitive $2m$-th root of 1, and $r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A character from $\mathfrak{X}(H)$ is determined by a couple of values $h \mapsto \varepsilon^k$, $r \mapsto i^l$ (where $i = \sqrt{-1}$).

Subregular semiinvariants are $f_{\mathrm{f}} = f_0 f_\infty$, $f_{\mathrm{e}} = f_0^m - (-if_\infty)^m$, $f_{\mathrm{v}} = f_0^m + (-if_\infty)^m$, with respective biweights $(2\omega, (1,-1))$, $(m\omega, (-1, -i^m))$, $(m\omega, (-1, i^m))$.

Regular semiinvariants fill the two-dimensional subspace $k[G]^{(B\times H)}_{(2m\omega,(1,(-1)^m))} \ni f_{\mathrm{f}}^m, f_{\mathrm{e}}^2, f_{\mathrm{v}}^2,\ 4(-i)^m f_{\mathrm{f}}^m + f_{\mathrm{e}}^2 - f_{\mathrm{v}}^2 = 0$.

$\Gamma = \mathbb{Z}\cdot 2\omega$, $e_{2\omega} = \dfrac{f_{\mathrm{v}} f_{\mathrm{e}}}{f_{\mathrm{f}}^{m-1}}$.

The hyperspace looks like:



There are the following restrictions for the hypercones determining germs of type II: for the type A_1, $\ell_{x_1} < -1$ (if $x_1 = x_{\mathrm{f}}$), or $\ell_{x_1} < 1 - \frac{2}{m}$ (if $x_1 = x_{\mathrm{e}}$ or x_{v}); there are no types B_- and B_0.

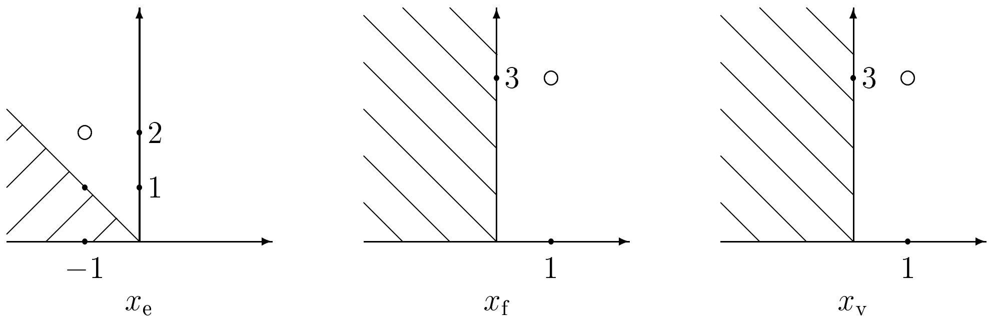
4.5. $G = SL_2$**,** $H = \widetilde{T}$ **is a binary tetrahedral group.** A character from $\mathfrak{X}(H)$ is determined by its value $h \mapsto \varepsilon^k$ (where ε is a primitive cubic root of 1) on an element $h \in \widetilde{T}$ of the 3-rd order rotating the tetrahedron by the angle $\frac{2\pi}{3}$ around the axis joining a given vertex and the center of the opposite face.

Subregular semiinvariants are f_{v}, f_{e}, f_{f}, with respective biweights $(4\omega, \varepsilon)$, $(6\omega, 1)$, $(4\omega, \varepsilon^{-1})$.

Regular semiinvariants fill the two-dimensional subspace $k[G]^{(B\times H)}_{(12\omega,1)} \ni f_{\mathrm{v}}^3, f_{\mathrm{e}}^2, f_{\mathrm{f}}^3,\ f_{\mathrm{v}}^3 + f_{\mathrm{e}}^2 + f_{\mathrm{f}}^3 = 0$.

$\Gamma = \mathbb{Z} \cdot 2\omega$, $e_{2\omega} = \dfrac{f_{\mathrm{v}} f_{\mathrm{f}}}{f_{\mathrm{e}}}$.

The hyperspace looks like:



There are no types B_- and B_0 and no other restrictions on hypercones.

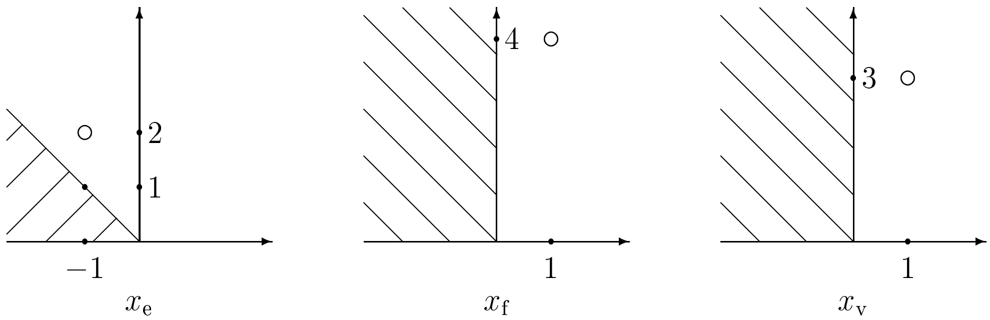
4.6. $G = SL_2$**,** $H = \widetilde{C}$ **is a binary cubic group.** A character from $\mathfrak{X}(H)$ is determined by its value $h \mapsto \pm 1$ on an element $h \in \widetilde{C}$ of the 4-th order rotating the cube by the angle π around the axis joining midpoints of two opposite edges.

Subregular semiinvariants are f_{v}, f_{e}, f_{f}, with respective biweights $(8\omega, 1)$, $(12\omega, -1)$, $(6\omega, -1)$.

Regular semiinvariants fill the two-dimensional subspace $k[G]^{(B\times H)}_{(24\omega,1)} \ni f_{\mathrm{v}}^3, f_{\mathrm{e}}^2, f_{\mathrm{f}}^4$, $f_{\mathrm{v}}^3 + f_{\mathrm{e}}^2 + f_{\mathrm{f}}^4 = 0$.

$\Gamma = \mathbb{Z} \cdot 2\omega$, $e_{2\omega} = \dfrac{f_{\mathrm{v}} f_{\mathrm{f}}}{f_{\mathrm{e}}}$.

The hyperspace looks like:



There are no types B_- and B_0 and no other restrictions on hypercones.

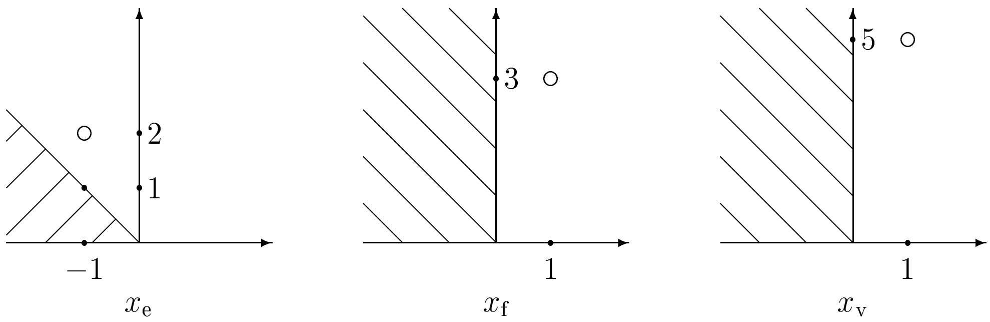
4.7. $G = SL_2$**,** $H = \widetilde{I}$ **is a binary icosahedral group.** $\mathfrak{X}(H) = \{1\}$.

Subregular semiinvariants are f_{v}, f_{e}, f_{f}, with respective biweights $(12\omega, 1)$, $(30\omega, 1)$, $(20\omega, 1)$.

Regular semiinvariants fill the two-dimensional subspace $k[G]^{(B\times H)}_{(60\omega,1)} \ni f_{\mathrm{v}}^5, f_{\mathrm{e}}^2, f_{\mathrm{f}}^3$, $f_{\mathrm{v}}^5 + f_{\mathrm{e}}^2 + f_{\mathrm{f}}^3 = 0$.

$\Gamma = \mathbb{Z}\cdot 2\omega$, $e_{2\omega} = \dfrac{f_{\mathrm{v}}f_{\mathrm{f}}}{f_{\mathrm{e}}}$.

The hyperspace looks like:



There are no types B_- and B_0 and no other restrictions on hypercones.

Remark 4.1. Cyclic and binary polyhedral groups exhaust, up to conjugacy, all finite subgroups $H \subset SL_2$ ($c(SL_2/H) = 1$ holds only for a finite subgroup). These examples were considered first in [M-J]. We also note that affine G-models exist only for cyclic H, in accordance with [Po].

Remark 4.2. From the above considerations, it is easy to derive a uniform description for polynomial invariants of finite subgroups $H \subset G = SL_2$ in its standard two-dimensional representation V, cf. [PV, 0.13]. We have an isomorphism of G-modules $V \cong \left(k[G]^{(B)}_{\omega}\right)^* \implies k[V] \cong k[G]^U$ (where G acts by right translations of an argument). Every (homogeneous) invariant $f \in k[V]^H$ decomposes uniquely in a product of indecomposable (homogeneous) H-semiinvariants. As we have seen, indecomposable homogeneous H-semiinvariants (i. e., indecomposable elements of $k[G]^{(B\times H)}$) are of two types: subregular (there are finitely many of them) and regular (filling a two-dimensional subspace, except a finite number of lines). Moreover, the subspace of regular semiinvariants is generated by powers of subregular semiinvariants. Hence a generating set for the submonoid of H-stable elements in the multiplicative monoid generated freely by subregular semiinvariants yields a generating set for the algebra $k[V]^H$ (in [PV], this fact is established a posteriori). Generally, this set is not minimal, but it is easy to choose a minimal subset. The latter always consists of 3 elements, hence, by $\operatorname{tr.deg} k[V]^H = 2$, there is just one relation, which is easily derived from the relation between subregular semiinvariants. In examples 4.3–4.7, subregular semiinvariants are denoted by f_β, where β is a letter or a digit. Denote $f_{\beta_1\dots\beta_s} = f_{\beta_1}\dots f_{\beta_s}$. Here are the precise answers:

H	Generators of $k[V]^H$ (their degrees)	Relation
Z_m	$f_{0^m},\ f_{\infty^m},\ f_{0\infty}$ $(\ m,\ \ m,\ \ 2\ )$	$f_{0^m}f_{\infty^m} - f_{0\infty}^m = 0$
$\widetilde{D}_m$ m odd	$f_{\mathrm{ff}},\ f_{\mathrm{ev}},\ f_{\mathrm{fee}}$ $(\ 4,\ 2m,\ 2m+2\ )$	$4(-i)^m f_{\mathrm{ff}}^{\frac{m+1}{2}} f_{\mathrm{fee}} + f_{\mathrm{fee}}^2 - f_{\mathrm{ff}}f_{\mathrm{ev}}^2 = 0$
$\widetilde{D}_m$ m even	$f_{\mathrm{ff}},\ f_{\mathrm{ee}},\ f_{\mathrm{fev}}$ $(\ 4,\ 2m,\ 2m+2\ )$	$4(-i)^m f_{\mathrm{ff}}^{\frac{m}{2}+1} f_{\mathrm{ee}} + f_{\mathrm{ff}}f_{\mathrm{ee}}^2 - f_{\mathrm{fev}}^2 = 0$
$\widetilde{T}$	$f_{\mathrm{vvv}},\ f_{\mathrm{e}},\ f_{\mathrm{vf}}$ $(\ 12,\ 6,\ 8\ )$	$f_{\mathrm{vvv}}^2 + f_{\mathrm{vvv}}f_{\mathrm{e}}^2 + f_{\mathrm{vf}}^3 = 0$
$\widetilde{C}$	$f_{\mathrm{v}},\ f_{\mathrm{ef}},\ f_{\mathrm{ff}}$ $(\ 8,\ 18,\ 12\ )$	$f_{\mathrm{v}}^3 f_{\mathrm{ff}} + f_{\mathrm{ef}}^2 + f_{\mathrm{ff}}^3 = 0$
$\widetilde{I}$	$f_{\mathrm{v}},\ f_{\mathrm{e}},\ f_{\mathrm{f}}$ $(\ 12,\ 30,\ 20\ )$	$f_{\mathrm{v}}^5 + f_{\mathrm{e}}^2 + f_{\mathrm{f}}^3 = 0$

In examples 4.8–4.14, we consider the case, where $G = SL_2 \times Z$ is a group of semisimple rank 1 (Z is a central torus). Respective G-varieties are "local models" for arbitrary varieties of complexity 1 with dense orbit in the same sense as toric varieties are "local models" for spherical ones [Bri].

Note that if G_1/H_1, G_2/H_2 are two homogeneous spaces with $c(G_1/H_1) = 1$, $c(G_2/H_2) = 0$, then the hyperspace for $G_1/H_1 \times G_2/H_2$ is obtained from the hyperspace for G_1/H_1 via direct product by $\mathcal{Q}_2 = \mathrm{Hom}(\Gamma(G_2/H_2), \mathbb{Q})$; the set of G-valuations is given by the direct product of the set $\mathcal{V}_1$ of G-valuations for $k(G_1/H_1)$ and the cone of G-valuations $\mathcal{V}_2 \subseteq \mathcal{Q}_2$; and the set of B-divisors is obtained by adding B-divisors of G_2/H_2 lying in $\mathcal{Q}_2$ to B-divisors of G_1/H_1. Thus it suffices to consider "indecomposable" pairs $H \subseteq G$.

What is the structure of a subgroup $H \subseteq G = G_1 \times G_2$? Denote by $\pi_i : G \to G_i$ the canonical projection. There are exact sequences

$$\begin{aligned} e \longrightarrow H_i = H \cap G_i \longrightarrow H \longrightarrow \pi_j(H) \longrightarrow e \\ e \longrightarrow H_1 \times H_2 \longrightarrow H \longrightarrow \overline{H} \cong \pi_j(H)/H_j \longrightarrow e \end{aligned} \qquad \{i,j\} = \{1,2\},$$

i. e., $G/H \cong (G_1/H_1 \times G_2/H_2)/\overline{H}$, where $\overline{H}$ acts diagonally.

In our case ($G_1 = SL_2$, $G_2 = Z$) we may assume $H_2 = \{e\} \implies H_1 \triangleleft H \cong \pi_1(H)$, $\overline{H} \cong H/H_1 \cong \pi_2(H)$ is abelian. Indecomposability means that $\overline{H}$ is not embedded in a proper subtorus of Z. We shall see that this yields $\dim Z \leqslant 2$. The case $\dim Z = 0$, $\overline{H} = \{e\}$ is considered above (in examples 4.2–4.7).

We shall denote the function field, the weight group, semiinvariants, points on $\mathbb{P}^1$, etc. for G/H by the same letters as the respective objects for $G/H_1 = SL_2/H_1 \times Z$, but overlined. Let ω be the fundamental weight of SL_2, $\omega_1, \omega_2, \ldots$ be basic weights of Z. For the triple $(H_1, H, \overline{H})$, there are the following possibilities.

4.8. $(Z_k, Z_m, Z_{m/k})$**.** Subregular $(B\times H)$-semiinvariants are $\overline{f}_0 = f_0$, $\overline{f}_\infty = f_\infty$, with respective biweights (ω, ε), $(\omega, \varepsilon^{-1})$ (where ε is a primitive m-th root of 1). Put $\Phi = f_0^{\overline{k}}$.

Generators of $e(\Gamma)$ are: $e_\omega = \dfrac{\Phi}{(f_0 f_\infty)^{\frac{k-1}{2}}}$ (or $e_{2\omega} = \dfrac{\Phi^2}{(f_0 f_\infty)^{\frac{k}{2}-1}}$), e_{ω_1}, with respective H-weights ε^k, ε^{-k}.

Generators of $\overline{e}(\overline{\Gamma})$ are:

$$\overline{e}_\omega = e_\omega \times \frac{f_\infty^{\overline{k}\frac{m/k+1}{2}}}{f_0^{\overline{k}\frac{m/k+1}{2}}},\ \overline{e}_{\omega_1} = e_{\omega_1} \times \frac{f_0^{\overline{k}\frac{m/k+1}{2}}}{f_\infty^{\overline{k}\frac{m/k+1}{2}}}, \quad \text{whenever } 2 \nmid m;$$

$$\overline{e}_{2\omega} = e_{2\omega} \times \frac{f_\infty^{\overline{k}}}{f_0^{\overline{k}}},\ \overline{e}_{\omega+\omega_1} = e_{\omega+\omega_1},\ \overline{e}_{2\omega_1} = e_{2\omega_1} \times \frac{f_0^{\overline{k}}}{f_\infty^{\overline{k}}}, \quad \text{when } 2 \nmid k,\ 2 \mid m;$$

$$\overline{e}_{2\omega} = e_{2\omega} \times \frac{f_\infty^{\overline{k}}}{f_0^{\overline{k}}},\ \overline{e}_{\omega_1} = e_{\omega_1} \times \frac{f_0^{\overline{k}}}{f_\infty^{\overline{k}}}, \quad \text{whenever } 2 \mid k.$$

The hyperspace looks like:

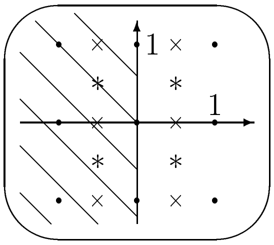
$\delta = 0$

Points of $\overline{\Gamma}^*$:

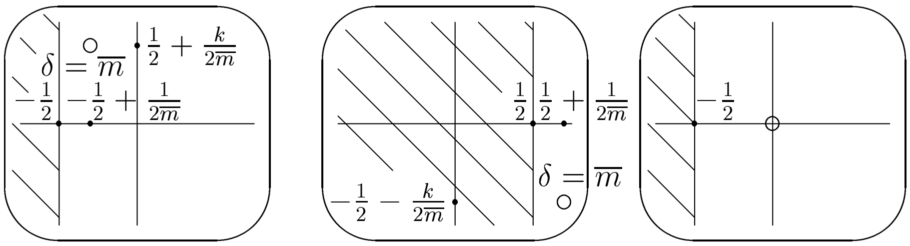
$\cdot$ for $2 \nmid m$

$\cdot, *$ for $2 \nmid k$, $2 \mid m$

$\cdot, \times$ for $2 \mid k$

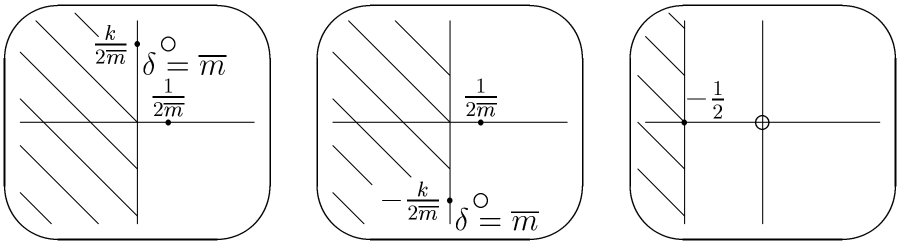


$\delta = 1$
$2 \nmid m$



x_0 $\qquad$ x_∞ $\qquad$ p. g. p.

$\delta = 1$
$2 \mid m$



x_0 $\qquad$ x_∞ $\qquad$ p. g. p.

4.9. $(Z_m, \widetilde{D}_m, Z_4)$**,** m **odd.** Subregular $(B \times H)$-semiinvariants are $\overline{f}_{\mathrm{f}} = f_0 f_\infty$, $\overline{f}_{\mathrm{e}} = f_0^m - (-if_\infty)^m$, $\overline{f}_{\mathrm{v}} = f_0^m + (-if_\infty)^m$, with respective biweights $(2\omega, (1,-1))$, $(m\omega, (-1, -i^m))$, $(m\omega, (-1, i^m))$.

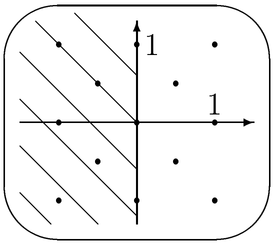
Generators of $e(\Gamma)$ are: $e_\omega = \dfrac{f_0^m + (-if_\infty)^m}{(f_0 f_\infty)^{\frac{m-1}{2}}}$, e_{ω_1}, with respective H-weights $(-1, i)$, $(-1, -i)$.

Generators of $\overline{e}(\overline{\Gamma})$ are: $\overline{e}_{2\omega} = e_{2\omega} \times \dfrac{f_0^m - (-if_\infty)^m}{f_0^m + (-if_\infty)^m}$, $\overline{e}_{\omega+\omega_1} = e_{\omega+\omega_1}$, $\overline{e}_{2\omega_1} = e_{2\omega_1} \times \dfrac{f_0^m + (-if_\infty)^m}{f_0^m - (-if_\infty)^m}$.

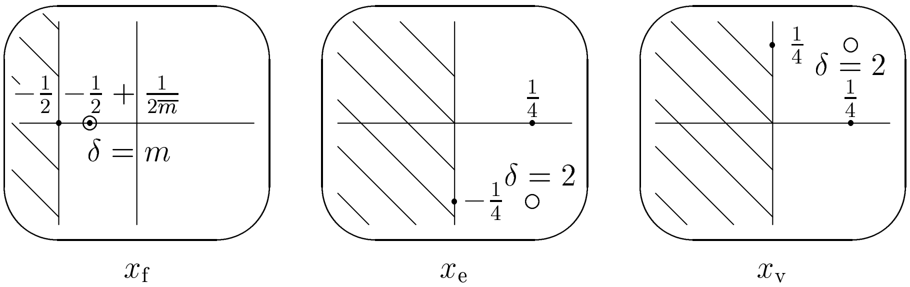
The hyperspace looks like:

$\delta = 0$

$\cdot$ are points of $\overline{\Gamma}^*$



$\delta = 1$



x_{f} $\qquad$ x_{e} $\qquad$ x_{v}

(the hyperplane $\mathcal{Q}_{x,1}$ over a point of general position has the same structure as in example 4.8, the picture is omitted).

4.10. $(Z_m, \widetilde{D}_m, V_4)$**,** m **even.** Subregular $(B \times H)$-semiinvariants are same as in 4.9.

Generators of $e(\Gamma)$ are: $e_{2\omega} = f_0 f_\infty$, e_{ω_1}, e_{ω_2}, with respective H-weights $(1,-1)$, $(1,-1)$, $(-1,1)$.

Generators of $\overline{e}(\overline{\Gamma})$ are:

$$\overline{e}_{2\omega} = e_{2\omega} \times \frac{(f_0^m - (-if_\infty)^m)(f_0^m + (-if_\infty)^m)}{f_0^m f_\infty^m},$$

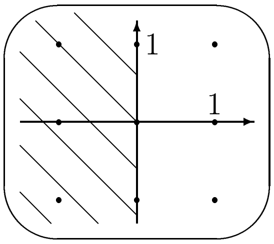
$$\overline{e}_{\omega_1} = e_{\omega_1} \times \frac{(f_0^m - (-if_\infty)^m)(f_0^m + (-if_\infty)^m)}{f_0^m f_\infty^m},$$

$$\overline{e}_{\omega_2} = e_{\omega_2} \times \frac{(f_0^{m/2} - i(-if_\infty)^{m/2})(f_0^{m/2} + i(-if_\infty)^{m/2})}{f_0^{m/2} f_\infty^{m/2}}.$$

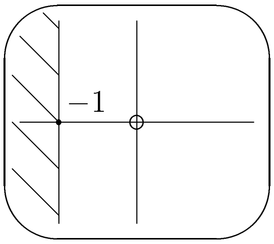
The hyperspace looks like:

$\delta = 0$

$\cdot$ are points of $\overline{\Gamma}^*$

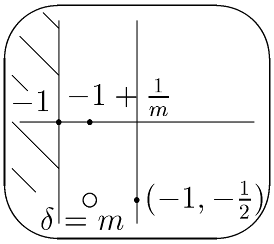


$\delta = 1$

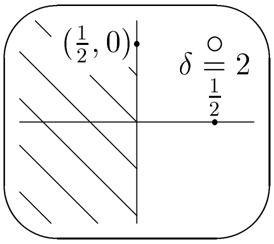


p. g. p.

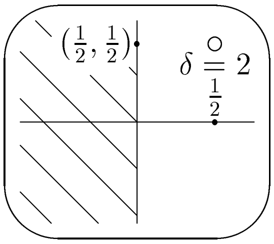
$\delta = 1$



x_{f}



x_{e}



x_{v}

An explanation is needed: in this (and only this) example, we have $\dim Z = 2$, hence the pictures are actually not two-dimensional but *three-dimensional*: the vertical axis is responsible for two coordinate functions ω_1, ω_2. In other cases from our series, hyperplanes $\mathcal{Q}$ and $\mathcal{Q}_{x,1}$ over points of general position have the same structure as in this example (but now they are two-dimensional), the pictures are omitted.

4.11. $(Z_{2m}, \widetilde{D}_m, Z_2)$**.** Subregular $(B \times H)$-semiinvariants are same as in 4.9.

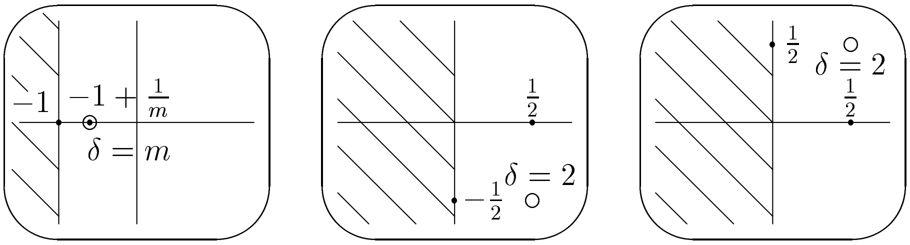
Generators of $e(\Gamma)$ are: $e_{2\omega} = f_0 f_\infty$, e_{ω_1}, with respective H-weights $(1,-1)$, $(1,-1)$.

Generators of $\overline{e}(\overline{\Gamma})$ are:

$$\overline{e}_{2\omega} = e_{2\omega} \times \frac{(f_0^m - (-if_\infty)^m)(f_0^m + (-if_\infty)^m)}{f_0^m f_\infty^m}, \quad \overline{e}_{\omega_1} = e_{\omega_1} \times \frac{f_0^m + (-if_\infty)^m}{f_0^m - (-if_\infty)^m}.$$

The hyperspace looks like:

$\delta = 1$



x_{f} x_{e} x_{v}

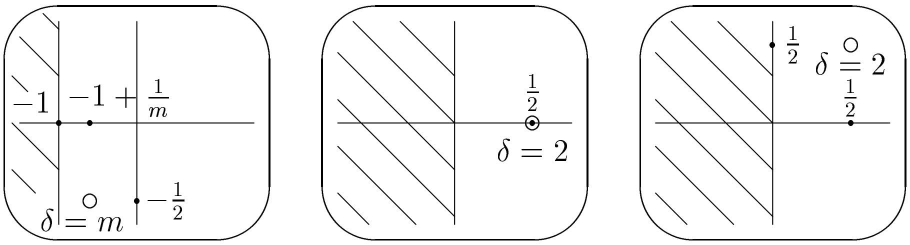
4.12. $(\widetilde{D}_{m/2}, \widetilde{D}_m, Z_2)$**,** m **even.** Subregular $(B \times H)$-semiinvariants are $\overline{f}_{\mathrm{f}} = f_{\mathrm{f}}$, $\overline{f}_{\mathrm{e}} = f_{\mathrm{e}} f_{\mathrm{v}}$, $\overline{f}_{\mathrm{v}} = f_{\mathrm{e}}^2 + f_{\mathrm{v}}^2$, with respective biweights $(2\omega, (1, -1))$, $(m\omega, (-1, -i^m))$, $(m\omega, (-1, i^m))$.

Generators of $e(\Gamma)$ are: $e_{2\omega} = \dfrac{f_{\mathrm{v}} f_{\mathrm{e}}}{f_{\mathrm{f}}^{m/2-1}}$, e_{ω_1}, with respective H-weights $(1, -1)$, $(1, -1)$.

Generators of $\overline{e}(\overline{\Gamma})$ are: $\overline{e}_{2\omega} = e_{2\omega} \times \dfrac{f_{\mathrm{e}}^2 + f_{\mathrm{v}}^2}{f_{\mathrm{f}}^{m/2}}$, $\overline{e}_{\omega_1} = e_{\omega_1} \times \dfrac{f_{\mathrm{e}}^2 + f_{\mathrm{v}}^2}{f_{\mathrm{f}}^{m/2}}$.

The hyperspace looks like:

$\delta = 1$



x_{f} x_{e} x_{v}

4.13. $(\widetilde{D}_2, \widetilde{T}, Z_3)$**.** Subregular $(B \times H)$-semiinvariants are $\overline{f}_{\mathrm{v}} = f_{\mathrm{f}}^2 + \varepsilon f_{\mathrm{v}}^2 + \varepsilon^2 f_{\mathrm{e}}^2$, $\overline{f}_{\mathrm{f}} = f_{\mathrm{f}}^2 + \varepsilon^2 f_{\mathrm{v}}^2 + \varepsilon f_{\mathrm{e}}^2$, $\overline{f}_{\mathrm{e}} = f_{\mathrm{f}} f_{\mathrm{e}} f_{\mathrm{v}}$ (where ε is a primitive cubic root of 1), with respective biweights $(4\omega, \varepsilon)$, $(4\omega, \varepsilon^{-1})$, $(6\omega, 1)$.

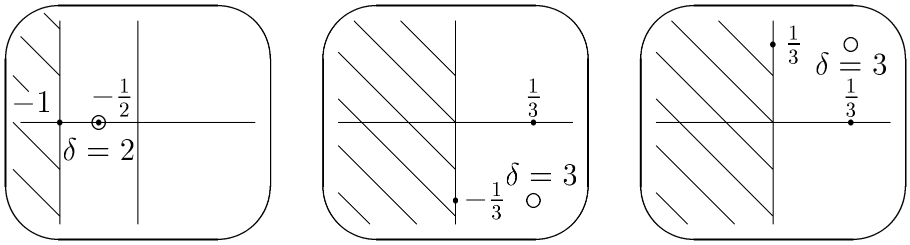
Generators of $e(\Gamma)$ are: $e_{2\omega} = \dfrac{f_{\mathrm{v}} f_{\mathrm{e}}}{f_{\mathrm{f}}}$ (not H-semiinvariant), e_{ω_1} (has the H-weight ε).

Generators of $\overline{e}(\overline{\Gamma})$ are:

$$\overline{e}_{2\omega} = e_{2\omega} \times \frac{(f_{\rm f}^2 + \varepsilon f_{\rm v}^2 + \varepsilon^2 f_{\rm e}^2)(f_{\rm f}^2 + \varepsilon^2 f_{\rm v}^2 + \varepsilon f_{\rm e}^2)}{f_{\rm v}^2 f_{\rm e}^2}, \quad \overline{e}_{\omega_1} = e_{\omega_1} \times \frac{f_{\rm f}^2 + \varepsilon f_{\rm v}^2 + \varepsilon^2 f_{\rm e}^2}{f_{\rm f}^2 + \varepsilon^2 f_{\rm v}^2 + \varepsilon f_{\rm e}^2}.$$

The hyperspace looks like:

$\delta = 1$



$x_{\rm e}$ $x_{\rm f}$ $x_{\rm v}$

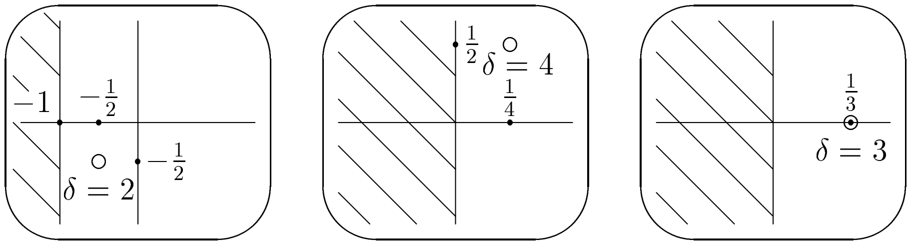
4.14. $(\widetilde{T}, \widetilde{C}, Z_2)$**.** Subregular $(B \times H)$-semiinvariants are $\overline{f}_{\rm v} = f_{\rm v} f_{\rm f}$, $\overline{f}_{\rm e} = f_{\rm v}^3 - f_{\rm f}^3$, $\overline{f}_{\rm f} = f_{\rm e}$, with respective biweights $(8\omega, 1)$, $(12\omega, -1)$, $(6\omega, -1)$.

Generators of $e(\Gamma)$ are: $e_{2\omega} = \dfrac{f_{\rm v} f_{\rm f}}{f_{\rm e}}$, e_{ω_1}, with respective H-weights -1, -1.

Generators of $\overline{e}(\overline{\Gamma})$ are: $\overline{e}_{2\omega} = e_{2\omega} \times \dfrac{f_{\rm e}^2}{f_{\rm v}^3 - f_{\rm f}^3}$, $\overline{e}_{\omega_1} = e_{\omega_1} \times \dfrac{f_{\rm e}^2}{f_{\rm v}^3 - f_{\rm f}^3}$.

The hyperspace looks like:

$\delta = 1$



$x_{\rm e}$ $x_{\rm f}$ $x_{\rm v}$

4.15. $G = SL_3$**,** $H = T$ **is a maximal torus.** The group G consists of matrices

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \qquad \det g = 1,$$

B consists of upper-triangular matrices, T of diagonal matrices. The group $\mathfrak{X}(B)$ is generated by the fundamental weights $\omega_1(g) = g_{11}$, $\omega_2(g) = g_{11} g_{22}$,

$\forall g \in B$. $\mathfrak{X}(H)$ is generated by the weights $\varepsilon_i(g) = g_{ii}$, $\forall g \in H$, $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 0$. Denote by $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$ simple roots, and let $\rho = \omega_1 + \omega_2 = \alpha_1 + \alpha_2$.

Subregular semiinvariants are $f_i(g) = g_{3i}$, $\tilde{f}_i(g) = \begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{vmatrix}$, where (i, j, k) is any cyclic permutation of $(1, 2, 3)$, with respective biweights $(\omega_2, \varepsilon_i)$, $(\omega_1, -\varepsilon_i)$.

Regular semiinvariants fill the two-dimensional subspace $k[G]^{(B\times H)}_{(\rho,0)} \ni f_i\tilde{f}_i,\ f_1\tilde{f}_1 + f_2\tilde{f}_2 + f_3\tilde{f}_3 = 0$.

Γ coincides with the root lattice, $e_{\alpha_1} = \dfrac{\tilde{f}_1\tilde{f}_2\tilde{f}_3}{F}$, $e_{\alpha_2} = \dfrac{f_1 f_2 f_3}{F}$.

Any G-valuation corresponding to a divisor with dense G-orbit is proportional to $v_{x(t)}$, where

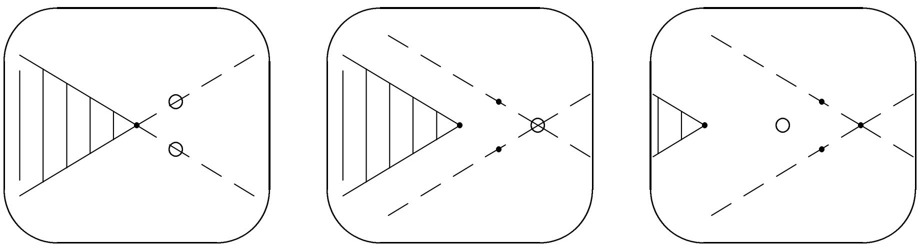
$$x(t) = \begin{pmatrix} 1 & t^p & u(t) \\ 0 & 1 & t^q \\ 0 & 0 & 1 \end{pmatrix},$$

and we may assume p, q, $d = v_t(u(t)) \leqslant 0$.

Computing values $v_{x(t)}(\alpha_1 f_1\tilde{f}_1 + \alpha_2 f_2\tilde{f}_2 + \alpha_3 f_3\tilde{f}_3)$ as in 4.2, we see that the set of G-valuations having the jump at x and corresponding to divisors with dense G-orbit is determined by inequalities $a_1, a_2 \leqslant 0 \leqslant \delta$ (if $x = x_i$ is the point corresponding to $f_i\tilde{f}_i$), or $a_1, a_2 \leqslant -2\delta \leqslant 0$ (if x is the distinguished point), or $a_1, a_2 \leqslant -\delta \leqslant 0$ (if x is a point of general position), and $\delta = 0 \implies a_1$ or $a_2 = 0$, where a_i are the coordinates of the central component w. r. t. the base $\omega_i^\vee \in \Gamma^*$ dual to α_i. Hence the cone $\mathcal{V}_x$ is determined by the same inequalities without any restrictions for $\delta = 0$.

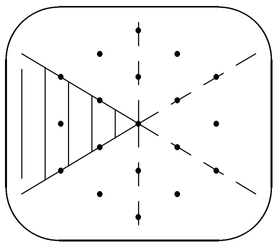
The hyperspace looks like:

$\delta = 1$



x_i, $i = 1, 2, 3$ p. g. p. dist. pt

$\delta = 0$



Looking at the picture, one can see, e. g., that there are no G-stable B-charts except one having $\mathcal{W} = \varnothing$, and the respective hypercone intersects $\mathcal{V}$ in $\{0\}$. This means that G/H cannot be embedded in an affine G-variety as a non-closed orbit (a well-known fact).

Remark 4.3. The set of invariant valuations for SL_3/T was computed in [Wa] (in somewhat different terms).

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