EQUIVARIANT EMBEDDINGS OF HOMOGENEOUS SPACES

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ABSTRACT. Homogeneous spaces of algebraic groups naturally arise in various problems of geometry and representation theory. The same reasons that motivate considering projective spaces instead of affine spaces (e.g. solutions "at infinity" of systems of algebraic equations) stimulate the study of compactifications or, more generally, equivariant embeddings of homogeneous spaces.

The embedding theory of a homogeneous space is governed by a certain numerical invariant called complexity. We discuss the geometric and representation-theoretic meaning and methods to compute this invariant.

Homogeneous spaces of complexity zero are called spherical. They can be characterized by a number of remarkable equivalent conditions and have an elegant and well controlled theory of equivariant embeddings. We consider various applications of this theory. As a particular case, we study equivariant embeddings of reductive groups.

The embedding theory of spherical spaces is deduced from general results of Luna and Vust on embeddings of arbitrary homogeneous spaces and can be generalized to homogeneous spaces of the "next level of complexity"—complexity one.

INTRODUCTION

Homogeneous spaces of algebraic groups play an important rôle in various aspects of geometry and representation theory. We restrict our attention to linear, and even reductive, algebraic groups, because the most interesting interplay between geometric and representation-theoretic aspects occurs for this class of algebraic groups.

Classical examples of algebraic homogeneous spaces:

(1) The affine space $\mathbb{A}^n$ is homogeneous under GA_n, the general affine group;
(2) The projective space $\mathbb{P}^n$ is homogeneous under GL_{n+1};
(3) The sphere $S^{n-1} = \mathrm{SO}_n/\mathrm{SO}_{n-1}$;
(4) Grassmannians $\mathrm{Gr}_k(\mathbb{P}^n)$ $(k \le n)$ and flag varieties are homogeneous under GL_{n+1}.
(5) The space of non-degenerate quadrics $Q_n = \mathrm{PGL}_{n+1}/\mathrm{PO}_{n+1}$;
(6) The space $\mathrm{Mat}^{(r)}_{m,n}$ of $(m \times n)$-matrices of rank r is homogeneous under $\mathrm{GL}_m \times \mathrm{GL}_n$.

The relations of algebraic homogeneous spaces to representation theory have their origin in the Borel–Weil theorem, realizing all simple modules of reductive groups as spaces of sections of line bundles on (generalized)

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flag varieties. This geometric approach to representation theory by realizing representations in spaces of sections of line bundles on homogeneous spaces (or on their embeddings) is rather fruitful, and it also raises an interesting problem of describing higher cohomology groups of line bundles (generalizing the Borel–Weil–Bott theorem).

As another motivation for studying embeddings of homogeneous spaces, consider enumerative geometry.

A classical enumerative problem: How many plane conic curves are tangent to five given conics in general position?

The natural approach is to compactify Q_2 by degenerate conics, and consider the compact embedding space $\mathbb{P}^5$, where each tangency condition determines a hypersurface of degree 6. However, the answer $6^5 = 7776$ suggested by the Bézout theorem is wrong! The reason is that these hypersurfaces do not intersect the boundary of Q_2 properly: each of them contains all double lines.

Approach of Halphen–De Concini–Procesi. More generally, consider a number of closed subvarieties $Z_1, \dots, Z_s$ of a homogeneous space G/H (typically, varieties of geometric objects satisfying certain conditions) such that $\sum \operatorname{codim} Z_i = \dim G/H$. If Z_i are in sufficiently general position w.r.t. each other, then it is natural to expect that $Z_1 \cap \dots \cap Z_s$ is finite. By Kleiman's transversality theorem [Har, Thm. III.10.8], the translates $g_i Z_i$ are in general position for generic $g_1, \dots, g_s \in G$, and the number $(Z_1, \dots, Z_s) = |g_1 Z_1 \cap \dots \cap g_s Z_s|$, called the *intersection number*, does not depend on the g_i.

To compute the intersection number, one tries to embed G/H as an open orbit in a compact G-variety X with finitely many orbits, so that $\operatorname{codim}_Y(\overline{Z_i} \cap Y) = \operatorname{codim} Z_i$ for any G-orbit $Y \subseteq X$. If such an X exists, then $g_1\overline{Z_1} \cap \dots \cap g_s\overline{Z_s} \subset G/H$ for generic g_i, whence $(Z_1, \dots, Z_s) = [\overline{Z_1}] \cdots [\overline{Z_s}]$, the product in $H^*(X)$. It is now clear that in order to solve enumerative problems on homogeneous spaces, one needs to have a good control on their compactifications or, more generally, equivariant embeddings.

The geometry of embeddings of a homogeneous space G/H under a reductive group G is governed by its complexity, which is the codimension of generic orbits of a Borel subgroup $B \subseteq G$. The complexity has also a representation-theoretic meaning: it characterizes the growth of multiplicities of simple G-modules in the spaces of sections of line bundles on G/H, see 1.5. Another important numerical invariant is the rank of a homogeneous space. Complexity and rank are discussed in Section 1.

A method for computing complexity and rank was developed by Knop and Panyushev. It involves equivariant symplectic geometry of the cotangent bundle $T^*(G/H)$ and gives formulæ for these numbers in terms of the coisotropy representation, see 1.3. Panyushev showed that the computations can be reduced to representations of reductive groups, see 1.4. Other contribution of Panyushev are formulæ for complexity and rank of double flag varieties, which are considered in 1.6. Double flag varieties arise in the problem of decomposing tensor products of simple G-modules, cf. 3.6.

There are two distinct approaches to embedding theory of homogeneous spaces. The first one is based on explicit constructions of embeddings in ambient spaces (determinantal varieties, complete quadrics, wonderful compactifications of de Concini–Procesi, projective compactifications of reductive groups, see 3.4, etc.) In Section 2, we discuss the second, intrinsic, approach to equivariant embeddings of arbitrary homogeneous spaces, due to Luna, Vust, and the author. An important rôle in the local description of embeddings is played by B-stable divisors and respective discrete valuations of $\mathbb{C}(G/H)$. However, the Luna–Vust theory provides a complete and transparent description of equivariant embeddings only for homogeneous spaces of complexity ≤ 1.

Homogeneous spaces of complexity 0 are called spherical. They are characterized by a number of particularly nice properties (Theorem 18). Many classical homogeneous varieties are in fact spherical: for instance, all above examples, except the first one, are spherical. Normal embeddings of spherical homogeneous spaces are called spherical varieties. For spherical varieties, the Luna–Vust theory provides an elegant description in terms of certain objects of combinatorial convex geometry (coloured cones and fans). The well-known theory of toric varieties is in fact a particular case. We study spherical varieties in Section 3.

The group G itself may be considered as a spherical homogeneous space $(G \times G)/\operatorname{diag} G$. We study its embeddings in 3.3–3.4. As an application, we obtain a classification of reductive algebraic semigroups due to Vinberg and Rittatore. We also study natural projective compactifications of G obtained by closing the image of G in the space of operators of a projective representation of G.

Divisors and line bundles on spherical varieties are discussed in 3.5. Following Brion, we describe the Picard group of a spherical variety and give criteria for a divisor to be Cartier, base point free, or ample. We also describe the G-module structure for the space of sections of a line bundle on a spherical variety in terms of lattice points of certain polytopes.

An interesting application of the divisor theory on spherical varieties is a geometric way to decompose certain tensor products of simple G-modules considered in 3.6. The idea is to view these simple modules as spaces of sections of line bundles on flag varieties. Then their tensor product is the space of sections of a line bundle on a double flag variety, cf. 1.6, and the above description of its G-module structure enters the game.

Another application is a formula for the degree of an ample divisor on a projective spherical variety, which leads to an intersection theory of divisors and to the "Bézout theorem" on spherical homogeneous spaces, see 3.7.

Finally, we discuss the embedding theory for homogeneous spaces of complexity 1, due to the author. It is developed from the general Luna–Vust theory in a way parallel to the spherical case. However, the description of embeddings is more complicated. We try to emphasize the common features and the distinctions from the spherical case.

The aim of this survey is to introduce a reader to equivariant embeddings of homogeneous spaces under reductive groups, and to show how this subject links together algebra, geometry, and representation theory. There are several excellent monographs and surveys devoted to some of the topics discussed in these notes, see e.g. [Kn2], [Bri5] for spherical varieties, and [Pan5] for complexity and rank. However, in this paper we hope to gather some useful results, which are scattered in the literature and never appeared in survey papers before, paying special attention to practical computation of important invariants of homogeneous spaces and to the general embedding theory.

For its introductory character, this survey does not cover all topics in this area, and some results are not considered in full generality, as well as the list of references is by no means complete. Also we tried to avoid long and complicated proofs, so that *Proof* in the text often means rather *Sketch of a proof*, or even *Hints to a proof.*

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Notation and terminology. All algebraic varieties and groups are considered over the base field $\mathbb{C}$ of complex numbers. Lowercase gothic letters always denote Lie algebras of respective "uppercase" algebraic groups.

The unipotent radical of an algebraic group H is denoted by U_H. The centralizer in H or $\mathfrak{h}$ of an element or subset of H or $\mathfrak{h}$ is denoted by $Z(\cdot)$ or $\mathfrak{z}(\cdot)$, respectively. The *character group* $\Lambda(H)$ consists of homomorphisms $\chi : H \to \mathbb{C}^\times$ and is written additively. It is a finitely generated abelian group, and even a lattice if H is connected. Any action of H on a set M is denoted by $H : M$, and M^H is the set of H-fixed points. If $H : M$ is a linear representation, then $M^{(H)}_\chi$ denotes the set of H-eigenvectors of eigenweight $\chi \in \Lambda(H)$.

Throughout the paper, G is a connected reductive group. We often fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. $U \subseteq B$ is the maximal unipotent subgroup, and B^- is the *opposite* Borel subgroup (i.e., such that $B^- \cap B = T$), with the maximal unipotent radical U^-. Denote by V_λ the simple G-module of B-dominant highest weight λ. If G is semisimple simply connected, then the character lattice $\Lambda(B) = \Lambda(T)$ is generated by the *fundamental weights* ω_i, $i = 1, \dots, \operatorname{rk} G$, dual to the simple coroots w.r.t. B, and the dominant weights are the positive linear combinations of the ω_i.

$\mathbb{C}[X]$ is the coordinate algebra of a quasiaffine variety X, and $\mathbb{C}(X)$ is the field of rational functions on any variety X. The line bundle associated with a Cartier divisor δ on X is denoted by $\mathcal{O}(\delta)$. The divisor of a rational section s of $\mathcal{O}(\delta)$ is denoted by $\operatorname{div}_X s$, and s_δ is the canonical rational section with $\operatorname{div}_X s_\delta = \delta$.

An *H-line bundle* on an H-variety X is a line bundle equipped with a fiberwise linear H-action compatible with the projection onto the base. If X is normal and H is connected, then any line bundle on X can be $\widetilde{H}$-linearized for some finite cover $\widetilde{H} \to H$ [KKLV]. Hence a sufficiently big power of any line bundle can be H-linearized.

If $H \subset G$ is a closed subgroup, then $G \times^H X$ denotes the *homogeneous fibration* over G/H with fiber X, i.e., the quotient variety $(G \times X)/H$ modulo the action $h(g, x) = (gh^{-1}, hx)$, $\forall g \in G$, $h \in H$, $x \in X$. The image of (g, x) in $G \times^H X$ is denoted by $g * x$.

We shall frequently speak of *generic* points (or orbits) in X assuming thereby that we consider points (orbits) from a certain (sufficiently small for our purposes) dense open subset of X.

We use the notation $\operatorname{conv} \mathcal{C}$, $\operatorname{int} \mathcal{C}$ for the convex hull and the relative interior of a subset $\mathcal{C}$ in a vector spece E over $\mathbb{Q}$ or $\mathbb{R}$. If $\mathcal{C} \subseteq E$ is a convex polyhedral cone, then $\mathcal{C}^\vee \subseteq E^*$ denotes the dual cone.

Our general references are: [Har] for algebraic geometry, [Hum], [Jan] for linear algebraic groups, and [Kr], [PV] for algebraic transformation groups and Invariant Theory.

1. Complexity and rank

There are two numerical invariants of a homogeneous space G/H, which proved their importance in its embedding theory as well as in other geometric, representation-theoretic and invariant-theoretic problems on G/H. Roughly speaking, the first one, the *complexity*, says whether the geometry and embedding theory of G/H can be well controlled. The second invariant, the *rank* (or more subtly, the weight lattice) of G/H provides an environment for certain combinatorial objects used in the description of equivariant embeddings and in the representation theory related to G/H.

Actually, these invariants can be defined for an arbitrary G-variety.

Definition 1. Let X be an (irreducible) algebraic variety equipped with a G-action. The *complexity* $c(X)$ is by definition the codimension of a generic B-orbit in X, or equivalently, $\operatorname{tr.deg} \mathbb{C}(X)^B$.

If we denote by $d_H(X)$ the *generic modality* of X under an action of an algebraic group H, i.e., the codimension of generic H-orbits, then $c(X) = d_B(X)$.

The set of all weights of rational B-eigenfunctions on X forms the *weight lattice* $\Lambda(X) \subseteq \Lambda(B)$. The *rank* of X is $r(X) = \operatorname{rk} \Lambda(X)$.

If X is quasiaffine, then we have the isotypic decomposition of its coordinate algebra $\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_+(X)} \mathbb{C}[X]_{(\lambda)}$, where $\mathbb{C}[X]_{(\lambda)}$ is the sum of all simple G-submodules of highest weight λ (w.r.t. B), and $\Lambda_+(X) = \{\lambda \mid \mathbb{C}[X]_{(\lambda)} \neq 0\}$ is the *weight semigroup* of X. Every rational B-eigenfunction can be represented as a ratio of two regular B-eigenfunctions, whence $\Lambda_+(X)$ generates $\Lambda(X)$.

1.1. **Local structure.** The complexity, rank, and weight lattice are visible in terms of the "local structure" of the action $G : X$ described by Brion, Luna, and Vust [BLV].

We start with the following simple situation. Let $G : V$ be a rational finite-dimensional representation, $v \in V$ a lowest weight vector, and $v^* \in V^*$ a highest weight vector such that $\langle v, v^* \rangle \neq 0$. Let $P \supseteq B$ be the projective stabilizer of v^* with a Levi decomposition $P = L \cdot U_P$, so that the opposite parabolic subgroup $P^- = L \cdot U_P^-$ is the projective stabilizer of v. Put $\mathring{V} = V \setminus \langle v^* \rangle^\perp$, $W = (\mathfrak{u}_P^- v^*)^\perp$, and $\mathring{W} = W \cap \mathring{V}$. (Here $^\perp$ denotes the annihilator in the dual space.)

Theorem 1 ([BLV]). *There is a natural P-equivariant isomorphism $\mathring{V} \simeq U_P \times \mathring{W} \simeq P \times^L \mathring{W}$.*

Proof. First note that $V = \mathfrak{u}_P v \oplus W$. Indeed, by dimension count it suffices to prove $\mathfrak{u}_P v \cap W = 0$. Otherwise there would exist a root vector $e_\alpha \in \mathfrak{u}_P$ such that $e_\alpha v \in W$, in particular, $\langle e_\alpha v, e_{-\alpha} v^* \rangle = \langle [e_\alpha, e_{-\alpha}] v, v^* \rangle = 0$, hence $[e_\alpha, e_{-\alpha}] v = 0$ and α is a root of L, a contradiction.

Also note that $W = \langle v \rangle \oplus W_0$, where $W_0 = (\mathfrak{g} v^*)^\perp$. The hyperplanes $V_c = \{x \in V \mid \langle x, v^* \rangle = c\} = \mathfrak{u}_P v + cv + W_0$ as well as W_0 are U_P-stable. Now it suffices to prove that U_P acts on V_c/W_0 transitively and freely, $\forall c \neq 0$.

Clearly, $cv \bmod W_0$ has a dense U_P-orbit in V_c/W_0 and trivial stabilizer. Being an affine space, this orbit cannot be embedded into another affine space as a proper open subset. (Otherwise the boundary is a hypersurface, and its equation yields an invertible regular function on the orbit, a contradiction.) This proves the required assertion. □

This theorem applies to describing the structure of an open subset of sufficiently general points in any G-variety X.

Theorem 2 ([BLV]). *There exist a parabolic subgroup $P = L \cdot U_P \supseteq B$, an intermediate subgroup $[L, L] \subseteq L_0 \subseteq L$, an open P-stable subset $\mathring{X} \subseteq X$, and a closed subset $C \subseteq \mathring{X}^{L_0}$ such that $\mathring{X} \simeq U_P \times A \times C \simeq P \times^{L_0} C$, where $A = L/L_0$ is the quotient torus.*

Proof. Replace X by a birationally isomorphic projective G-variety in $\mathbb{P}(V)$. In the notation of Theorem 1, put $\mathring{X} = \mathbb{P}(\mathring{V}) \cap X$, $Z = \mathbb{P}(\mathring{W}) \cap X$, then $\mathring{X} \simeq U_P \times Z \simeq P \times^L Z$. If the kernel L_0 of the action $L : Z$ contains $[L, L]$, then the effectively acting group is the torus $A = L/L_0$, and we may replace $\mathring{X}$ and Z by open subsets such that $Z \simeq A \times C$.

In order to arrive to this situation, take a B-stable hypersurface $D \subset X$ such that the parabolic subgroup $P(D) = \{g \in G \mid gD = D\}$ is the smallest possible one. Adding new components if necessary, we may assume that D is given by one equation in projective coordinates. Applying the Veronese embedding, we may assume that $D = \mathbb{P}(\langle v^* \rangle^\perp) \cap X$ is a hyperplane section, where $v^* \in V^*$ is a highest weight vector. Then $\mathring{X} = X \setminus D$, $P = P(D)$, and each $(B \cap L)$-stable hypersurface in Z is L-stable. Thus each $(B \cap L)$-eigenvector in $\mathbb{C}[Z]$ is an L-eigenvector, whence L-isotypic components of $\mathbb{C}[Z]$ are 1-dimensional, and $[L, L]$ acts trivially on $\mathbb{C}[Z]$ and on Z. □

Corollary. *In the notation of Theorem 2, we have $c(X) = \dim C$, $r(X) = \dim A$, and $\Lambda(X) = \Lambda(A)$.*

1.2. **Horospherical varieties.** The local structure theorems of Brion–Luna–Vust describe the action of a certain parabolic $P \subseteq G$ on a certain open subset of X. There is a remarkable class of G-varieties, which in particular admit a local description of the G-action itself and have a number of other nice properties.

Definition 2. A subgroup of G containing a maximal unipotent subgroup is called *horospherical.* A G-variety X is *horospherical* if the stabilizers of all points of X are horospherical.

The terminology, due to Knop [Kn1], is explained by the following

Example 1. Let L^n be the Lobachevsky space modelled as the upper pole of the hyperboloid $\{x \in \mathbb{R}^{n+1} \mid (x,x) = 1\}$ in an $(n+1)$-dimensional pseudoeuclidean space of signature $(1,n)$. A horosphere in L^n (i.e., a hypersurface perpendicular to a pencil of parallel lines) is defined by the equation $(x,y) = 1$, where $y \in \mathbb{R}^{n+1}$ is a nonzero isotropic vector. The space of horospheres is homogeneous under the connected isometry group $\mathrm{SO}^+_{1,n}$ of L^n and is isomorphic to the upper pole of the isotropic cone $\{y \in \mathbb{R}^{n+1} \mid (y,y) = 0\}$. Its complexification is the space of highest weight vectors for $\mathrm{SO}_{n+1}(\mathbb{C}) : \mathbb{C}^{n+1}$, which is a horospherical variety in the sense of the above definition.

Horospherical subgroups have an explicit description. Up to conjugacy, we may assume that a horospherical subgroup $S \subseteq G$ contains the "lower" maximal unipotent subgroup U^-. By the Chevalley theorem, S is the stabilizer of a line $\langle v\rangle$ in a representation $G : V$. Then $v = v_{\lambda_1} + \cdots + v_{\lambda_m}$ is the sum of lowest weight vectors v_{λ_i} of weights λ_i. Let $P^-(\lambda_i)$ be the projective stabilizer of v_{λ_i}, $P^- = \bigcap_i P^-(\lambda_i) = L \cdot U_P^-$ (a Levi decomposition), $T_0 = \bigcap_{i,j} \mathrm{Ker}(\lambda_i - \lambda_j) \subseteq T$, and $L_0 = [L,L]T_0$. Then $S = L_0 \cdot U_P^-$ (a Levi decomposition).

The local structure of horospherical varieties is quite simple.

Theorem 3. *Each horospherical G-variety X contains an open G-stable subset $\mathring{X} \simeq (G/S) \times C$, where $S \subseteq G$ is horospherical and G acts on $\mathring{X}$ via the first factor.*

Proof. We have $X = GX^{U^-}$. By the structure of horospherical subgroups, for each $x \in X^{U^-}$ there exists a parabolic $P^- = L \cdot U_P^- \supseteq G_x \supseteq [L,L]U_P^-$. There are finitely many choices for P^-, hence X^{U^-} is covered by finitely many closed subsets of $[L,L]U_P^-$-fixed points. It follows that there exists the smallest P^- and a dense open subset $\mathring{X}^{U^-} \subseteq X^{U^-}$ such that $P^- \supseteq G_x \supseteq [L,L]U_P^-$, $\forall x \in \mathring{X}^{U^-}$. Then $\mathring{X} = G\mathring{X}^{U^-} \simeq G \times^{P^-} \mathring{X}^{U^-}$, and the P^--action on $\mathring{X}^S$ factors through the effective action of the torus $A = L/L_0 = P^-/S$, $L \supseteq L_0 \supseteq [L,L]$, $S = L_0 \cdot U_P^-$. Shrinking $\mathring{X}$ if necessary, we may assume that $\mathring{X}^{U^-} \simeq (P^-/S) \times C$, whence the desired assertion. □

Affine (or quasiaffine) horospherical varieties are characterized in terms of the multiplication law in their coordinate algebras.

Theorem 4 ([Po]). *A quasiaffine G-variety X is horospherical iff the isotypic decomposition of $\mathbb{C}[X]$ is in fact an algebra grading, i.e., $\mathbb{C}[X]_{(\lambda)} \cdot \mathbb{C}[X]_{(\mu)} \subseteq \mathbb{C}[X]_{(\lambda+\mu)}$, $\forall \lambda, \mu \in \Lambda_+(X)$.*

Proof. Assume that X is horospherical. In the notation of Theorem 3, $\mathbb{C}[X] \subseteq \mathbb{C}[\mathring{X}] = \mathbb{C}[G/S] \otimes \mathbb{C}[C]$, hence it suffices to consider $X = G/S$. The torus $A = P^-/S$ acts on G/S by G-automorphisms ("translations from the right"), so that $\mathbb{C}[G/S]_{(\lambda)}$ is the eigenspace of weight $-\lambda$. Indeed, for any highest weight vector $f_\lambda \in \mathbb{C}[G/S]_{(\lambda)}$ which is an eigenvector of A, we have $f_\lambda(eS) \neq 0$ (because the U-orbit of eS is dense in G/S) and $f_\lambda(eS \cdot t) = f_\lambda(tS) = \lambda(t^{-1})f_\lambda(eS)$, $\forall t \in T$. Therefore the isotypic decomposition respects the multiplication.

Conversely, suppose that the isotypic decomposition is an algebra grading. We may assume that X is affine. It suffices to show that GX^{U^-} is dense in X, because it is closed being the image of the natural proper morphism $G \times^{B^-} X^{U^-} \hookrightarrow G \times^{B^-} X \simeq (G/B^-) \times X \to X$. In other words, the ideal I of X^{U^-} in $\mathbb{C}[X]$ may not contain nonzero G-submodules or, equivalently, may not contain highest weight vectors $f_\lambda \in \mathbb{C}[X]$, $\lambda \in \Lambda_+(X)$ (because the orbit $U^- f_\lambda$ spans a G-submodule).

But I is generated by $gf - f$, $g \in U^-$, $f \in \mathbb{C}[X]$. (It even suffices to take for f the restrictions of the coordinate functions in an affine embedding of X.) If $I \ni f_\lambda = \sum_i p_i(g_i f_i - f_i)$, $p_i \in \mathbb{C}[X]_{(\lambda_i)}$, $f_i \in \mathbb{C}[X]_{(\mu_i)}$, $g_i \in U^-$, then $\lambda = \lambda_i + \mu_i$ and p_i, $g_i f_i - f_i$ must be highest weight vectors of weights λ_i, μ_i, which never occurs for $g_i f_i - f_i$, a contradiction. □

The above theorems provide an evidence that horospherical varieties have relatively simple structure. Remarkably, every G-variety degenerates to a horospherical one.

Theorem 5 ([Po], [Kn1]). *Given a G-variety X, there exists a smooth $(G \times \mathbb{C}^\times)$-variety E and a smooth $(G \times \mathbb{C}^\times)$-equivariant morphism $\pi : E \to \mathbb{A}^1$ (here G acts on $\mathbb{A}^1$ trivially and $\mathbb{C}^\times$ acts by homotheties) such that $X_t = \pi^{-1}(t)$ is G-isomorphic to an open smooth G-stable subset of X whenever $t \neq 0$, X_0 is a smooth horospherical variety, and all fields $\mathbb{C}(X_t)^U$ are B-isomorphic. In particular, all X_t have the same complexity, rank, and weight lattice as X.*

Proof. By the standard techniques of passing to an open G-stable subset and taking the affine cone over a projective variety, the theorem is reduced to the affine case handled by Popov [Po]. So we may assume X to be affine.

We define the *height* of any weight λ by decomposing $\lambda = \sum_i c_i \alpha_i + \lambda_0$, where α_i are the simple roots and $\lambda_0 \perp \alpha_i$, $\forall i$, and by putting $\operatorname{ht} \lambda = 2\sum_i c_i$. (The multiplier 2 forces ht to take integer values. Namely, $\operatorname{ht}\lambda$ is the inner product of λ with the sum of the positive coroots.) It follows from the structure of T-weights of simple G-modules that $\mathbb{C}[X]_{(\lambda)} \cdot \mathbb{C}[X]_{(\mu)} \subseteq \mathbb{C}[X]_{(\lambda+\mu)} \oplus \bigoplus_i \mathbb{C}[X]_{(\nu_i)}$, where $\operatorname{ht}\nu_i < \operatorname{ht}\lambda + \operatorname{ht}\mu$.

Now $R = \bigoplus_{\operatorname{ht}\lambda \leq k} \mathbb{C}[X]_{(\lambda)} t^k$ is a $(G \times \mathbb{C}^\times)$-algebra of finite type generated by $f_1 t^{\operatorname{ht}\lambda_1}, \dots, f_m t^{\operatorname{ht}\lambda_m}, t$, where $f_i \in \mathbb{C}[X]_{(\lambda_i)}$ are generators of $\mathbb{C}[X]$. Then $R = \mathbb{C}[E]$ is the coordinate algebra of an affine $(G \times \mathbb{C}^\times)$-variety E, and the morphism $\pi : E \to \mathbb{A}^1$ corresponds to the inclusion $R \supseteq \mathbb{C}[t]$.

It is easy to see that all $\mathbb{C}[X_t]$ are canonically isomorphic to $\mathbb{C}[X]$ as G-modules. In fact, all algebras $\mathbb{C}[X_t]^U$ are canonically isomorphic to $\mathbb{C}[X]^U$, and $\mathbb{C}[X_t] \simeq \mathbb{C}[X]$ whenever $t \neq 0$. But the multiplication law in $\mathbb{C}[X_0]$ is obtained from that in $\mathbb{C}[X]$ by "forgetting" isotypic components of lower height. Hence, by Theorem 4, X_0 is horospherical. By [Kr, III.3] all fibers X_t are reduced and irreducible, hence the smooth locus of E meets X_t. Passing to open subsets completes the proof. □

Example 2. In Example 1, the "horospherical contraction" X_0 of the Lobachevsky space $X = L^n$ is the space of horospheres, the total space of deformation E being given by the "upper pole" of $\{(x,t) \in \mathbb{R}^{n+1} \times \mathbb{R} \mid (x,x) = t\}$. More precisely, we have to complexify the whole picture, so that X is a sphere in $\mathbb{C}^{n+1}$ and X_0 is the isotropic cone.

1.3. **Relation to symplectic geometry.** There is a deep connection between the geometry of G/H and the equivariant symplectic geometry of its cotangent bundle.

Recall that the cotangent bundle T^*X of any smooth variety X is equipped with a natural symplectic structure given by the 2-form $\omega = d\ell$, where ℓ is the *action 1-form* defined by $\ell_\alpha(\xi) = \langle \alpha, d\pi(\xi) \rangle$, $\forall \alpha \in T^*X,\ \xi \in T_\alpha(T^*X)$, and $\pi : T^*X \to X$ is the canonical projection. In local coordinates $q_1, \dots, q_n$ on X, which determine the dual coordinates $p_1, \dots, p_n$ in cotangent spaces, one has $\ell = \sum p_i\, dq_i$ and $\omega = \sum dp_i \wedge dq_i$.

If X is a G-variety, then G acts on T^*X by symplectomorphisms, and the velocity fields $\alpha \mapsto \xi\alpha$ of $\forall \xi \in \mathfrak{g}$ have global Hamiltonians $H_\xi(\alpha) = \ell_\alpha(\xi\alpha)$. Furthermore, the action $G : T^*X$ is Poisson, i.e., the map $\xi \mapsto H_\xi$ is a homomorphism of $\mathfrak{g}$ to the algebra of functions on T^*X equipped with the Poisson bracket. The dual morphism $\Phi : T^*X \to \mathfrak{g}^*$ given by $\langle \Phi(\alpha), \xi \rangle = H_\xi(\alpha) = \langle \alpha, \xi x \rangle$, $\forall \alpha \in T^*_x X,\ \xi \in \mathfrak{g}$, is called the *moment map*.

It is easy to see that the moment map is G-equivariant, and $\langle d_\alpha \Phi(\nu), \xi \rangle = \omega_\alpha(\nu, \xi\alpha)$, $\forall \nu \in T_\alpha(T^*X),\ \xi \in \mathfrak{g}$. It follows that $\operatorname{Ker} d_\alpha \Phi = (\mathfrak{g}\alpha)^\angle$, $\operatorname{Im} d_\alpha \Phi = (\mathfrak{g}_\alpha)^\perp$, where $^\angle$ and $^\perp$ denote the skew-orthocomplement and the annihilator in $\mathfrak{g}^*$, respectively. Let $M_X = \overline{\operatorname{Im} \Phi}$ be the closure of the image of the moment map. It follows that $\dim M_X = \dim G\alpha$ for generic $\alpha \in T^*X$.

For $X = G/H$ we have $T^*(G/H) \simeq G \times^H \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp = (\mathfrak{g}/\mathfrak{h})^*$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. The moment map is given by $\Phi(g * \alpha) = g\alpha$ (with the coadjoint g-action on the r.h.s.). Indeed, the formula is true for $g = e$ since $\langle \Phi(\alpha), \xi \rangle = \langle \alpha, \xi(eH) \rangle$ and $\xi(eH)$ identifies with $\xi \bmod \mathfrak{h}$, and we conclude by G-equivariance. Moreover, for any G-variety X, the moment map of its cotangent bundle restricted to an orbit $Gx \subseteq X$ factors through the moment map of T^*Gx.

Remark. We may (and will) identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via a G-invariant inner product (given, e.g., by the trace form for any faithful representation of G). Then $\mathfrak{h}^\perp$ identifies with the orthocomplement of $\mathfrak{h}$.

The algebra homomorphism dual to Φ can be defined both in the commutative and in the non-commutative setting. Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{D}(X)$ be the algebra of differential operators on X. Each $\xi \in \mathfrak{g}$ determines a vector field on X, i.e., a differential operator of order 1, and this assignment extends to a homomorphism $\Phi^* : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}(X)$. The map Φ^* preserves the natural filtrations, and the associated graded map

$$\operatorname{gr} \Phi^* : \operatorname{gr} \mathcal{U}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*] \longrightarrow \operatorname{gr} \mathcal{D}(X) \subseteq \mathbb{C}[T^*X], \qquad \xi \mapsto H_\xi, \quad \forall \xi \in \mathfrak{g}$$

is the pull-back of functions w.r.t. Φ. Here the isomorphism $\operatorname{gr} \mathcal{U}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$ is provided by the Poincaré–Birkhoff–Witt theorem, and the embedding $\operatorname{gr} \mathcal{D}(X) \subseteq \mathbb{C}[T^*X]$ is the symbol map.

We have already seen that the complexity, rank and weight lattice are preserved by the "horospherical contraction". The same is true for the closure of the image of the moment map.

Theorem 6 ([Kn1]). *In the notation of Theorem 5, $M_X = M_{X_0}$.*

Proof. The assertion can be reformulated in algebraic terms: put $I_X = \operatorname{Ker} \operatorname{gr} \Phi^*$, then $I_X = I_{X_0}$. We deduce this equality from its non-commutative analogue: put $\mathcal{I}_X = \operatorname{Ker} \Phi^*$, then $\mathcal{I}_X = \mathcal{I}_{X_0}$.

The latter equality is obvious in the affine case, because $\mathcal{I}_X$ depends only on the G-module structure of $\mathbb{C}[X]$. The general case is reduced to the affine one by standard techniques [Kn1, 5.1].

Put $\mathcal{M}_X = \operatorname{Im} \Phi_X^* \subseteq \mathcal{D}(X)$. By the above, $\mathcal{M}_X \simeq \mathcal{M}_{X_0}$, but the filtrations by the order of differential operators on X and on X_0 are *apriori* different. It suffices to show that in fact they coincide.

There is even a third filtration, the quotient one induced from $\mathcal{U}(\mathfrak{g})$. Let $\operatorname{ord}_X \partial$, $\operatorname{ord} \partial$ denote the order of $\partial \in \mathcal{M}_X$ as a differential operator on X and w.r.t. the quotient filtration, respectively. It is clear that $\operatorname{ord} \geq \operatorname{ord}_X$.

First note that, in the notation of Theorem 5, there are obvious isomorphic restriction maps $\mathcal{M}_E \to \mathcal{M}_{X_t}$, which do not rise the order of differential operators and even preserve it whenever $t \neq 0$, because then $E \setminus X_0 \simeq X_t \times (\mathbb{A}^1 \setminus \{0\})$. Thus $\operatorname{ord}_X \geq \operatorname{ord}_{X_0}$.

Secondly, $\operatorname{gr} \mathcal{M}_{X_0}$ is a finite $\mathbb{C}[\mathfrak{g}^*]$-module. To prove this, we may assume by Theorem 3 that $X_0 = (G/S) \times C$. Hence $\mathcal{M}_{X_0} \simeq \mathcal{M}_{G/S}$ and $\operatorname{ord}_{X_0} = \operatorname{ord}_{G/S}$. We use the notation of 1.2. The torus $A = P^-/S$ acts on G/S by G-automorphisms, whence $\mathcal{M}_{G/S} \subseteq \mathcal{D}(G/S)^A$. Therefore $\operatorname{gr} \mathcal{M}_{G/S} \subseteq \mathbb{C}[T^*(G/S)]^A = \mathbb{C}[G \times^{P^-} \mathfrak{s}^\perp]$. But the natural morphism $G \times^{P^-} \mathfrak{s}^\perp \to \mathfrak{g}^*$ is proper with finite generic fibres by Lemma 1 below. It follows that $\mathbb{C}[G \times^{P^-} \mathfrak{s}^\perp]$, and hence $\operatorname{gr} \mathcal{M}_{G/S}$, is a finite $\mathbb{C}[\mathfrak{g}^*]$-module.

Now let $\partial_1, \dots, \partial_m \in \mathcal{M}_{X_0}$ represent generators of $\operatorname{gr} \mathcal{M}_{X_0}$ over $\mathbb{C}[\mathfrak{g}^*]$, $d_i = \operatorname{ord}_{X_0} \partial_i$, and $d = \max_i \operatorname{ord} \partial_i$. If $\operatorname{ord}_{X_0} \partial = n$, then $\partial = \sum_i u_i \partial_i$ for some $u_i \in \mathcal{U}(\mathfrak{g})$, $\operatorname{ord} u_i \leq n - d_i$, hence $\operatorname{ord} \partial \leq n + d$. But if $\operatorname{ord} \partial > \operatorname{ord}_{X_0} \partial$, then $\operatorname{ord} \partial^{d+1} > \operatorname{ord}_{X_0} \partial^{d+1} + d$, a contradiction. Therefore $\operatorname{ord} = \operatorname{ord}_X = \operatorname{ord}_{X_0}$, and we are done. □

Thus in the study of the image of the moment map, we may assume that X is horospherical and even $X = G/S$, where S is a horospherical subgroup containing U^-. In the sequel, we use the notation of 1.2. The moment map $\Phi : T^*(G/S) \simeq G \times^S \mathfrak{s}^\perp \to \mathfrak{g}^* \simeq \mathfrak{g}$ factors through $\overline{\Phi} : G \times^{P^-} \mathfrak{s}^\perp \to \mathfrak{g}$. We have the decomposition $\mathfrak{g} = \mathfrak{u}_P \oplus \mathfrak{a} \oplus \mathfrak{l}_0 \oplus \mathfrak{u}_P^-$, where $\mathfrak{a}$ embeds into $\mathfrak{l}$ as the orthocomplement of $\mathfrak{l}_0$, so that $\mathfrak{s}^\perp = \mathfrak{a} \oplus \mathfrak{u}_P^-$. The following helpful result is essentially due to Richardson:

Lemma 1 ([Kn1, 4.1]). *The morphism $\overline{\Phi} : G \times^{P^-} \mathfrak{s}^\perp \to \mathfrak{g}$ is proper with finite generic fibres.*

Another nice consequence of "horospherical contraction" is the conjugacy of the stabilizers of generic points in cotangent bundles [Kn1, §8]. We consider only the quasiaffine case.

Theorem 7 ([Kn1]). *In the notation of Theorem 2, suppose X is quasi-affine; then the stabilizers in G of generic points in T^*X are all conjugate to L_0.*

Corollary. *We have $\Lambda(X) = \Lambda(T/(T \cap G_\alpha))$ for some sufficiently general point $\alpha \in T^*X$ such that G_α is an intermediate subgroup between a standard Levi subgroup and its commutator subgroup.*

Remark. Intermediate subgroups between standard Levi subgroups and their commutator subgroups, as well as embeddings onto such subgroups in G, will be called *standard.* Thus corollary says that a standard embedding of G_α for generic $\alpha \in T^*X$ yields the weight lattice of X. However, in applying this corollary for computing the weight lattice, one should be cautious, because G_α might have different conjugate standard embeddings into G. Some additional argument may be required to specify the weight lattice, see Example 5.

Proof. We prove the theorem for horospherical varieties. The general case can be deduced with the aid of "horospherical contraction" using some additional reasoning [Kn1, 8.1].

We may assume $X = G/S$. As X is quasiaffine, $G/S \simeq Gv$ is an orbit in a representation $G : V$. Then $v = v_{\lambda_1} + \cdots + v_{\lambda_m}$ is the sum of lowest weight vectors, $P^- = \bigcap_i P^-(\lambda_i) = L \cdot U_P^-$, and $S = L_0 \cdot U_P^-$, where $L_0 = [L, L]T_0$, $T_0 = \bigcap_i \operatorname{Ker} \lambda_i \subseteq T$.

Note that $Z(\mathfrak{a}) = L$. Indeed, β is a root of $Z(\mathfrak{a})$ iff $\beta|_\mathfrak{a} = 0$ iff $\beta \perp \lambda_1, \dots, \lambda_m$ iff β is a root of L.

We have $T^*(G/S) \simeq G \times^S \mathfrak{s}^\perp$, whence the stabilizers in G of generic points in $T^*(G/S)$ are, up to conjugacy, the stabilizers in S of generic points in $\mathfrak{s}^\perp = \mathfrak{a} \oplus \mathfrak{u}_P^-$. If $\xi \in \mathfrak{a}$ is a sufficiently general point (it suffices to have $\beta(\xi) \neq 0$ for all roots β of G that are not roots of L), then $\mathfrak{z}(\mathfrak{a}) = \mathfrak{l}$ yields $[\mathfrak{s}, \xi] = \mathfrak{u}_P^-$. Since the projection map $\pi : \mathfrak{s}^\perp \to \mathfrak{a}$ is S-invariant, $S\xi$ is dense in (in fact, coincides with) $\pi^{-1}(\xi) = \xi + \mathfrak{u}_P^-$. Therefore the stabilizers of generic points in $\mathfrak{s}^\perp$ are conjugate to $S_\xi = L_0$. □

The following fundamental result of Knop interprets complexity and rank in terms of equivariant symplectic geometry.

Theorem 8 ([Kn1]). *Let X be a G-variety with $\dim X = n$, $c(X) = c$, $r(X) = r$. Then*

$$\dim M_X = 2n - 2c - r \tag{1}$$

$$d_G(T^*X) = 2c + r \tag{2}$$

$$d_G(M_X) = r \tag{3}$$

Proof. We may assume that X is horospherical and even $X = G/S \times C$. By Lemma 1, $\dim M_X = \dim(G \times^{P^-} \mathfrak{s}^\perp) = \dim G/P^- + \dim \mathfrak{s}^\perp = 2 \dim G/S - \dim A = 2(n - c) - r$ and $d_G(M_X) = d_G(G \times^{P^-} \mathfrak{s}^\perp) = d_{P^-}(\mathfrak{s}^\perp)$. The projection map $\pi : \mathfrak{s}^\perp \to \mathfrak{a}$ is P^--invariant, and P^- has a dense orbit in $\pi^{-1}(0) = \mathfrak{u}_P^-$ (the Richardson orbit). By semicontinuity of orbit and fibre dimensions, generic (in fact, all) fibres of π contain dense P^--orbits, whence $d_{P^-}(\mathfrak{s}^\perp) = \dim \mathfrak{a} = r$. Thus we have proved (1) and (3), and (2) stems from (1) and from $d_G(T^*X) = 2n - \dim M_X$. □

In particular, for $X = G/H$ we obtain formulæ for complexity and rank in terms of the coisotropy representation $(H : \mathfrak{h}^\perp)$:

Theorem 9 (Knop [Kn1], Panyushev [Pan1]).

$$2c(G/H) + r(G/H) = \operatorname{codim}_{\mathfrak{h}^\perp} H\alpha = \dim G - 2\dim H + \dim H_\alpha \tag{4}$$

$$r(G/H) = \dim G_\alpha - \dim H_\alpha \tag{5}$$

where $\alpha \in \mathfrak{h}^\perp$ is a generic point. For reductive H, Formula (5) amounts to

$$r(G/H) = \operatorname{rk} G - \operatorname{rk} H_\alpha \tag{6}$$

and also

$$\Lambda(G/H) = \Lambda(T/(T \cap H_\alpha)) \tag{7}$$

Proof. The isomorphism $T^*(G/H) \simeq G \times^H \mathfrak{h}^\perp$ yields $d_G(T^*(G/H)) = d_H(\mathfrak{h}^\perp)$, whence (4). Further, $d_G(M_{G/H}) = \dim M_{G/H} - \dim G\alpha = \dim(G * \alpha) - \dim G\alpha = \dim G_\alpha - \dim H_\alpha$ implies (5). Finally, if H is reductive, then G/H is affine, and (6)–(7) stem from Theorem 7 and its corollary. □

Examples:

3. Consider the space of quadrics $Q_n = \mathrm{PGL}_{n+1}/\mathrm{PO}_{n+1}$. Here the coisotropy representation identifies with the natural representation of PO_{n+1} in the space $S_0^2\mathbb{C}^{n+1}$ of traceless symmetric matrices. The stabilizer of a generic point is $\mathbf{Z}_2^n = \{\operatorname{diag}(\pm1, \dots, \pm1)\}/\{\pm E\}$. The weight lattice of the (standard, diagonal) maximal torus $T \subset \mathrm{PGL}_{n+1}$ is the root lattice Λ_{ad} of PGL_{n+1}, whence $\Lambda(Q_n) = 2\Lambda_{\mathrm{ad}}$, and $r(Q_n) = n$. Finally, $2c(Q_n) + r(Q_n) = (n+1)(n+2)/2 - 1 - n(n+1)/2 = n$ yields $c(Q_n) = 0$. The latter equality can be seen directly since $B \cdot \mathrm{PO}_{n+1}$ is open in PGL_{n+1}, where $B \subseteq \mathrm{PGL}_{n+1}$ is the standard Borel subgroup of upper-triangular matrices. (The Gram–Schmidt orthogonalization.)

4. Let $G/H = \mathrm{Sp}_n/\mathrm{Sp}_{n-2}$. As the adjoint representation $G : \mathfrak{g}$ identifies with $\mathrm{Sp}_n : S^2\mathbb{C}^n$, the symmetric square of the standard representation, and similarly for $H : \mathfrak{h}$, we have $\mathfrak{h}^\perp \simeq \mathbb{C}^{n-2} \oplus \mathbb{C}^{n-2} \oplus \mathbb{C}^3$, where $\mathrm{Sp}_{n-2} : \mathbb{C}^{n-2}$ is the standard representation, and $\mathrm{Sp}_{n-2} : \mathbb{C}^3$ a trivial one. It follows that $H_\alpha = \mathrm{Sp}_{n-4}$. There exists a unique standard embedding $\mathrm{Sp}_{n-4} \hookrightarrow \mathrm{Sp}_n$ as a subgroup generated by all the simple roots except the first two. Therefore $\Lambda = \langle \omega_1, \omega_2 \rangle$, where ω_i are the fundamental weights, and $r = 2$. We also have $2c + r = 2(n-2) + 3 - (n-2)(n-1)/2 + (n-4)(n-3)/2 = 4$, whence $c = 1$.

5. Let $G/H = \mathrm{GL}_n/(\mathrm{GL}_1 \times \mathrm{GL}_{n-1})$. Here $\mathfrak{h}^\perp \simeq \big(\mathbb{C}^1 \otimes (\mathbb{C}^{n-1})^*\big) \oplus \big((\mathbb{C}^1)^* \otimes \mathbb{C}^{n-1}\big)$, where $\mathbb{C}^k$ is the standard representation of GL_k $(k = 1, n-1)$. It is easy to find that $H_\alpha = \{\operatorname{diag}(t, A, t) \mid t \in \mathbb{C}^\times,\ A \in \mathrm{GL}_{n-2}\}$. Therefore $r = 1$, and $2c + r = 2(n-1) - 1 - (n-1)^2 + 1 + (n-2)^2 = 1$, whence $c = 0$.

However H_α has three different standard embeddings into G obtained by permuting the diagonal blocks. To choose the right one, note that $\mathfrak{sl}_n$ contains a vector with stabilizer $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$. Hence $G/H \hookrightarrow \mathfrak{sl}_n$ and the restriction of the highest weight covector yields a highest weight function in $\mathbb{C}[G/H]$ of highest weight $\omega_1 + \omega_{n-1}$ (the highest root). Thus $\Lambda = \langle \omega_1 +$

$\omega_{n-1}\rangle$, and H_α indeed embeds into G as above. (The simple roots of H_α are the simple roots of G except the first and the last one.)

6. The space of twisted (i.e., irreducible non-planar) cubic curves in $\mathbb{P}^2$ is isomorphic to $G/H = \mathrm{PGL}_4/\mathrm{PGL}_2$, where $\mathrm{GL}_2 \hookrightarrow \mathrm{GL}_4$ is given by the representation $\mathrm{GL}_2 : V_3$. Here V_d denotes the space of binary d-forms. Indeed, each twisted cubic is the image of a Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.

From the H-isomorphisms $\mathfrak{gl}_2 \simeq V_1 \otimes V_1^*$, $\mathfrak{gl}_4 \simeq V_3 \otimes V_3^*$, and the Clebsch–Gordan formula, it is easy to deduce that $\mathfrak{h}^\perp \simeq (V_6 \otimes \det^{-3}) \oplus (V_4 \otimes \det^{-2})$. It follows that $H_\alpha = \{E\}$, $r = 3$, $2c + r = 7 + 5 - 3 = 9$, whence $c = 3$.

If we replace G by PSp_4 in the above computations, then $\mathfrak{h}^\perp \simeq V_6 \otimes \det^{-3}$, still $H_\alpha = \{E\}$, but $r = 2$, $2c + r = 7 - 3 = 4$, whence $c = 1$.

On the other hand, replacing H by PSp_4 yields $\mathfrak{h}^\perp \simeq \bigwedge_0^2 \mathbb{C}^4$, the space of bivectors having zero contraction with the symplectic form. We obtain $H_\alpha = \mathrm{P}(\mathrm{SL}_2 \times \mathrm{SL}_2)$, whence $\Lambda = \langle 2\omega_2 \rangle$, $r = 1$, $2c + r = 5 - 10 + 6 = 1$, and $c = 0$.

In the notation of Theorem 2, there is an open embedding $U_P \times A \hookrightarrow \mathrm{PGL}_4/\mathrm{PSp}_4$. Since $\Lambda(A) \subset \Lambda(T)$ is generated by an indivisible vector, A embeds in T as a subtorus, and $U_P \cdot A \hookrightarrow \mathrm{PGL}_4$. This yields an open embedding $\mathrm{PGL}_4/PGL_2 \simeq \mathrm{PGL}_4 \times^{\mathrm{PSp}_4} \mathrm{PSp}_4/\mathrm{PGL}_2 \hookleftarrow U_P \times A \times \mathrm{PSp}_4/\mathrm{PGL}_2$. Applying Theorem 2 to $\mathrm{PSp}_4/\mathrm{PGL}_2$ this time, we obtain open embeddings $\mathrm{PSp}_4/\mathrm{PGL}_2 \hookleftarrow U_{P_0} \times A_0 \times C$, $\mathrm{PGL}_4/\mathrm{PGL}_2 \hookleftarrow U_P \times A \times U_{P_0} \times A_0 \times C$, where P_0 is a parabolic in PSp_4, A_0 is its quotient torus, and C is a rational curve by the Lüroth theorem. This proves the theorem of Piene–Schlessinger on rationality of the space of twisted cubics. (See [Bri3, §3] for another proof using homogeneous spaces.)

1.4. **Reduction to representations.** Theorem 9 yields computable formulæ for complexity and rank of affine homogeneous spaces reducing everything to computing stabilizers of general position for representations of reductive groups, which is an accessible problem. Panyushev [Pan4] performed a similar reduction for arbitrary G/H. The idea is to consider a *regular* embedding $H \subseteq Q$ into a parabolic $Q \subseteq G$, i.e., such that there is also the inclusion of the unipotent radicals $U_H \subseteq U_Q$. The existence of a regular embedding into a parabolic subgroup was first proved by Weisfeiler, see e.g. [Hum, 30.3].

Let $H = K \cdot U_H$, $Q = M \cdot U_Q$ be Levi decompositions. We may assume $K \subseteq M$. The space $G/H \simeq G \times^Q Q/H$ is a homogeneous fibre space with generic fibre $Q/H \simeq M \times^K (U_Q/U_H) \simeq M \times^K (\mathfrak{u}_Q/\mathfrak{u}_H)$ a homogeneous vector bundle with affine base. The K-isomorphism $\mathfrak{u}_Q/\mathfrak{u}_H \xrightarrow{\sim} U_Q/U_H$ is proved in [Mon] essentially in the same way as in the non-equivariant setting, using a normal K-stable series $U_Q = U_0 \rhd \cdots \rhd U_m = U_H$, considering K-stable decompositions $\mathfrak{u}_{i-1} = \mathfrak{u}_i \oplus \mathfrak{m}_i$, and mapping $x = x_1 + \cdots + x_m \mapsto (\exp x_1) \cdots (\exp x_m)$, $\forall x_i \in \mathfrak{m}_i$. Up to conjugacy, we may assume $Q \supseteq B^-$, $M \supseteq T$. Let K_0 denote the stabilizer of a generic point in the coisotropy representation $K : \mathfrak{k}^\perp$, with its standard embedding into M.

Theorem 10 ([Pan4]).

$$c(G/H) = c(M/K) + c(\mathfrak{u}_Q/\mathfrak{u}_H) \tag{8}$$

$$r(G/H) = r(M/K) + r(\mathfrak{u}_Q/\mathfrak{u}_H) \tag{9}$$

and there is an exact sequence of weight lattices

$$0 \longrightarrow \Lambda(M/K) \longrightarrow \Lambda(G/H) \longrightarrow \Lambda(\mathfrak{u}_Q/\mathfrak{u}_H) \longrightarrow 0 \tag{10}$$

Here complexities, ranks, and weight lattices are considered for the homogeneous spaces G/H, M/K, and for the linear representation $K_0 : \mathfrak{u}_Q/\mathfrak{u}_H$.

Proof. As $U(eQ)$ is dense in G/Q and $B \cap M$ is a Borel subgroup of M, the complexities and the weight lattices of G/H and of $M : Q/H$ coincide. We may assume that $eK \in M/K$ is a generic point w.r.t the $(B \cap H)$-action. Then by Theorems 2,7, $B \cap K = B \cap K_0$ is a Borel subgroup of K_0, and the stabilizers in $B \cap M$ of generic points in M/K are conjugate to $B \cap K_0$. Now an easy computation of orbit dimensions implies (8).

By Theorem 2, the stabilizers of generic points for the actions $B : G/H$, $(B \cap M) : Q/H$, $(B \cap K_0) : \mathfrak{u}_Q/\mathfrak{u}_H$ are conjugate to $B \cap L_0$. It follows that $\Lambda(G/H) = \Lambda(T/(T \cap L_0))$, $\Lambda(M/K) = \Lambda(T/(T \cap K_0))$, and $\Lambda(\mathfrak{u}_Q/\mathfrak{u}_H) = \Lambda((T \cap K_0)/(T \cap L_0))$. This yields (10), and (9) stems from (10). □

Example 7. Let $G = \mathrm{Sp}_n$ and H be the stabilizer of three ordered generic vectors in the symplectic space $\mathbb{C}^n$. Without loss of generality, we may assume that these vectors are e_1, e_{n-1}, e_n, where $e_1, \dots, e_n$ is a symplectic basis of $\mathbb{C}^n$ such that the symplectic form has an antidiagonal matrix in this basis. Take for Q the stabilizer in Sp_n of the isotropic plane $\langle e_{n-1}, e_n \rangle$. Then $M \simeq \mathrm{GL}_2 \times \mathrm{Sp}_{n-4}$ consists of symplectic operators preserving the decomposition $\mathbb{C}^n = \langle e_1, e_2 \rangle \oplus \langle e_3, \dots, e_{n-2} \rangle \oplus \langle e_{n-1}, e_n \rangle$, $K \simeq \mathrm{Sp}_{n-4}$, $\mathfrak{u}_Q$ consists of skew-symmetric (w.r.t. the symplectic form) operators mapping $\mathbb{C}^n \to \langle e_3, \dots, e_n \rangle \to \langle e_{n-1}, e_n \rangle \to 0$, and $\mathfrak{u}_H$ is the annihilator of e_1 in $\mathfrak{u}_Q$.

For M/K we have: $\mathfrak{k}^{\perp} \simeq \mathbb{C}^4$ is a trivial representation of K, whence $K_0 = K = \mathrm{Sp}_{n-4}$, $\Lambda = \langle \omega_1, \omega_2 \rangle$, $r = 2$, $2c + r = 4$, $c = 1$.

Further, $\mathfrak{u}_Q \simeq \mathbb{C}^{n-4} \oplus \mathbb{C}^{n-4} \oplus \mathbb{C}^3$, and $\mathfrak{u}_H \simeq \mathbb{C}^{n-4} \oplus \mathbb{C}^1$, where $\mathbb{C}^{n-4}$ is the standard representation of Sp_{n-4} and $\mathbb{C}^3$, $\mathbb{C}^1$ are trivial ones. Therefore $\mathfrak{u}_Q/\mathfrak{u}_H \simeq \mathbb{C}^{n-4} \oplus \mathbb{C}^2$. One easily finds that the stabilizers of generic points in $T^*(\mathfrak{u}_Q/\mathfrak{u}_H) = \mathfrak{u}_Q/\mathfrak{u}_H \oplus (\mathfrak{u}_Q/\mathfrak{u}_H)^* \simeq \mathfrak{u}_Q/\mathfrak{u}_H \oplus \mathfrak{u}_Q/\mathfrak{u}_H$ (i.e., of generic pairs of vectors) are conjugate to Sp_{n-6}. It follows that $\Lambda = \langle \overline{\omega_3} \rangle$ is generated by the 1-st fundamental weight of Sp_{n-4}, which is the restriction of the 3-rd one for Sp_n. Hence $r = 1$, $2c + r = 2(n-3+2) - (n-4)(n-3)/2 + (n-6)(n-5)/2 = 5$, $c = 2$.

By Theorem 10, we conclude that $c(G/H) = r(G/H) = 3$ and $\Lambda(G/H) = \langle \omega_1, \omega_2, \omega_3 \rangle$.

1.5. **Complexity and growth of multiplicities.** The complexity of a homogeneous space has a nice representation-theoretic meaning: it provides asymptotics of the growth of multiplicities of simple G-modules in representation spaces of regular functions or global sections of line bundles.

For any G-module M, let $\operatorname{mult}_\lambda M$ denote the multiplicity of a simple G-module of highest weight λ in M. Equivalently, $\operatorname{mult}_\lambda M = \dim M^U_{(\lambda)}$, where $M_{(\lambda)}$ is the respective isotypic component of M.

Theorem 11. *The complexity $c(G/H)$ is the minimal integer c such that $\operatorname{mult}_{n\lambda} H^0(G/H, \mathcal{L}^{\otimes n}) = O(n^c)$ over all dominant weights λ and all G-line bundles $\mathcal{L} \to G/H$. If G/H is quasiaffine, then it suffices to consider $\operatorname{mult}_{n\lambda} \mathbb{C}[G/H]$.*

Proof. We may identify $\mathcal{L}$ with $G \times^H \mathbb{C}_\chi$, where H acts on $\mathbb{C}_\chi = \mathbb{C}$ by the character χ. Then $H^0(G/H, \mathcal{L})$ is the H-eigenspace of $\mathbb{C}[G]$ of weight $-\chi$, where H acts on G from the right. From the structure of $\mathbb{C}[G]$ as a $(G \times G)$-module (see 3.3) we see that $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}) = \dim V^*_{\lambda,-\chi}$, where $V^*_{\lambda,-\chi} \subseteq V^*_\lambda$ is the H-eigenspace of weight $-\chi$.

Put $c = c(G/H)$. Replacing H by a conjugate, we may assume that $\operatorname{codim} B(eH) = c$. If $c > 0$, then there exists a minimal parabolic $P_1 \supseteq B$ which does not stabilize $\overline{B(eH)}$. Therefore $\operatorname{codim} P_1(eH) = c - 1$. Continuing in the same way, we construct a sequence of minimal parabolics $P_1, \dots, P_c \supset B$ such that $\overline{P_c \cdots P_1(eH)} = G/H$, i.e., $P_c \cdots P_1 H$ is dense in G. It follows that $\dim P_1 \cdots P_c/B = c$, whence $S_w = \overline{BwB}/B = P_1 \dots P_c/B \subseteq G/B$ is the Schubert variety corresponding to an element w of the Weyl group W with reduced decomposition $w = s_1 \cdots s_c$, where $s_i \in W$ are the simple reflections corresponding to P_i.

The B-submodule $V_{\lambda,w} \subseteq V_\lambda$ generated by wv_λ is called a *Demazure module*. We have $V_{\lambda,w} = \langle P_1 \cdots P_c v_\lambda \rangle = H^0(S_w, \mathcal{L}_{-\lambda})^*$, where $\mathcal{L}_{-\lambda} = G \times^B \mathbb{C}_{-\lambda}$ [Jan].

Lemma 2. *The pairing between V^*_λ and V_λ provides an embedding $V^*_{\lambda,-\chi} \hookrightarrow (V_{\lambda,w})^*$. Consequently $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}) \le \dim V_{\lambda,w}$.*

Proof of the Lemma. If a nonzero $v^* \in V^*_{\lambda,-\chi}$ vanishes on $V_{\lambda,w}$, then it vanishes on $P_1 \cdots P_c v_\lambda$, i.e., $\langle P_c \cdots P_1 v^* \rangle = \langle G v^* \rangle = V^*_\lambda$ vanishes on v_λ, a contradiction. □

Replacing λ by $n\lambda$ and $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$ means that we replace $V_{\lambda,w}$ by $V_{n\lambda,w} = H^0(S_w, \mathcal{L}^{\otimes n}_{-\lambda})^*$. As S_w is a projective variety of dimension c, the dimension of the r.h.s. space of sections grows as $O(n^c)$.

On the other hand, let $f_1, \dots, f_c$ be a transcendence base of $\mathbb{C}(G/H)^B$. There exists a line bundle $\mathcal{L}$ and B-eigenvectors $s_0, \dots, s_c \in H^0(G/H, \mathcal{L})$ of the same weight λ such that $f_i = s_i/s_0$, $\forall i = 1, \dots, c$. (Indeed, $\mathcal{L}$ and s_0 may be determined by any B-stable effective divisor dominating the poles of all f_i.) These $s_0, \dots, s_c$ are algebraically independent in $R = \bigoplus_{n \ge 0} H^0(G/H, \mathcal{L}^{\otimes n})^U_{(n\lambda)}$, hence $\operatorname{mult}_{n\lambda} H^0(G/H, \mathcal{L}^{\otimes n}) = \dim R_n \ge \binom{n+c}{c} \sim n^c$. Therefore the exponent c in the estimate for the multiplicity cannot be made smaller.

Finally, if G/H is quasiaffine, then there even exist $s_0, \dots, s_c \in \mathbb{C}[G/H]$ with the same properties. □

For homogeneous spaces of small complexity much more precise information can be obtained.

Theorem 12. *In the above notation,*

(1) *If* $c(G/H) = 0$*, then* $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}) \leq 1$ *for all* λ *and* $\mathcal{L}$.
(2) *If* $c(G/H) = 1$*, then there exists a* G*-line bundle* $\mathcal{L}_0$ *and a dominant weight* λ_0 *such that* $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}) = n + 1$*, where* n *is the maximal integer such that* $\mathcal{L} = \mathcal{L}_0^n \otimes \mathcal{M}$*,* $\lambda = n\lambda_0 + \mu$*,* $H^0(G/H, \mathcal{M})_{(\mu)} \neq 0$.

Proof. In the case $c = 0$, assuming the contrary yields two non-proportional B-eigenvectors $s_0, s_1 \in H^0(G/H, \mathcal{L})$ of the same weight. Hence $f = s_1/s_0 \in \mathbb{C}(G/H)^B$, $f \neq \text{const}$, a contradiction.

In the case $c = 1$, we have $c(G/H)^B \simeq \mathbb{C}(\mathbb{P}^1)$ by the Lüroth theorem. Consider the respective rational map $\pi : G/H \dashrightarrow \mathbb{P}^1$, whose generic fibres are (the closures of) generic B-orbits. In a standard way, π is given by two B-eigenvectors $s_0, s_1 \in H^0(G/H, \mathcal{L}_0)$ of the same weight λ_0 for a certain line bundle $\mathcal{L}_0$. Moreover, s_0, s_1 are algebraically independent, and each $f \in \mathbb{C}(G/H)^B$ can be represented as a homogeneous rational fraction in s_0, s_1 of degree 0.

Now fix $s_\mu \in H^0(G/H, \mathcal{M})^U_{(\mu)}$ and take any $s_\lambda \in H^0(G/H, \mathcal{L})^U_{(\lambda)}$. Then $f = s_\lambda/s_0^n s_\mu \in \mathbb{C}(G/H)^B$, whence $f = F_1/F_0$ for some m-forms F_0, F_1 in s_0, s_1. We may assume the fraction to be reduced and decompose $F_1 = L_1 \dots L_m$, $F_0 = M_1 \dots M_m$, as products of linear forms, with all L_i distinct from all M_j. Then $s_\lambda M_1 \dots M_m = s_\mu s_0^n L_1 \dots L_m$. Being fibres of π, the divisors of s_0, L_i, M_j on G/H either coincide or have no common components. By the definition of $\mathcal{M}$, the divisor of s_μ does not dominate any one of M_j. Therefore $M_1 = \dots = M_m = s_0$, $m \leq n$, and s_λ/s_μ is an n-form in s_0, s_1. The assertion follows. □

Remark. The algebraic interpretation of complexity in terms of growth of multiplicities is well-known, see versions of Theorem 11 in [Pan2] (multiplicities in $\mathbb{C}[G/H]$ for G/H quasiaffine and $\mathbb{C}[G/H]$ finitely generated) and [Bri5, 1.3] (multiplicities in coordinate algebras for affine varieties and in section spaces of line bundles for projective varieties). Part 1 of Theorem 12 is due to Vinberg and Kimelfeld [VK], and Part 2 for finitely generated coordinate algebras of quasiaffine homogeneous spaces was handled by Panyushev [Pan2].

1.6. **Double flag varieties.** We illustrate the method of computing complexity and rank at double flag varieties, which are of importance in representation theory (cf. 3.6).

Let $P, Q \subseteq G$ be two parabolics. The product $X = G/P \times G/Q$ of the two respective (generalized) flag varieties is called a *double flag variety*. We may assume that P, Q are the projective stabilizers of lowest weight vectors v, w in G-modules V, W, respectively. Consider the Levi decompositions $P = L \cdot U_P$, $Q = M \cdot U_Q$ such that $L, M \supseteq T$. The following theorem is due to Panyushev.

Theorem 13 ([Pan3]). *Let* S *be the stabilizer in* $L \cap M$ *of a generic point in* $(\mathfrak{l} + \mathfrak{m})^\perp \simeq (\mathfrak{u}_P \cap \mathfrak{u}_Q) \oplus (\mathfrak{u}_P \cap \mathfrak{u}_Q)^*$*. Then*

$$2c(X) + r(X) = 2\dim(U_P \cap U_Q) - \dim(L \cap M) + \dim S \tag{11}$$
$$= \dim G - \dim L - \dim M + \dim S$$

$$r(X) = \operatorname{rk} G - \operatorname{rk} S \tag{12}$$

and also

$$\Lambda(X) = \Lambda(T/(T \cap S)) \tag{13}$$

provided $S \hookrightarrow L \cap M$ *is the standard embedding.*

Proof. Let $U_P^+, U_Q^+, U_{P\cap Q}^+$ be the unipotent radicals of the parabolics opposite to $P, Q, P \cap Q$. We have a decomposition $U_{P\cap Q}^+ = (U_P^+ \cap U_Q^+) \cdot (L \cap U_Q^+) \cdot (U_P^+ \cap M)$.

Consider the Segre embedding $X \simeq G\langle v\rangle \times G\langle w\rangle \subseteq \mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W)$. Choose highest weight covectors $v^* \in V^*$, $w^* \in W^*$ such that $\langle v, v^*\rangle, \langle w, w^*\rangle \neq 0$. By 1.1, we may restrict our attention to $\mathring{X} = X \setminus \mathbb{P}(\langle v^* \otimes w^*\rangle^\perp) = U_P^+\langle v\rangle \times U_Q^+\langle w\rangle$.

By the above decomposition, $\mathring{X} \simeq U_{P\cap Q}^+ \times (U_P^+ \cap U_Q^+)\langle v \otimes w\rangle$ is an $(L \cap M)$-equivariant isomorphism. (This is nothing else but the local structure of X provided by Theorem 1.) Therefore the complexity, rank and weight lattice for the actions $G : X$ and $(L \cap M) : (U_P^+ \cap U_Q^+)$ are the same. The latter action is isomorphic to the linear representation of $L \cap M$ in $(\mathfrak{u}_P \cap \mathfrak{u}_Q)^*$, and we may apply Theorems 2,7, and their corollaries. This yields (12), (13), and the first equality in (11), whereupon the second equality is derived by a simple dimension count. $\square$

Examples:

8. Let $G = \mathrm{GL}_n(\mathbb{C})$, $P = Q =$ the stabilizer of a line in $\mathbb{C}^n$; we may assume this line to be spanned by e_n, the last vector of the standard basis. Then $X = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. Here $L = M = \mathrm{GL}_{n-1} \times \mathbb{C}^\times$, and $(\mathfrak{l} + \mathfrak{m})^\perp \simeq (\mathbb{C}^{n-1})^* \oplus \mathbb{C}^{n-1}$, where GL_{n-1} acts on $\mathbb{C}^{n-1}$ in the standard way and $\mathbb{C}^\times$ acts by homotheties.

One easily finds $S = \{\mathrm{diag}(A, t, t) \mid A \in \mathrm{GL}_{n-2},\ t \in \mathbb{C}^\times\}$. (We choose one of the two possible standard embeddings $S \hookrightarrow L \cap M$ by observing the existence of a highest weight linear function on $(\mathfrak{u}_P \cap \mathfrak{u}_Q)^* \simeq \mathbb{C}^{n-1}$ of weight $-\epsilon_{n-1} + \epsilon_n$, where ϵ_i are the T-weights of the standard basic vectors e_i.) It follows that $\Lambda(X) = \langle \epsilon_{n-1} - \epsilon_n\rangle$ is generated by the last simple root, $r(X) = 1$, and $2c(X) + r(X) = n^2 - 2((n-1)^2 + 1) + (n-2)^2 + 1 = 1$, whence $c(X) = 0$.

9. Let $G = \mathrm{Sp}_n(\mathbb{C})$, P be the stabilizer of a line in $\mathbb{C}^n$, and Q be the stabilizer of a Lagrangian subspace in $\mathbb{C}^n$. Choose a symplectic basis $e_1, \dots, e_n$ such that $(e_i, e_j) = \mathrm{sgn}(j - i)$ whenever $i + j = n + 1$, and 0, otherwise. We may assume that the above line is $\langle e_n\rangle$, and the Lagrangian subspace is $\langle e_{l+1}, \dots, e_n\rangle$, $n = 2l$. Then $X = \mathbb{P}^{n-1} \times \mathrm{LGr}(\mathbb{C}^n)$, where LGr denotes the Lagrangian Grassmannian. Here $L \cap M = \mathrm{GL}_1 \times \mathrm{GL}_{l-1}$, and $(\mathfrak{l} + \mathfrak{m})^\perp \simeq \big((\mathbb{C}^1 \otimes \mathbb{C}^{l-1}) \oplus (\mathbb{C}^1)^{\otimes 2}\big)^* \oplus \big((\mathbb{C}^1 \otimes \mathbb{C}^{l-1}) \oplus (\mathbb{C}^1)^{\otimes 2}\big)$, where $\mathbb{C}^k$ is the standard representation of GL_k $(k = 1, l-1)$.

Now the same reasoning as in Example 5 shows that $S = \{\pm\,\mathrm{diag}(1, A, 1) \mid A \in \mathrm{GL}_{l-2}\} \subset M = \mathrm{GL}_l$. It follows that $\Lambda(X) = \langle \epsilon_1 + \epsilon_l, \epsilon_1 - \epsilon_l\rangle$, where ϵ_i are the eigenweights of e_i, $i = 1, \dots, l$, w.r.t. the standard diagonal maximal torus $T \subset \mathrm{Sp}_n$. Therefore $r(X) = 2$, and $2c(X) + r(X) = 2l - (l-1)^2 - 1 + (l-2)^2 = 2$, whence $c(X) = 0$.

2. Embedding theory

The general theory of equivariant embeddings of homogeneous spaces was constructed by Luna and Vust in the seminal paper [LV]. It is rather abstract, and we present here only the most important results, required in the sequel, skipping complicated and/or technical proofs. In our exposition, we follow [Tim1], where the Luna–Vust theory is presented in a more compact way (and generalized to non-homogeneous varieties).

Further on, a (G-*equivariant*) *embedding* of G/H is a *normal* algebraic variety X equipped with a G-action and containing an open dense orbit isomorphic to G/H. More precisely, we fix an open embedding $G/H \hookrightarrow X$.

2.1. **Uniform study of embeddings.** The first thing to do is to patch together all embeddings of G/H in a huge prevariety $\mathbb{X}$. Geometrically, we patch any two embeddings X_1, X_2 of G/H along their largest isomorphic G-stable open subsets $\mathring{X}_1 \simeq \mathring{X}_2$. Algebraically, we consider the collection of all local rings $(\mathcal{O}, \mathfrak{m})$ that are localizations at maximal ideals of $\mathfrak{g}$-stable finitely generated subalgebras $R \subset \mathbb{C}(G/H)$ with $\operatorname{Quot} R = \mathbb{C}(G/H)$. We identify these local rings with points of $\mathbb{X}$. The Zariski topology is given by basic affine open subsets formed by all $(\mathcal{O}, \mathfrak{m})$ that are localizations of a given R, with the obvious structure sheaf. From this point of view, an embedding of G/H is just a Noetherian separated G-stable open subset $X \subset \mathbb{X}$.

Next important thing is to observe that an embedding $X \hookleftarrow G/H$ is uniquely determined by the collection of germs of G-stable subvarieties in X. To make this assertion precise, introduce a natural equivalence relation on the set of G-stable subvarieties in $\mathbb{X}$: $Y_1 \sim Y_2$ if $\overline{Y_1} = \overline{Y_2}$. Considering a subvariety up to equivalence means that we are interested only in its generic points. Equivalence classes are called G-*germs* (of embeddings along subvarieties). G-germs (of embeddings X along subvarieties Y) are determined by the local rings $\mathcal{O}_{X,Y}$, which are just G- and $\mathfrak{g}$-stable local rings of finite type in $\mathbb{C}(G/H)$. It is clear that X is determined by the collection of G-germs along subvarieties intersecting X.

2.2. **Invariant valuations and colours.** Germs along G-stable prime divisors $D \subset \mathbb{X}$ are of particular importance. The respective local rings $\mathcal{O}_{\mathbb{X},D} = \mathcal{O}_v$ are discrete valuation rings corresponding to G-invariant discrete geometric valuations v of $\mathbb{C}(G/H)$. (A valuation is said to be *geometric* if its valuation ring is the local ring of a prime divisor.) For $v = \operatorname{ord}_D$ the value group is $\mathbb{Z}$, but sometimes it is convenient to multiply v by a positive rational constant. The set of G-*valuations* (= G-invariant discrete $\mathbb{Q}$-valued geometric valuations) of $\mathbb{C}(G/H)$ is denoted by $\mathcal{V}$.

B-stable prime divisors of G/H are also called *colours*. The set of colours is denoted by $\mathcal{D}$. We say that the pair $(\mathcal{V}, \mathcal{D})$ is the *coloured data of* G/H. It is in terms of coloured data that embeddings of G/H are described.

Lemma 3. *G-valuations are uniquely determined by restriction to B-semi-invariant functions.*

Proof. We prove it in the quasiaffine case. The general case is more or less reduced to the quasiaffine one, cf. [LV, 7.4]. For quasiaffine G/H, any

$v \in \mathcal{V}$ is determined by a G-stable decreasing filtration $\mathbb{C}[G/H]_{v \geq c} = \{f \in \mathbb{C}[G/H] \mid v(f) \geq c\}$, $c \in \mathbb{Q}$, of the coordinate algebra.

Take any $w \in \mathcal{V}$, $w \neq v$. Without loss of generality we may assume that $\mathbb{C}[G/H]_{v \geq c} \not\subseteq \mathbb{C}[G/H]_{w \geq c}$ for a certain c. Consider a G-stable decomposition $\mathbb{C}[G/H]_{v \geq c} = \mathbb{C}[G/H]_{v,w \geq c} \oplus M$, $M \neq 0$, and choose a highest weight vector $f \in M$. Then $v(f) \geq c > w(f)$, q.e.d. □

Clearly, the value $v(f)$ of a geometric valuation at a function does not change if we multiply f by a constant. Thus G-valuations are determined by their restrictions to the multiplicative group $\mathcal{A}$ of B-semiinvariant rational functions on G/H regarded up to a scalar multiple.

Similarly, colours are mapped (by restriction of the respective valuation) to additive functions on $\mathcal{A}$, but this map is no longer injective in general.

It is natural to think of G-valuations and (the images of) colours as elements of the "linear dual" of $\mathcal{A}$. We shall see evidences of this principle in Sections 3,4, and reflect it in the notation by writing $\langle v, f\rangle = v(f)$, $\langle D, f\rangle = \operatorname{ord}_D(f)$, $\forall v \in \mathcal{V}$, $D \in \mathcal{D}$, $f \in \mathcal{A}$.

The following result of Knop is helpful in studying properties of G-valuations and colours by restricting to $\mathcal{A}$.

Lemma 4 ([Kn3])**.** *Fix $v \in \mathcal{V}$. For any $f \in \mathbb{C}(G/H)$ having B-stable divisor of poles, there exists $\tilde{f} \in \mathcal{A}$ such that:*

$$\begin{cases} \langle v, \tilde{f}\rangle = v(f) & \\ \langle w, \tilde{f}\rangle \geq w(f), & \forall w \in \mathcal{V} \\ \langle D, \tilde{f}\rangle \geq \operatorname{ord}_D(f), & \forall D \in \mathcal{D} \end{cases}$$

2.3. ***B*-charts.** In the study of manifolds it is natural to utilize coverings by "simple" local charts. In our situation, this principle leads to the following

Definition 3. A *B-chart* is a B-stable affine open subvariety $\mathring{X} \subset \mathbb{X}$. An embedding $X \hookleftarrow G/H$ is said to be *simple* if $X = G\mathring{X}$.

The ubiquity of B-charts is justified by the following

Lemma 5. *Given a normal G-variety X and a G-stable subvariety $Y \subseteq X$, there exists a B-stable affine open subvariety $\mathring{X} \subseteq X$ meeting Y.*

Proof. By Sumihiro's theorem (see e.g. [KKLV]), Y intersects a G-stable quasiprojective open subset of X. Shrinking X if necessary, we may assume it to be quasiprojective. Passing to the projective closure, we may assume without loss of generality that $X \subseteq \mathbb{P}(V)$ is a projective variety and $Y = G\langle v\rangle$ is the (closed) projectivized orbit of a lowest weight vector. Now in the notation of 1.1, it suffices to take $\mathring{X} = X \cap \mathbb{P}(\mathring{V})$. □

Theorem 14.

1. *Any B-chart $\mathring{X}$ determines a simple embedding $X = G\mathring{X} \subset \mathbb{X}$.*
2. *Any embedding is covered by finitely many simple embeddings.*

Proof. For (1) it suffices to verify that X is Noetherian and separated. Being the image of $G \times \mathring{X}$ under the action morphism, X is Noetherian. Assuming X is not separated, i.e., $\operatorname{diag} X$ is not closed in $X \times X$, we take a G-orbit in $Y \subseteq \overline{\operatorname{diag} X} \setminus \operatorname{diag} X$. Then Y intersects the two open subsets $\mathring{X} \times X$ and

$X \times \mathring{X}$ of $X \times X$. But $Y \cap (\mathring{X} \times X) \cap (X \times \mathring{X}) = Y \cap (\mathring{X} \times \mathring{X}) = \emptyset$ since $\mathring{X}$ is separated, a contradiction.

For (2) it suffices to note that any G-stable subvariety $Y \subset X$ intersects a certain B-chart, whence X is covered by simple embeddings, and it remains to choose a finite subcover. □

Being a normal affine variety, a B-chart $\mathring{X}$ is determined by its coordinate algebra $R = \mathbb{C}[\mathring{X}]$, so that

$$R = \bigcap_{D,\ BD \neq D} \mathcal{O}_D \cap \bigcap_{D \in \mathcal{F}} \mathcal{O}_D \cap \bigcap_{w \in \mathcal{W}} \mathcal{O}_w \tag{14}$$

is a finitely generated Krull ring with $\operatorname{Quot} R = \mathbb{C}(G/H)$. Here $\mathcal{W}$ is the set of G-valuations corresponding to G-stable prime divisors intersecting $\mathring{X}$, $\mathcal{F}$ is the set of colours intersecting $\mathring{X}$, and the first intersection runs over all non-B-stable prime divisors in G/H. The pair $(\mathcal{W}, \mathcal{F})$ is said to be the *coloured data of* $\mathring{X}$.

Conversely, consider arbitrary subsets $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{F} \subseteq \mathcal{D}$, and introduce an equivalence relation on the set of pairs: $(\mathcal{W}, \mathcal{F}) \sim (\mathcal{W}', \mathcal{F}')$ if $\mathcal{W}$ differs from $\mathcal{W}'$ and $\mathcal{F}$ from $\mathcal{F}'$ by finitely many elements. Clearly, the coloured data of all B-charts lie in a distinguished equivalence class, denoted by **CD**.

Theorem 15. *Suppose* $(\mathcal{W}, \mathcal{F}) \in \mathbf{CD}$*; then:*

(1) *The algebra R defined by Formula (14) is a Krull ring.*
(2) $\operatorname{Quot} R = \mathbb{C}(G/H)$ *iff*

(C) $\forall \mathcal{W}_0 \subseteq \mathcal{W},\ \mathcal{F}_0 \subseteq \mathcal{F},\ \mathcal{W}_0, \mathcal{F}_0$ *finite*,
$$\exists f \in \mathcal{A},\ \langle \mathcal{W}, f \rangle \geq 0,\ \langle \mathcal{F}, f \rangle \geq 0,\ \langle \mathcal{W}_0, f \rangle > 0,\ \langle \mathcal{F}_0, f \rangle > 0$$

(3) *R is finitely generated iff*

(F) $$R^U \text{ is finitely generated}$$

(4) *A valuation $v \in \mathcal{W}$ is essential for R iff*

(W) $$\exists f \in \mathcal{A},\ \langle \mathcal{W} \setminus \{v\}, f \rangle \geq 0,\ \langle \mathcal{F}, f \rangle \geq 0,\ \langle v, f \rangle < 0$$

(5) *All the valuations* ord_D *corresponding to $D \in \mathcal{F}$ are essential for R.*

Corollary. $(\mathcal{W}, \mathcal{F})$ *is the coloured data of a B-chart iff the conditions* (C), (F), (W) *are satisfied.*

Proof. Claim (1) stems from the simple observation that the set of defining valuations for R differs from that of $\mathbb{C}[\mathring{X}]$ by finitely many elements, where $\mathring{X}$ is any B-chart.

(2) If $\operatorname{Quot} R = \mathbb{C}(G/H)$, then there exists $f \in R$ such that $w(f) > 0$, $\operatorname{ord}_D(f) > 0, \forall w \in \mathcal{W}_0,\ D \in \mathcal{F}_0$. Replacing f by $\tilde{f}$ from Lemma 4 yields (C).

Conversely, suppose that (C) holds, and take any $h \in \mathbb{C}(G/H)$. We have $h = h_1/h_0$ for some $h_i \in \mathbb{C}[\mathring{X}]$, where $\mathring{X}$ is an arbitrary B-chart. Let $\mathcal{W}_0$ be the set of valuations that are negative at h_0, and $\mathcal{F}_0$ given by the poles of h_0. Then $h_0 f^N \in R$ for $N \gg 0$; similarly for h_1. Thus $h \in \operatorname{Quot} R$.

Claim (3) is well known in the case $\mathcal{F} = \mathcal{D}$, i.e., whenever R is G-stable [Kr, III.3.1–2]. The general case is reduced to this one by a tricky argument [Tim1, 1.4].

(4) If v is essential, then there exists $f \in \mathbb{C}(G/H)$ with B-stable poles such that $v(f) < 0$, $w(f) \geq 0$, $\operatorname{ord}_D f \geq 0$, $\forall w \in \mathcal{W} \setminus \{v\}$, $D \in \mathcal{F}$. Replacing f by $\tilde{f}$ from Lemma 4 yields (W).

Conversely, if (W) holds, then obviously v cannot be removed from the l.h.s. of Formula (14), i.e., it is essential for R.

(5) Take a G-line bundle $\mathcal{L} \to G/H$ and a section $s \in H^0(G/H, \mathcal{L})$, whose divisor is a multiple of D. Put $f = gs/s$, where $g \in G$, $gD \neq D$. Then $\operatorname{ord}_{D'} f \geq 0$, $\forall D' \subset G/H$, $D' \neq D$, and $v(f) = 0$, $\forall v \in \mathcal{V}$ (because v can be extended G-invariantly to sections of line bundles [LV, 3.2], [Kn3, §3]), but $\operatorname{ord}_D f < 0$. Thus D cannot be removed from Formula (14). ☐

2.4. **G-germs.** Now we study G-germs of a simple embedding $X = G\mathring{X}$, i.e., G-germs intersecting the B-chart $\mathring{X}$. Let $(\mathcal{W}, \mathcal{F})$ be the coloured data of $\mathring{X}$.

Definition 4. The *support* $\mathcal{S}_Y$ of a G-germ along Y is the set of G-valuations having centre Y.

The support is nonempty, which can be seen by blowing up Y, normalizing, and taking the valuation corresponding to a component of the exceptional divisor. Each G-subvariety $Y \subset \mathbb{X}$ intersects a certain simple embedding X, and any valuation has at most one centre in X by the separation axiom, hence the G-germ along Y is determined by the triple $(\mathcal{W}, \mathcal{F}, \mathcal{S}_Y)$.

There is also an intrinsic way to characterize G-germs regardless of simple embeddings. Let $\mathcal{V}_Y$ be the set of G-valuations corresponding to G-stable divisors containing Y, and $\mathcal{D}_Y = \{D \in \mathcal{D} \mid \overline{D} \supset Y\}$. The pair $(\mathcal{V}_Y, \mathcal{D}_Y)$ is said to be the *coloured data of the G-germ.* Clearly, $\mathcal{V}_Y \subseteq \mathcal{W}$, $\mathcal{D}_Y \subseteq \mathcal{F}$.

Theorem 16. (1) *A G-valuation $v \in \mathcal{S}_Y$ for some $Y \subseteq X$ iff*

(V) $$\langle \mathcal{W}, f \rangle \geq 0,\ \langle \mathcal{F}, f \rangle \geq 0 \implies \langle v, f \rangle \geq 0, \qquad \forall f \in \mathcal{A}$$

(2) *Suppose $v \in \mathcal{S}_Y$, $w \in \mathcal{W}$, $D \in \mathcal{F}$; then:*

- *$D \in \mathcal{D}_Y$ iff*

(D′) $$\langle \mathcal{W}, f \rangle \geq 0,\ \langle \mathcal{F}, f \rangle \geq 0,\ \langle v, f \rangle = 0 \implies \langle D, f \rangle = 0, \qquad \forall f \in \mathcal{A}$$

- *$w \in \mathcal{V}_Y$ iff*

(V′) $$\langle \mathcal{W}, f \rangle \geq 0,\ \langle \mathcal{F}, f \rangle \geq 0,\ \langle v, f \rangle = 0 \implies \langle w, f \rangle = 0, \qquad \forall f \in \mathcal{A}$$

(3) *$v \in \mathcal{S}_Y$ iff*

(S) $$\langle \mathcal{V}_Y, f \rangle \geq 0,\ \langle \mathcal{D}_Y, f \rangle \geq 0 \implies \langle v, f \rangle \geq 0, \qquad \forall f \in \mathcal{A}$$
and $\langle v, f \rangle > 0$ whenever some of the l.h.s. inequalities are stirct

(4) *G-germs are uniquely determined by their coloured data.*

Proof. (1) A G-valuation v has a centre in X iff it has a centre in $\mathring{X}$ iff it is nonnegative on $\mathbb{C}[\mathring{X}]$, which implies (V). Conversely, if there exists $f \in \mathbb{C}[\mathring{X}]$, $v(f) < 0$, then replacing f by $\tilde{f}$ from Lemma 4 we see that (V) fails.

(2) By assumption, $\mathcal{O}_v$ dominates $\mathcal{O}_Y$. Assume $\overline{D} \supset Y$, and take $f \in \mathcal{A}$ satisfying the l.h.s. of (D′). Then f is invertible in $\mathcal{O}_v$, whence in $\mathcal{O}_Y$, and in $\mathcal{O}_D$ as well. This implies (D′).

On the other hand, if $\overline{D} \not\supset Y$, then $\exists f \in \mathbb{C}[\mathring{X}]$, $f = 0|_D$, $f \neq 0|_Y$, hence $v(f) = 0$. Applying Lemma 4, we see that (D′) fails.

A similar reasoning proves the second equivalence.

(3) Assume $v \in \mathcal{S}_Y$. If the l.h.s. inequalities hold, then the poles of f do not contain Y, whence $f \in \mathcal{O}_Y$ and $\langle v, f\rangle \geq 0$. If one of these inequalities is strict, then the zeroes of f contain Y, whence $\langle v, f\rangle > 0$. This implies (S).

Conversely, if $v \notin \mathcal{S}_Y$, then there exists $f \in \mathcal{O}_Y$ such that either $v(f) < 0$ or $f|_Y = 0$, $v(f) = 0$. Applying Lemma 4 again, we see that (S) fails.

(4) Consider the algebra R defined by Formula (14) with $\mathcal{W} = \mathcal{V}_Y$, $\mathcal{F} = \mathcal{D}_Y$. Then $\mathcal{O}_Y$ is the localization of R at the ideal given by the condition $v > 0$, $\forall v \in \mathcal{S}_Y$. But $\mathcal{S}_Y$ is determined by $(\mathcal{V}_Y, \mathcal{D}_Y)$. ☐

2.5. **Résumé.** Summing up, we can construct all embeddings $X \hookleftarrow G/H$ in the following way:

- Take a finite collection of coloured data $(\mathcal{W}_i, \mathcal{F}_i)$ satisfying (C), (F), (W). These coloured data determine B-charts $\mathring{X}_i$ and simple embeddings $X_i = G\mathring{X}_i$.
- Compute the coloured data $(\mathcal{V}_Y, \mathcal{D}_Y)$ of G-germs $Y \subseteq X_i$ using the conditions (V), (V′), (D′).
- Compute the supports $\mathcal{S}_Y$ using (S).
- Finally, simple embeddings X_i can be pasted together in an embedding X iff the supports $\mathcal{S}_Y$ are all disjoint, which stems from the following version of the valuative criterion of separation.

Theorem 17. *An open G-stable subset $X \subset \mathbb{X}$ is separated iff each G-valuation has at most one centre in X.*

Proof. If X is not separated, and $Y \subseteq \overline{\operatorname{diag} X} \setminus \operatorname{diag} X$ is a G-orbit, then the projections Y_i of Y to the copies of X $(i = 1, 2)$ are disjoint. Now any G-valuation having centre Y in $\overline{\operatorname{diag} X}$ has at least two centres Y_1, Y_2 in X. The converse implication stems from the usual valuative criterion of separation (involving all valuations). ☐

The above "combinatorial" description of embeddings looks rather cumbersome and inaccessible for practical use. However, we shall see in the sequel, that for homogeneous spaces of small complexity, this theory looks much nicer.

3. Spherical varieties

3.1. **Spherical homogeneous spaces.** The most elegant and deep theory can be developed for *spherical* homogeneous spaces, namely those of complexity 0. A homogeneous space G/H is spherical iff B has an open orbit in G/H. It should be noted that a number of classical varieties are in fact spherical: e.g. all examples in the introduction (except the first one), flag varieties, varieties of matrices of given rank, of complexes, symmetric spaces etc. Also the class of spherical homogeneous spaces is stable under degeneration.

The importance of this class of homogeneous spaces is also justified by a number of particularly nice properties characterizing them. Some of these properties are listed in

Theorem 18. *The following conditions are equivalent:*

(1) *B acts on G/H with an open orbit.*
(2) $\mathbb{C}(G/H)^B = \mathbb{C}$
(3) $\exists g \in G: \ \mathfrak{g} = \mathfrak{b} + \mathrm{Ad}(g)\mathfrak{h}$
(4) *For any G-line bundle $\mathcal{L} \to G/H$, the representation $G : H^0(G/H, \mathcal{L})$ is multiplicity free.*
(5) *(For quasiaffine G/H) The representation $G : \mathbb{C}[G/H]$ is multiplicity free.*

Proof. (1) $\iff$ (2) This holds by Rosenlicht's theorem.

(1) $\iff$ (3) $\mathfrak{b} + \mathrm{Ad}(g)\mathfrak{h}$ is the tangent space at e of $BgHg^{-1} \subseteq G$, the latter being a translate of the preimage of $B(gH) \subseteq G/H$.

(2) $\iff$ (4) $\iff$ (5) This follows from Theorems 11,12(1). ☐

3.2. **Embedding theory** [LV, 8.10], [Kn2], [Bri5], [Tim1, 1.7].

Definition 5. A *spherical variety* is an algebraic variety G-isomorphic to an embedding of a spherical homogeneous space G/H, i.e., a normal algebraic G-variety X containing an open orbit isomorphic to G/H.

We are going to apply the theory of Section 2 to spherical varieties.

As $\mathbb{C}(G/H)^B = \mathbb{C}$, any B-semiinvariant rational function of G/H is determined by its weight uniquely up to a scalar multiple. Therefore $\mathcal{A} = \Lambda(G/H)$, and G-valuations $v \in \mathcal{V}$ may be regarded as vectors in $\Lambda^*_{\mathbb{Q}} = \mathrm{Hom}(\Lambda, \mathbb{Q})$ given by $\langle v, \lambda \rangle = v(f_\lambda)$, $\forall \lambda \in \Lambda$, where f_λ is a function of weight λ. Colours $D \in \mathcal{D}$ are also mapped to vectors $v_D \in \Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$ given by $\langle v_D, \lambda \rangle = \mathrm{ord}_D(f_\lambda)$. Colours are just the components of the complement of the open B-orbit in G/H, whence $\mathcal{D}$ is finite.

Theorem 19. *G-valuations form a solid convex polyhedral cone $\mathcal{V} \subseteq \Lambda^*_{\mathbb{Q}}$* (valuation cone)*.*

Proof. We consider the quasiaffine case, the general case being reduced to this one. Since the G-module $\mathbb{C}[G/H]$ is multiplicity free, there is a unique G-stable complement of each G-stable subspace. Thus for $\forall v \in \mathcal{V}$, the filtration $\mathbb{C}[G/H]_{v \geq c}$ comes from a unique G-stable grading of $\mathbb{C}[G/H]$, the latter being given by the vector $v \in \Lambda^*_{\mathbb{Q}}$, so that $v(\mathbb{C}[G/H]_{(\lambda)}) = \langle v, \lambda \rangle$, $\forall \lambda \in \Lambda_+$.

Conversely, each $v \in \Lambda^*_{\mathbb{Q}}$ determines a G-stable grading and a decreasing filtration of $\mathbb{C}[G/H]$, and $v \in \mathcal{V}$ iff this filtration respects the multiplication. We have $\mathbb{C}[G/H]_{(\lambda)} \cdot \mathbb{C}[G/H]_{(\mu)} = \mathbb{C}[G/H]_{(\lambda+\mu)} \oplus \bigoplus_i \mathbb{C}[G/H]_{(\lambda+\mu-\beta_i)}$, $\forall \lambda, \mu \in \Lambda_+(G/H)$, where β_i are positive linear combinations of positive roots. Thus $v \in \mathcal{V}$ iff $\langle v, \beta_i \rangle \leq 0$, $\forall \lambda, \mu, \beta_i$.

These inequalities define a convex cone containing the image of the antidominant Weyl chamber. Brion and Pauer proved that $\mathcal{V}$ is polyhedral by constructing a projective "colourless" embedding, i.e., $X \hookleftarrow G/H$ such that $\mathcal{D}_Y = \emptyset$, $\forall Y \subset X$, see e.g. [Kn2, 5], [Bri5, 2.4]. (Then $\mathcal{V}$ is generated by finitely many vectors corresponding to G-stable divisors in X by Theorem 21(3) below.) Brion [Bri2] proved that $\mathcal{V}$ is even cosimplicial and is in fact a fundamental chamber of a certain crystallographic reflection group, called the *little Weyl group* of G/H. A nice geometric interpretation for this group in the spirit of 1.3 was found by Knop [Kn4]. □

Example 10. If G/H is horospherical, then $\mathcal{V} = \Lambda^*_{\mathbb{Q}}$. In particular, this is the case if $G = T$ is a torus. In the toric case, there are no colours, and we may also assume $H = \{e\}$ without loss of generality.

Now we reorganize coloured data in a more convenient way.

The class **CD** consists of the pairs of finite subsets. Take $(\mathcal{W}, \mathcal{F}) \in$ **CD** and consider the polyhedral cone $\mathcal{C}$ generated by $\mathcal{W}$ and (the image of) $\mathcal{F}$.

Condition (C) means that $\mathcal{C}$ is strictly convex, and no $D \in \mathcal{F}$ maps to 0.

Condition (F) is automatically satisfied, because R^U is just the semigroup algebra of $\mathcal{C}^\vee \cap \Lambda$, the semigroup of lattice points in the dual cone, which is finitely generated by Gordan's lemma.

Condition (W) says that $\mathcal{W}$ is recovered from $(\mathcal{C}, \mathcal{F})$ as the set of generators of those edges of $\mathcal{C}$ which do not intersect $\mathcal{F}$.

Definition 6. A *coloured cone* is a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{C}$ is a strictly convex cone generated by $\mathcal{F} \subseteq \mathcal{D}$ and by finitely many vectors of $\mathcal{V}$, and $\mathcal{F} \not\ni 0$. The coloured cone is said to be *supported* if $(\operatorname{int} \mathcal{C}) \cap \mathcal{V} \neq \emptyset$.

Thus B-charts are in bijection with coloured cones. Let us consider G-germs of the simple embedding X spanned by the B-chart $\mathring{X}$ given by a coloured cone $(\mathcal{C}, \mathcal{F})$.

Condition (V) means simply that $v \in \mathcal{C}$.

Conditions (V$'$) and (D$'$) say that $\mathcal{V}_Y$, $\mathcal{D}_Y$ consist of those elements of $\mathcal{W}$, $\mathcal{F}$, respectively, which lie in the face $\mathcal{C}_Y \subseteq \mathcal{C}$ such that $v \in \operatorname{int} \mathcal{C}_Y$.

Condition (S) means that $v \in \mathcal{V} \cap \operatorname{int} \mathcal{C}_Y$.

Thus G-germs are in bijection with supported coloured cones.

Definition 7. A *face* of a coloured cone $(\mathcal{C}, \mathcal{F})$ is a coloured cone $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$, and $\mathcal{F}' = \mathcal{F} \cap \mathcal{C}'$.

A *coloured fan* is a finite collection of supported coloured cones which is closed under passing to supported faces and such that different cones intersect along faces inside $\mathcal{V}$.

The arguments of 2.5 yield

Theorem 20. *Spherical embeddings are in bijection with coloured fans.*

Amazingly, a lot of geometry of a spherical variety can be read off its coloured fan. We illustrate this principle by the following result.

Theorem 21. *Let X be a spherical variety.*

(1) *The G-orbits $Y \subseteq X$ are in bijection with the coloured cones in the respective coloured fan. Moreover, $Y \subset \overline{Y'}$ iff $(\mathcal{C}_{Y'}, \mathcal{D}_{Y'})$ is a face of $(\mathcal{C}_Y, \mathcal{D}_Y)$.*
(2) *X is affine iff its fan is formed by all supported faces of a coloured cone $(\mathcal{C}, \mathcal{D})$.*
(3) *X is complete iff its fan covers the valuation cone.*

Proof. (1) It follows from the above that there are finitely many germs along G-subvarieties in X, whence each G-subvariety contains a dense orbit. If $Y \subset \overline{Y'}$, then $\mathcal{V}_Y \supseteq \mathcal{V}_{Y'}$, $\mathcal{D}_Y \supseteq \mathcal{D}_{Y'}$, hence $(\mathcal{C}_{Y'}, \mathcal{D}_{Y'})$ is a face of $(\mathcal{C}_Y, \mathcal{D}_Y)$.

Conversely, suppose $Y \not\subseteq \overline{Y'}$, and take $v \in \mathcal{S}_Y = (\operatorname{int} \mathcal{C}_Y) \cap \mathcal{V}$. There exists $f \in \mathbb{C}[\mathring{X}]$ such that $f|_{Y'} = 0$, $f|_Y \neq 0$, whence $v(f) = 0$. Applying Lemma 4, we replace f by a B-eigenfunction f_λ, and obtain $\langle v, \lambda \rangle = 0$, whence $\langle \mathcal{C}_Y, \lambda \rangle = 0$, but $\langle v', \lambda \rangle > 0$, $\forall v' \in (\operatorname{int} \mathcal{C}_{Y'}) \cap \mathcal{V}$. Therefore $\mathcal{C}_{Y'}$ is not a face of $\mathcal{C}_Y$.

(2) X is affine iff X is a G-stable B-chart, i.e., $\mathcal{D}$ is the set of colours of X.

(3) If the fan of X does not cover $\mathcal{V}$, then it is easy to construct an open embedding $X \hookrightarrow \overline{X}$ by adding more coloured cones (e.g. one ray in $\mathcal{V}$) to the fan. Conversely, if X is non-complete, we choose a G-equivariant completion $X \hookrightarrow \overline{X}$ and take any orbit $Y \subseteq \overline{X} \setminus X$. Then $\mathcal{S}_Y$ is not covered by the fan of X. $\square$

Corollary (Servedio). *Any spherical variety has finitely many orbits.*

It is instructive to deduce this assertion directly from the multiplicity-free property, see e.g. [Bri5, 2.1].

Examples:

- The (well-known) toric varieties [Dan], [Ful] are nothing else but spherical embeddings of algebraic tori. Since there are no colours in this case, toric varieties are classified by usual fans, i.e., collections of strictly convex rational polyhedral cones intersecting along faces, which are closed under passing to faces.
- Complete symmetric varieties [CP1], [CP2] are certain compact embeddings of homogeneous symmetric spaces.
- Determinantal varieties are affine embeddings of spaces of matrices with given rank.

Example 11. Consider the space of plane conics Q_2 acted on by $G = PGL_3$. The smooth conics in $\mathbb{P}^2$ are represented by non-degenerate symmetric (3×3)-matrices of the respective quadratic forms: a matrix q determines a conic by the equation $x^\top q x = 0$ (x is a vector of projective coordinates). Let $\Delta_i(q)$ be the upper-left corner i-minor of q ($i = 1, 2, 3$).

We have seen in Example 3 that Q_2 is spherical and $\Lambda = 2\Lambda_{\text{ad}} = \langle 2\alpha_1, 2\alpha_2 \rangle$, where α_i are the simple roots. We may take $f_{2\alpha_1} = \Delta_1^2/\Delta_2$, $f_{2\alpha_2} = \Delta_2^2/\Delta_1\Delta_3$. There are the two colours: D_1 consists of conics passing through the B-fixed point, and D_2 of those tangent to the B-stable line, D_i being given by the equation $\Delta_i = 0$, whence $v_{D_i} = \alpha_i^\vee/2$, where $\alpha_i^\vee$ are the simple coroots.

Consider the embedding $Q_2 \hookrightarrow \mathbb{P}^5 = \{\text{all conics in } \mathbb{P}^2\}$. The boundary is the G-stable prime divisor D of singular conics, given by the equation $\Delta_3 = 0$, whence $v_D = -\omega_2^\vee/2$, where $\omega_i^\vee$ are the fundamental coweights. There are 3 orbits: the open one Q_2, the closed one $Y = \{\text{double lines}\}$, and $D \setminus Y = \{\text{pairs of distinct lines}\}$. We have $\mathcal{V}_Y = \{v_D\}$, $\mathcal{D}_Y = \{D_2\}$, hence $\mathcal{C}_Y$ is generated by $-\omega_2^\vee/2, \alpha_2^\vee/2$.

The dual embedding $Q_2 \hookrightarrow (\mathbb{P}^5)^* = \{\text{all conics in } (\mathbb{P}^2)^*\}$ is given by mapping each smooth conic in $\mathbb{P}^2$ to the dual one in $(\mathbb{P}^2)^*$ consisting of all lines tangent to the given conic. In coordinates, $q \mapsto q^\vee$, the adjoint matrix formed by the cofactors of the entries in q. All above considerations can be repeated, but the indices 1, 2 are interchanged. In particular, there is a unique G-stable divisor $D' \subset (\mathbb{P}^5)^*$ with $v_{D'} = -\omega_1^\vee/2$, and a unique closed orbit Y' with $\mathcal{C}_{Y'}$ generated by $-\omega_1^\vee/2, \alpha_1^\vee/2$.

By Theorem 21(3), $\mathcal{C}_Y, \mathcal{C}_{Y'} \supseteq \mathcal{V}$, whence $\mathcal{V} = \mathcal{C}_Y \cap \mathcal{C}_{Y'}$ is generated by $-\omega_1^\vee/2, -\omega_2^\vee/2$, i.e., $\mathcal{V}$ is the antidominant Weyl chamber.

Now consider the diagonal embedding $Q_2 \hookrightarrow \mathbb{P}^5 \times (\mathbb{P}^5)^*$ and let $X = \overline{Q_2}$ be the closure of its image. It is given by the equation $q \cdot q^* = \lambda E$ ($\lambda \in \mathbb{C}$), where q, q^* are nonzero symmetric (3×3)-matrices. It is easy to see that there are four orbits $Y_{ij} \subset X$ given by $(\operatorname{rk} q, \operatorname{rk} q^*) = (i, j) = (3,3)$, $(2,1)$, $(1,2)$, $(1,1)$, respectively. Differentiating the equation at a point of the unique closed orbit Y_{11}, one verifies that X is smooth. Since Y_{11} projects onto Y, Y', we have $\mathcal{C}_{Y_{11}} \subseteq \mathcal{C}_Y \cap \mathcal{C}_{Y'} = \mathcal{V}$, $\mathcal{D}_{Y_{11}} \subseteq \mathcal{D}_Y \cap \mathcal{D}_{Y'} = \emptyset$. But X is a complete simple embedding of Q_2, whence $(\mathcal{C}_{Y_{11}}, \mathcal{D}_{Y_{11}}) = (\mathcal{V}, \emptyset)$ by Theorem 21(3). The space X, called the space of *complete conics*, was first considered by Chasles (1864).

3.3. **Algebraic semigroups.** A nice application of the embedding theory 3.2 is the classification of reductive algebraic monoids, i.e., linear algebraic semigroups with unity whose groups of invertibles are reductive. The general study of algebraic semigroups was undertaken by Putcha and Renner, particular cases were classified by them. A complete classification of normal reductive monoids was developed by Vinberg [Vin]. It soon became clear that this classification can be easily derived from the embedding theory of spherical varieties. Rittatore [Rit] made this last step.

The point is that a reductive monoid X with unit group $G \subseteq X$ can be considered as a $(G \times G)$-variety, where the factors act by left/right multiplication. From this viewpoint, X is a $(G \times G)$-equivariant embedding of $G = (G \times G)/\operatorname{diag} G$.

Theorem 22 ([Vin], [Rit]). *X is an affine embedding of G. Conversely, any affine $(G \times G)$-embedding of G carries a structure of algebraic monoid with unit group G.*

Proof. The actions of the left and right copy of $G \times G$ on X define coactions $\mathbb{C}[X] \to \mathbb{C}[G] \otimes \mathbb{C}[X]$ and $\mathbb{C}[X] \to \mathbb{C}[X] \otimes \mathbb{C}[G]$, which are the restrictions to $\mathbb{C}[X] \subseteq \mathbb{C}[G]$ of the comultiplication $\mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$. Hence the image of $\mathbb{C}[X]$ lies in $(\mathbb{C}[G] \otimes \mathbb{C}[X]) \cap (\mathbb{C}[X] \otimes \mathbb{C}[G]) = \mathbb{C}[X] \otimes \mathbb{C}[X]$, and we have a comultiplication in $\mathbb{C}[X]$. Now G is open in X and consists of invertibles. For any invertible $x \in X$, we have $xG \cap G \neq \emptyset$, hence $x \in G$. $\square$

To apply 3.2, we have to determine the coloured data for $(G \times G)/\operatorname{diag} G$. This was done by Vust [Vu] in the more general context of symmetric spaces.

First, the isotypic decomposition of the coordinate algebra has the form $\mathbb{C}[G] = \bigoplus_{\lambda \in \Lambda_+} \mathbb{C}[G]_{(\lambda)}$, where Λ_+ is the set of dominant weights, and $\mathbb{C}[G]_{(\lambda)} \cong V_\lambda^* \otimes V_\lambda$ is the linear span of the matrix entries of the representation $G : V_\lambda$. It is convenient to choose the Borel subgroup $B^- \times B$ in $G \times G$. Thus Λ is naturally identified with $\Lambda(B)$.

Secondly, the valuation cone $\mathcal{V} \subseteq \Lambda_{\mathbb{Q}}^*$ is identified with the antidominant Weyl chamber. To see it, we recall the proof of Theorem 19. A vector

$v \in \Lambda^*_{\mathbb{Q}}$ determines a G-valuation iff $\langle v, \beta_i \rangle \le 0$ for all β_i which occur in the decompositions $\mathbb{C}[G]_{(\lambda)} \cdot \mathbb{C}[G]_{(\mu)} = \mathbb{C}[G]_{(\lambda+\mu)} \oplus \bigoplus_i \mathbb{C}[G]_{(\lambda+\mu-\beta_i)}$, $\forall \lambda, \mu \in \Lambda_+$. But $\mathbb{C}[G]_{(\lambda)} \cdot \mathbb{C}[G]_{(\mu)}$ is the linear span of the matrix entries of $G : V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus \bigoplus_i V_{\lambda+\mu-\beta_i}$, and all simple roots occur among β_i for generic λ, μ.

The colours are the Schubert subvarieties $D_j = \overline{B^- s_j B} \subset G$ of codimension 1, where s_j is the reflection along the simple root α_j in the Weyl group W. It is easy to see (e.g. from [Bri5, 3.1]) that $v_{D_j} = \alpha_j^\vee$ are the simple coroots.

From Theorems 21(2), 22 and other results of 3.2, we then deduce

Theorem 23. *Normal reductive monoids X are in bijection with strictly convex cones $\mathcal{C}(X) \subset \Lambda^*_{\mathbb{Q}}$ generated by all simple coroots and finitely many antidominant vectors. The set $\mathcal{C}(X)^\vee \cap \Lambda$ of lattice points in the dual cone consists of all highest weights of $\mathbb{C}[X]$, and determines $\mathbb{C}[X] \subseteq \mathbb{C}[G]$ completely.*

Remark. This is in terms of highest weights of the coordinate algebra that the classification of Vinberg was initially presented. The semigroup $\mathcal{C}(X)^\vee \cap \Lambda$ is formed by the highest weights of the representations $G \to \mathrm{GL}(V_\lambda)$ extendible to X. If we are interested in non-normal reductive monoids, then we have to replace $\mathcal{C}(X)^\vee \cap \Lambda$ by any finitely generated subsemigroup $S \subseteq \Lambda_+$ such that $\mathbb{Z}S = \Lambda$ and $\bigoplus_{\lambda \in S} \mathbb{C}[G]_{(\lambda)} \subseteq \mathbb{C}[G]$ is closed under multiplication, i.e., all highest weights $\lambda + \mu - \beta$ of $V_\lambda \otimes V_\mu$ belong to S whenever $\lambda, \mu \in S$. X is normal iff S is the semigroup of all lattice vectors in a polyhedral cone.

Definition 8. We say that $\lambda_1, \dots, \lambda_m$ *G-generate* S if S consists of all highest weights $k_1\lambda_1 + \dots + k_m\lambda_m - \beta$ of G-modules $V(\lambda_1)^{\otimes k_1} \otimes \dots \otimes V(\lambda_m)^{\otimes k_m}$, $k_1, \dots, k_m \in \mathbb{Z}_+$. (In particular any generating set G-generates S.)

It is easy to see that $X \hookrightarrow \operatorname{End} V$ iff the highest weights $\lambda_1, \dots, \lambda_m$ of $G : V$ G-generate S.

Lemma 6 ([Tim3, §2]). *$\mathbb{Q}_+ S = (\mathbb{Q}_+ W\{\lambda_1, \dots, \lambda_m\}) \cap C$, where $C = \mathbb{Q}_+\Lambda_+$ is the dominant Weyl chamber. (In other words, a multiple of each dominant vector in the weight polytope eventually occurs as a highest weight in a tensor power of V.)*

If $V = V_\lambda$ is irreducible, then the center of G acts by homotheties, whence $G = \mathbb{C}^\times \cdot G_0$, where G_0 is semisimple, $\Lambda \subseteq \mathbb{Z} \oplus \Lambda_0$ is a cofinite sublattice, Λ_0 being the weight lattice of G_0, and $\lambda = (1, \lambda_0)$. By Lemma 6, $\mathbb{Q}_+S$ is the intersection of $\mathbb{Q}_+(W\lambda)$ with the dominant Weyl chamber. Recently de Concini showed that $\mathbb{Q}_+(W\lambda) \cap \Lambda_+$ is G-generated by $(\operatorname{conv} W\lambda) \cap \Lambda_+$ [Con]. It follows that X is normal iff λ_0 is a minuscule weight [Con], [Tim3, §12].

Example 12. Let $G = \mathrm{GL}_n$, and $X = \mathrm{Mat}_n$ be the full matrix algebra. For B take the standard Borel subgroup of upper-triangular matrices. We have $\Lambda = \langle \epsilon_1, \dots, \epsilon_n \rangle$, where the ϵ_i are the diagonal matrix entries of B. We identify Λ with Λ^* via the inner product such that the ϵ_i form an orthonormal basis. Let $(k_1, \dots, k_n)$ denote the coordinates of $\lambda \in \Lambda_{\mathbb{Q}}$ w.r.t. this basis.

The upper-left corner i-minors Δ_i are highest weight vectors in $\mathbb{C}[X]$, and their weights $\epsilon_1 + \dots + \epsilon_i$ generate Λ. Put $D_i = \{x \in X \mid \Delta_i(x) = 0\}$. Then $\mathcal{D} = \{D_1, \dots, D_{n-1}\}$, $v_{D_i} = \epsilon_i - \epsilon_{i+1}$, $\forall i < n$, and D_n is the unique G-stable prime divisor, $v_{D_n} = \epsilon_n$. Therefore $\mathcal{C}(X) = \mathbb{Q}_+v_{D_1} + \dots + \mathbb{Q}_+v_{D_n} =$

$\{k_1 + \dots + k_i \geq 0,\ i = 1, \dots, n\}$, and $\mathcal{C}(X)^\vee = \{k_1 \geq \dots \geq k_n \geq 0\}$. Lattice vectors of $\mathcal{C}(X)^\vee$ are exactly the dominant weights of polynomial representations, and $S = \mathcal{C}(X)^\vee \cap \Lambda$ is generated by $\epsilon_1 + \dots + \epsilon_i$, $i = 1, \dots, n$, and G-generated by ϵ_1.

3.4. **Projective group compactifications.** Given a faithful representation $G : V$, we obtain a reductive monoid $X = \overline{G} \subseteq \operatorname{End} V$, whose weight semigroup S is G-generated by the highest weights of V. The projective counterpart of this situation is studied in [Tim3]: given a faithful projective representation $G : \mathbb{P}(V)$ with highest weights $\lambda_0, \dots, \lambda_m$, we examine the geometry of $X = \overline{G} \subseteq \mathbb{P}(\operatorname{End} V)$ in terms of the weight polytope $\mathcal{P} = \operatorname{conv} W\{\lambda_0, \dots, \lambda_m\}$ of V. Without loss of generality we may assume $V = V_{\lambda_0} \oplus \dots \oplus V_{\lambda_m}$. The affine situation can be regarded as a particular case of the projective one, since $\operatorname{End} V \hookrightarrow \mathbb{P}(\operatorname{End}(V \oplus \mathbb{C}))$ is an affine chart. To a certain extent, the projective case reduces to the affine case by taking the affine cone.

Theorem 24 ([Kap], [Tim3]). *$(G \times G)$-orbits $Y \subset X$ are in bijection with the faces $\Gamma \subseteq \mathcal{P}$ such that $(\operatorname{int}\Gamma) \cap C \neq \emptyset$. They are represented by $y = \langle e_\Gamma \rangle$, where e_Γ is the projector of V onto the sum of T-eigenspaces of weights in Γ. The cone $\mathcal{C}_Y$ is dual to the cone of $\mathcal{P} \cap C$ at the face $\Gamma \cap C$, and $\mathcal{D}_Y$ consists of simple coroots orthogonal to $\langle \Gamma \rangle$.*

Remark. One can also describe the stabilizers $(G \times G)_y$ [Tim3, §9].

Proof. It is easy to see that the points $y = \langle e_\Gamma \rangle$ are limits of 1-parameter subgroups in T, whence $y \in \overline{T}$. Moreover, one deduces from elementary toric geometry that wy ($w \in W$) represent all T-orbits in $\overline{T}$, because $w\Gamma$ run over all faces of $\mathcal{P}$.

Recall the Cartan decomposition $G = KTK$, where $K \subset G$ is a maximal compact subgroup. Hence $X = K\overline{T}K$, and therefore y represent all $(G \times G)$-orbits $Y \subset X$. In particular, closed $(G \times G)$-orbits $Y_i \subset X$ correspond to the dominant vertices $\lambda_i \in \mathcal{P}$, and the representatives are $y_i = \langle v_{\lambda_i} \otimes v^*_{-\lambda_i} \rangle$, where $v_{\lambda_i} \in V$ is a highest weight vector, and $v^*_{-\lambda_i} \in V^*$ the dual lowest weight vector.

Take one of these vertices, say λ_0, and consider the parabolic $P = P(\lambda_0) = L \cdot U_P$. There is an L-stable decomposition $V = \langle v_{\lambda_0} \rangle \oplus V_0$. Let $\mathring{X} = X \cap \mathbb{P}\big((\operatorname{End} V) \setminus \langle v^*_{-\lambda_0} \otimes v_{\lambda_0} \rangle^\perp\big)$. Here is a (projectivized) version of Theorem 1:

Lemma 7. *$\mathring{X} \simeq U_P^- \times U_P \times Z$, where $Z \simeq \overline{L} \subseteq \operatorname{End}(V_0 \otimes \mathbb{C}_{-\lambda_0})$, and $y_0 \in \mathring{X}$ corresponds to $0 \in Z$.*

Proof of the Lemma. By Theorem 1, the affine chart $\mathring{X}$ has the above structure with $Z = X \cap \mathbb{P}\big(\mathbb{C}^\times (v_{\lambda_0} \otimes v^*_{-\lambda_0}) \oplus W_0\big)$, where $W_0 = (\mathfrak{g} \times \mathfrak{g})\big(v^*_{-\lambda_0} \otimes v_{\lambda_0}\big)^\perp = \big(\mathfrak{g}v^*_{-\lambda_0} \otimes v_{\lambda_0} + v^*_{-\lambda_0} \otimes \mathfrak{g}v_{\lambda_0}\big)^\perp \supseteq V_0 \otimes V_0^* = \operatorname{End} V_0$. Hence $Z = \overline{(L \times L)e} = \overline{L} \subseteq \mathbb{P}\big(\mathbb{C}^\times (v_{\lambda_0} \otimes v^*_{-\lambda_0}) \oplus \operatorname{End} V_0\big) \simeq \operatorname{End}(V_0 \otimes \mathbb{C}_{-\lambda_0})$. □

By Lemma 6, $\mathbb{Q}_+ S = (C \cap \mathcal{P})_{\lambda_0}$ is the cone of $C \cap \mathcal{P}$ at λ_0, and $\mathcal{C}_{Y_0} = (\mathbb{Q}_+ S)^\vee$ by 3.2. It is also clear that $D_j \ni y_0$ iff $\alpha_j \perp \lambda_0$. Thus Theorem 24 is proven for closed orbits, and the assertion for other orbits is deduced by passing to coloured faces, see details in [Tim3, §9]. □

Example 13. $X = \mathbb{P}(\mathrm{Mat}_n)$ is a projective embedding of $G = \mathrm{PGL}_n$. In the notation of Example 12, we have $\mathcal{P} = \mathrm{conv}\{\epsilon_1, \dots, \epsilon_n\}$, $\mathcal{P} \cap C = \{k_1 \geq \dots \geq k_n \geq 0,\ k_1 + \dots + k_n = 1\} = \mathrm{conv}\{(\epsilon_1 + \dots + \epsilon_i)/i \mid i = 1, \dots, n\}$, $\Gamma = \mathrm{conv}\{\epsilon_1, \dots, \epsilon_i\}$ $(i = 1, \dots, n)$, e_Γ is the projector onto the span of the first i basic vectors of $V = \mathbb{C}^n$, and $Y = \mathbb{P}(\text{matrices of rank } i)$ are the $(G \times G)$-orbits in X.

Finally, we give criteria of normality and smoothness of X. It clearly suffices to look at singularities at points of closed orbits.

Theorem 25. *In the above notation,*

1. *X is normal at points of Y_0 iff the weights $\lambda_1 - \lambda_0, \dots, \lambda_m - \lambda_0$ and negative simple roots $-\alpha_j \not\perp \lambda_0$ L-generate $\Lambda \cap (\mathcal{P} \cap C)_{\lambda_0}$.*
2. *X is smooth at points of Y_0 iff $L \simeq \mathrm{GL}_{n_1} \times \dots \times \mathrm{GL}_{n_p}$, the representation $(L : V_0 \otimes \mathbb{C}_{-\lambda_0})$ is polynomial and contains the minimal representations $(\mathrm{GL}_{n_i} : \mathbb{C}^{n_i})$ of factors of L.*

Proof. (1) X is normal along Y_0 iff Z is normal at 0 iff $\Lambda \cap (\mathcal{P} \cap C)_{\lambda_0}$ is L-generated by the highest weights $\mu_1, \dots, \mu_s$ of $(L : V_0 \otimes \mathbb{C}_{-\lambda_0})$. The weights $\lambda_1 - \lambda_0, \dots, \lambda_m - \lambda_0$, $-\alpha_j$ occur among them, being the highest weights of $v_{\lambda_1}, \dots, v_{\lambda_m}, e_{-\alpha_j} v_{\lambda_0} \in V_0$, where $e_{-\alpha_j} \in \mathfrak{g}$ are root vectors. But

$$V = \sum_{k,i} \underbrace{\mathfrak{p}^- \cdots \mathfrak{p}^-}_{k} v_{\lambda_i} = \sum_{n,i,j_1,\dots,j_n} \mathfrak{g}_{L,-\alpha_{j_1}} \cdots \mathfrak{g}_{L,-\alpha_{j_n}} \cdot V_{L,\lambda_i}$$

where $V_{L,\lambda_i} \subseteq V$, $\mathfrak{g}_{L,-\alpha_j} \subseteq \mathfrak{g}$ are simple L-modules generated by v_{λ_i}, $e_{-\alpha_j}$, respectively. The summands on the r.h.s. are quotients of $\mathfrak{g}_{L,-\alpha_{j_1}} \otimes \dots \otimes \mathfrak{g}_{L,-\alpha_{j_n}} \otimes V_{L,\lambda_i}$. Hence $\lambda_i - \lambda_0, -\alpha_j$ L-generate all remaining μ_k.

(2) Again it suffices to consider the smoothness of Z at 0. Z naturally embeds into $\bigoplus_{i=1}^s \mathrm{End}\, V_{L,\mu_i}$ and $T_0 Z = \bigoplus_{i=1}^p \mathrm{End}\, V_{L,\mu_i}$, $p \leq s$, after reordering μ_i. If Z is smooth, then the L-equivariant projection $Z \to T_0 Z$ is étale at 0 and in fact isomorphic by a weak version of Luna's fundamental lemma from the étale slice theory, see [Tim3, §3]. Now it is easy to conclude that $L \simeq \mathrm{GL}_{n_1} \times \dots \times \mathrm{GL}_{n_p}$, $Z \simeq \mathrm{Mat}_{n_1} \times \dots \times \mathrm{Mat}_{n_p}$, and μ_i $(i \leq p)$ are the highest weights of $(\mathrm{GL}_{n_i} : \mathbb{C}^{n_i})$, whence all the required conditions hold. The converse implication is obvious. □

Examples:

14. Take $G = \mathrm{Sp}_4$, with simple roots $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = 2\epsilon_2$, $\pm\epsilon_i$ being the weights of the minimal representation $\mathrm{Sp}_4 : \mathbb{C}^4$. Let $\lambda_0 = 3\epsilon_1$, $\lambda_1 = 2(\epsilon_1 + \epsilon_2)$ be the highest weights of V. We have $\alpha_1 \not\perp \lambda_0 \perp \alpha_2$ and $L \simeq \mathrm{SL}_2 \times \mathbb{C}^*$, so that α_2 is the simple root of SL_2, and ϵ_1 is a generator of $\Lambda(\mathbb{C}^*)$. The Clebsch–Gordan formula implies that $\lambda_1 - \lambda_0 = 2\epsilon_2 - \epsilon_1$, $-\alpha_1 = \epsilon_2 - \epsilon_1$ L-generate all lattice points in the cone $\mathbb{Q}_+\{2\epsilon_2 - \epsilon_1, -\epsilon_1\}$ except $-\epsilon_1$. Thus X is non-normal along Y_0. But if we increase V by adding V_{λ_2}, $\lambda_2 = 2\epsilon_1$, then X becomes normal.

15. Suppose $G = \mathrm{SO}_{2l+1}$, and $V = V_{\omega_i}$ is a fundamental representation. We have a unique closed orbit $Y_0 \subset X$. If $i < l$, then $L \not\simeq \mathrm{GL}_{n_1} \times \dots \times \mathrm{GL}_{n_p}$, hence X is singular. But for $i = l$, $L \simeq \mathrm{GL}_l$ is the common stabilizer of two transversal maximal isotropic subspaces in $\mathbb{C}^{2l+1}$. It follows e.g. from the realization of the spinor representation in the Clifford algebra that $V_{\omega_l}\otimes\mathbb{C}_{-\omega_l}$ is L-isomorphic to $\bigwedge^\bullet\mathbb{C}^l$. Here all the conditions of Theorem 25(2) are satisfied, whence X is smooth.

3.5. **Divisors and line bundles.** The theory of divisors on spherical varieties is due to Brion [Bri1]. The starting point is to show that each divisor on a spherical variety is rationally equivalent to a combination of colours and of G-stable prime divisors.

Theorem 26. *Each Weil divisor δ on a spherical variety X is rationally equivalent to a B-stable Weil divisor δ'.*

Proof. Let $\mathring{X}$ be the B-chart, corresponding to the coloured cone $(0,\emptyset)$, i.e., just the open B-orbit in G/H. Since $\mathring{X}$ is a factorial variety, $\delta|_{\mathring{X}}=\operatorname{div}_{\mathring{X}}f$ for some $f\in\mathbb{C}(\mathring{X})$. Now take $\delta'=\delta-\operatorname{div}_X f$. □

Remark. This assertion is a particular case of a more general result [FMSS] stating that each effective algebraic cycle on a B-variety is rationally equivalent to a B-stable effective one. The idea here is to apply Borel's fixed point theorem to Chow varieties of cycles.

Next, we describe the relations between the B-stable generators of the divisor class group $\operatorname{Cl}X$, i.e., between colours and G-stable divisors on X.

Theorem 27. *There is a finite presentation*

$$\operatorname{Cl}X=\left\langle D_1,\dots,D_n\right\rangle \Big/ \left\langle \sum_{i=1}^n\langle v_i,\lambda\rangle D_i \;\middle|\; \lambda\in\Lambda\right\rangle$$

where D_i are all the B-stable divisors on X, represented by indivisible vectors $v_i\in\Lambda^*$. (Of course, it suffices to take λ from a basis of Λ.)

Proof. Just note that B-stable principal divisors are of the form $\operatorname{div}f_\lambda$, and $\operatorname{ord}_{D_i}f_\lambda=\langle v_i,\lambda\rangle$. □

There are transparent combinatorial criteria in terms of coloured data for a B-stable divisor to be Cartier, base point free, or ample.

Theorem 28. *Let $\delta=\sum m_iD_i$ be a B-stable divisor on X.*

(1) *δ is Cartier iff for any G-orbit $Y\subseteq X$, $\exists\lambda_Y\in\Lambda^*$ such that $m_i=\langle v_i,\lambda_Y\rangle$ whenever $\overline{D_i}\supseteq Y$.*
(2) *δ is base point free iff these λ_Y can be chosen in such a way that $\lambda_Y\ge\lambda_{Y'}|_{\mathcal{C}_Y}$ and $m_i\ge\langle v_i,\lambda_Y\rangle$, $\forall Y,Y'\subseteq X$, $\forall D_i\in\mathcal{D}\setminus\bigcup_{Y\subseteq X}\mathcal{D}_Y$.*
(3) *δ is ample iff λ_Y can be chosen in such a way that $\lambda_Y>\lambda_{Y'}|_{\mathcal{C}_Y\setminus\mathcal{C}'_Y}$ and $m_i>\langle v_i,\lambda_Y\rangle$, $\forall Y,Y'\subseteq X$, $\forall D_i\in\mathcal{D}\setminus\bigcup_{Y\subseteq X}\mathcal{D}_Y$.*

Remark. Theorem 28 says that a Cartier divisor is determined by a piecewise linear function on the fan, and it is base point free, resp. ample, iff this function is convex, resp. strictly convex w.r.t. the fan, with some additional positivity condition on the coefficients at the colours which do not contain G-orbits in their closures.

Proof. Note that δ is Cartier outside a G-stable subvariety in $\operatorname{supp}\delta$ [Kn4, 2.2], because $g\delta \sim \delta$, $\forall g \in G$.

(1) If δ satisfies the condition, then $\operatorname{supp}(\delta - \operatorname{div} f_{\lambda_Y}) \not\supseteq Y$, whence δ is Cartier on an open subset $\mathring{X} \subseteq X$, $\mathring{X} \cap Y \neq \emptyset$. By the above remark, δ is Cartier on X.

Conversely, suppose δ is Cartier. By Sumihiro's theorem, we may assume that X is quasiprojective and δ is very ample, since each Cartier divisor on a quasiprojective variety is the difference of two very ample divisors. Then there exists a B-eigenvector $s_Y \in H^0(X, \mathcal{O}(\delta))$, $s_Y \neq 0|_Y$, and $\delta = \operatorname{div}(f_{\lambda_Y} s_Y)$ for some $\lambda_Y \in \Lambda$, which obviously satisfies the required condition.

(2) δ is base point free iff for any G-orbit $Y \subseteq X$, $\exists s_Y \in H^0(X, \mathcal{O}(\delta))$, $s_Y \neq 0|_Y$. We may assume s_Y to be a B-eigenvector. Then $\delta = \operatorname{div}(f_{\lambda_Y} s_Y)$ for some $\lambda_Y \in \Lambda$ satisfying the required condition.

(3) If δ is ample, then, replacing δ by a multiple, we may assume that $\delta' = \delta - \sum_{\overline{D_i} \not\supseteq Y} D_i$ is base point free for a given $Y \subseteq X$ and apply the argument from the previous paragraph to δ' in order to obtain the required λ_Y.

Conversely, assume that the condition on λ_Y is satisfied. Then $\delta = \operatorname{div}(f_{\lambda_Y} s_Y)$, where $s_Y \in H^0(X, \mathcal{O}(\delta))$ has the zero locus $X \setminus \mathring{X}$, $\mathring{X}$ being the B-chart given by $(\mathcal{C}_Y, \mathcal{D}_Y)$. Then clearly $\mathbb{C}[\mathring{X}] = \bigcup_{m \geq 0} H^0(X, \mathcal{O}(m\delta))/s_Y^m$. Replacing δ by a multiple, we may assume that $H^0(X, \mathcal{O}(\delta))/s_Y$ contains generators of $\mathbb{C}[\mathring{X}]$, $\forall Y \subseteq X$. Furthermore, we may replace $H^0(X, \mathcal{O}(\delta))$ here by a finite-dimensional G-submodule M containing all s_Y. Then the natural map $\phi : X \to \mathbb{P}(M^*)$ is well defined on $\mathring{X}$, whence on the whole X, $\mathring{X} = \phi^{-1}(\mathbb{P}(M^* \setminus \langle s_Y \rangle^\perp))$, and $\phi|_{\mathring{X}}$ is a closed embedding into $\mathbb{P}(M^* \setminus \langle s_Y \rangle^\perp)$, $\forall Y \subseteq X$. It follows that ϕ is a closed embedding, and δ is ample. $\square$

Now we describe the G-module structure of $H^0(X, \mathcal{O}(\delta))$ for a Cartier divisor δ.

Theorem 29. *In the notation of Theorem 28,*

$$H^0(X, \mathcal{O}(\delta)) \simeq \bigoplus_{\lambda \in \mathcal{P}(\delta) \cap \Lambda} V_{\lambda + \pi(\delta)}$$

where $\pi(\delta)$ *is the* B*-weight of the canonical rational section* s_δ *of* $\mathcal{O}(\delta)$ *with* $\operatorname{div} s_\delta = \delta$*, and*

$$\begin{aligned} \mathcal{P}(\delta) &= \{\lambda \mid \langle v_i, \lambda \rangle \geq -m_i,\ \forall i = 1, \dots, n\} \\ &= \bigcap_{Y \subseteq X} (-\lambda_Y + \mathcal{C}_Y^\vee) \cap \left\{ \lambda \;\middle|\; \langle v_i, \lambda \rangle \geq -m_i,\ \forall D_i \in \mathcal{D} \setminus \bigcap_{Y \subseteq X} \mathcal{D}_Y \right\} \end{aligned}$$

is the weight polytope *of* δ.

Proof. Since all simple G-modules occur in $H^0(X, \mathcal{O}(\delta))$ with multiplicities ≤ 1 by Theorem 18(4), it suffices to describe the set of highest weights. But $s = f_\lambda s_\delta$ is a highest weight section iff $\operatorname{div} f_\lambda \geq -\delta$ iff $\lambda \in \mathcal{P}(\delta) \cap \Lambda$. $\square$

Remark. In order to find $\pi(\delta)$, we may identify $\mathcal{O}(\delta)|_{G/H}$ with $G \times^H \mathbb{C}_\chi$, where H acts on $\mathbb{C}_\chi = \mathbb{C}$ by a character χ. Then rational sections of $\mathcal{O}(\delta)$

are identified with rational functions on G that are H-semiinvariant from the right with character $-\chi$, and $\pi(\delta)$ is the weight of the equation of the pull-back of δ to G, up to a shift by a character of G.

3.6. **Application: tensor product decompositions.** If $P, Q \subset G$ are two parabolics and $X = G/P \times G/Q$ is a spherical variety, then the geometry of X can be applied to finding decompositions of certain tensor products of simple modules. Namely, by the Borel–Weil–Bott theorem, the space of global sections of any line bundle on G/P or G/Q is a simple G-module (maybe zero). The tensor product of pull-backs to X of line bundles $\mathcal{L} \to G/P$, $\mathcal{M} \to G/Q$ equals $\mathcal{O}(\delta)$ for some B-stable Cartier divisor δ. Computing $\mathcal{P}(\delta)$ leads to a decomposition of $H^0(\mathcal{L}) \otimes H^0(\mathcal{M})$ into simple G-modules.

If P, Q stabilize the lines generated by lowest weight vectors $v_{-\lambda}, v_{-\mu}$ in two G-modules, respectively, and $\mathcal{L} = G \times^P \mathbb{C}_\lambda$, $\mathcal{M} = G \times^Q \mathbb{C}_\mu$ are pull-backs of ample line bundles on $G\langle v_{-\lambda}\rangle$, $G\langle v_{-\mu}\rangle$, then $H^0(\mathcal{L}) = V_\lambda$, $H^0(\mathcal{M}) = V_\mu$. All pairs of fundamental weights (λ, μ) such that X is spherical were classified by Littelmann [Lit] and the respective decompositions were computed. Recently all pairs of weights with spherical X were classified by Stembridge [St] and decompositions of $V_\lambda \otimes V_\mu$ were found in all cases.

Examples:

16. Consider the double flag variety $X = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of Example 8. We have seen that X is spherical and $\Lambda = \langle \epsilon_{n-1} - \epsilon_n \rangle \simeq \mathbb{Z}$, where ϵ_i are the diagonal matrix entries of B, the standard Borel subgroup of upper-triangular matrices. There are three B-stable divisors D, D', D'' given by equations

$$\Delta = \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} = 0, \qquad x_n = 0, \qquad y_n = 0$$

in homogeneous coordinates. Any B-eigenfunction is (up to a scalar multiple) an integer power of $f_{\epsilon_{n-1}-\epsilon_n}(x, y) = x_n y_n / \Delta$, whence D, D', D'' are represented by the vectors $v = -1$, $v' = v'' = 1$ in $\Lambda^* \simeq \mathbb{Z}$.

There are the two orbits in X: the closed one $Y = \operatorname{diag} \mathbb{P}^{n-1}$, and the open orbit $X \setminus Y$. We have $\mathcal{D}_Y = \{D\}$, $\mathcal{V}_Y = \emptyset$ (or vice versa for $n = 2$), hence $\mathcal{C}_Y = \mathbb{Q}_-$.

There is a relation $D = D' + D''$ in $\operatorname{Pic} X$, hence any divisor on X is equivalent to $\delta = pD' + qD''$. We have $H^0(X, \mathcal{O}(pD')) = H^0(\mathbb{P}^{n-1}, \mathcal{O}(p)) = \mathbb{C}[\mathbb{A}^n]_p \simeq V_{-p\epsilon_n}$, and similarly $H^0(X, \mathcal{O}(qD'')) = \mathbb{C}[\mathbb{A}^n]_q \simeq V_{-q\epsilon_n}$. On the other hand, it is easy to compute $\mathcal{P}(\delta) = \{k(\epsilon_{n-1} - \epsilon_n) \mid 0 \geq k \geq -p, -q\}$. Shifting by the highest weight $\pi(\delta) = -(p+q)\epsilon_n$ of the canonical section $s_\delta = x_n^p \otimes y_n^q$ yields a decomposition

$$\mathbb{C}[\mathbb{A}^n]_p \otimes \mathbb{C}[\mathbb{A}^n]_q = \bigoplus_{k=0}^{\min(p,q)} V_{(k-p-q)\epsilon_n - k\epsilon_{n-1}}$$

generalizing the Clebsch–Gordan formula.

17. Consider another spherical double flag variety $X = \mathbb{P}^{n-1} \times \operatorname{LGr}(\mathbb{C}^n)$ of Example 9. In the notation of that example, $\Lambda = \langle \epsilon_1 + \epsilon_l, \epsilon_1 - \epsilon_l \rangle$ w.r.t. the standard Borel subgroup of upper-triangular matrices in Sp_n. There are the two orbits in X: the closed one $Y = \{(\ell, F) \in X \mid \ell \subseteq F\}$, and the open orbit $X \setminus Y$. There are four B-stable divisors $D_1, \dots, D_4$ given by the conditions $\ell \perp \langle e_1 \rangle$, $F \cap \langle e_1, \dots, e_l \rangle \neq 0$, $(F + \ell) \cap \langle e_1, \dots, e_{l-1} \rangle \neq 0$, $(F + \ell) \cap \ell^{\perp} \cap \langle e_1, \dots, e_l \rangle \neq 0$, respectively. (One verifies it by proving that the complement of the union of the D_i is a single B-orbit.) Clearly $\mathcal{D}_Y = \{D_3, D_4\}$.

It is easy to see from the above description that the D_i can be determined by bihomogeneous equations F_i in projective coordinates of $\mathbb{P}^{n-1}$ and Plücker coordinates of $\mathrm{LGr}(\mathbb{C}^n)$ of bidegrees $(1,0)$, $(0,1)$, $(1,1)$, $(2,1)$, and B-eigenweights $\omega_1 = \epsilon_1$, $\omega_l = \epsilon_1 + \cdots + \epsilon_l$, $\omega_{l-1} = \epsilon_1 + \cdots + \epsilon_{l-1}$, ω_l, respectively. We have $f_{\epsilon_1+\epsilon_l} = F_1F_2/F_3$, $f_{\epsilon_1-\epsilon_l} = F_1F_3/F_4$, whence D_i are represented by the vectors $v_i \in \Lambda^*_{\mathbb{Q}}$, where $v_1 = \epsilon_1$, $v_2 = (\epsilon_1+\epsilon_l)/2$, $v_3 = -\epsilon_l$, $v_4 = (\epsilon_l - \epsilon_1)/2$, under the identification of $\Lambda_{\mathbb{Q}}$ with $\Lambda^*_{\mathbb{Q}}$ via the inner product such that the ϵ_1, ϵ_l form an orthonormal basis. In particular, $\mathcal{C}_Y$ is generated by $-\epsilon_l, (\epsilon_l - \epsilon_1)/2$.

Every divisor on X is rationally equivalent to $\delta = pD_1 + qD_2$. We have $H^0(X, \mathcal{O}(pD_1)) = V_{p\omega_1}$, $H^0(X, \mathcal{O}(qD_2)) = V_{q\omega_l}$. Computing $\mathcal{P}(\delta) = \{\lambda = a\epsilon_1 + b\epsilon_n \mid 0 \geq b \geq a \geq -p,\ a + b \geq -2q\}$ and shifting by $\pi(\delta) = p\omega_1 + q\omega_l$ finally yields a decomposition

$$V_{p\omega_1} \otimes V_{q\omega_l} = \bigoplus_{\substack{0 \leq b \leq a \leq p \\ a+b \leq 2q \\ a \equiv b \pmod 2}} V_{(p+q-a)\epsilon_1 + q\epsilon_2 + \cdots + q\epsilon_{l-1} + (q-b)\epsilon_l}$$

3.7. **Intersection theory.** The approach to enumerative problems on homogeneous spaces mentioned in the introduction leads to the definition of the *intersection ring* $C^*(G/H)$ [CP2]. It may be defined without use of compactifications, but one proves that $C^*(G/H) = \varinjlim H^*(X)$ over all smooth completions $X \supseteq G/H$.

In the simplest case, we have to compute the intersection number of divisors on G/H. Everything reduces to computing the self-intersection number (δ^d) for an effective divisor $\delta \subset G/H$, $d = \dim G/H$.

Translating δ by a generic element of G, we may assume that no colours are among the components of δ. Since the open B-orbit $\mathring{X} \subseteq G/H$ is a factorial variety, we may consider the equation $f \in \mathbb{C}[\mathring{X}]$ of $\delta|_{\mathring{X}}$.

Definition 9. The *Newton polytope* of δ is

$$\mathcal{N}(\delta) = \{\lambda \mid \langle v, \lambda \rangle \geq v(f),\ \langle v_D, \lambda \rangle \geq \mathrm{ord}_D(f),\ \forall v \in \mathcal{V},\ D \in \mathcal{D}\}$$

Example 18. Suppose G/H is quasiaffine and, for simplicity, $\delta = \operatorname{div} f$ is a principal divisor, $f = f_1 + \cdots + f_m$, $f_j \in \mathbb{C}[G/H]_{(\lambda_j)}$, $f_j \neq 0$. Then $v(f) = \min_j \langle v, \lambda_j \rangle$, $\forall v \in \mathcal{V}$, $\mathrm{ord}_D f = 0$, $\forall D \in \mathcal{D}$, and

$$\mathcal{N}(\delta) = (\mathrm{conv}\{\lambda_1, \dots, \lambda_m\} + \mathcal{V}^{\vee}) \cap \{\lambda \mid \langle v_D, \lambda \rangle \geq 0,\ \forall D \in \mathcal{D}\}$$

In particular, if G is a torus, then $\mathcal{N}(\delta) = \mathrm{conv}\{\lambda_1, \dots, \lambda_m\}$ is the usual Newton polytope of a Laurent polynomial f.

Theorem 30 ([Bri4])**.**

$$(\delta^d) = d! \int_{\mathcal{N}(\delta)} \prod_{\alpha \not\perp \Lambda + \langle \pi(\delta) \rangle} \frac{(\lambda + \pi(\delta), \alpha)}{(\rho, \alpha)} \, d\lambda \tag{15}$$

where α runs over positive roots, ρ is half the sum of positive roots, $\pi(\delta) = -\sum_{D\in\mathcal{D}}(\operatorname{ord}_D f)\pi(D)$, and the Lebesgue measure $d\lambda$ is normalized in such a way that the fundamental parallelepiped of Λ has volume 1.

Proof. Consider a smooth projective embedding $X \hookleftarrow G/H$. The divisor $\delta_X = \delta - \operatorname{div}_X f = -\sum_{i=1}^n(\operatorname{ord}_{D_i} f)D_i$ is B-stable, and $\mathcal{P}(\delta_X) = \{\lambda \mid \langle v_i,\lambda\rangle \geq \operatorname{ord}_{D_i} f,\ \forall i\}$. It is clear that $\mathcal{N}(\delta) = \bigcap_{X\hookleftarrow G/H}\mathcal{P}(\delta_X)$.

There exists X such that the closure of δ contains no G-orbits [CP2]. Then δ is base point free, $(\delta^d) = [\delta_X]^d \in H^{2d}(X)$, and $\mathcal{N}(\delta) = \mathcal{P}(\delta_X)$. Indeed, take any $\lambda \in \mathcal{P}(\delta_X)$ and $v \in \mathcal{V}$. Consider an embedding $\hat{X}$ obtained by subdividing the fan of X by v, and let $D \subset \hat{X}$ be the divisor corresponding to v. It is easy to see that there is a map $\hat{X} \to X$ contracting D to the center of v in X. For $k \gg 0$ we have $s = f_{k\lambda}s_{\delta_X}^k = f_{k\lambda}s_\delta^k/f^k \in H^0(X,\mathcal{O}(\delta)) \subseteq H^0(\hat{X},\mathcal{O}(\delta))$, whence $\operatorname{ord}_D s = \langle v,k\lambda\rangle + \operatorname{ord}_D s_\delta^k - v(f^k) \geq 0$. But $\operatorname{ord}_D s_\delta = 0$, hence $\langle v,\lambda\rangle \geq v(f)$, which yields $\lambda \in \mathcal{N}(\delta)$.

It remains to compute $[\delta_X]^d$. By [Har, Exer. II.7.5] base point free divisors lie in the closure of the ample cone in $(\operatorname{Pic} X)\otimes\mathbb{Q}$ (this is also visible from Theorem 28), and both sides of (15) depend continuously on δ_X. Therefore we may assume δ_X to be ample. Then $[\delta_X]^d = d!\cdot I$, where $\dim H^0(X,\mathcal{O}(k\delta_X)) = I\cdot k^d + \text{lower terms}$.

Recall Weyl's dimension formula $\dim V_\lambda = \prod_\alpha (\lambda+\rho,\alpha)/(\rho,\alpha)$ (over all positive roots α). By Theorem 29,

$$\begin{aligned}\dim H^0(X,\mathcal{O}(k\delta_X)) &= \sum_{\lambda\in\mathcal{P}(k\delta_X)\cap\Lambda}\prod_\alpha \frac{(\lambda+\pi(k\delta_X)+\rho,\alpha)}{(\rho,\alpha)}\\ &= \sum_{\lambda\in\mathcal{P}(\delta_X)\cap\Lambda/k}\prod_\alpha \frac{(k\lambda+k\pi(\delta)+\rho,\alpha)}{(\rho,\alpha)}\end{aligned}$$

The leading coefficient I equals the integral on the r.h.s. of (15). □

Theorem 30 can be regarded as a generalization of the classical Bézout theorem.

Examples:

19. If G is a torus, then $(\delta^d) = d!\operatorname{vol}\mathcal{N}(\delta)$. Polarization yields $(\delta_1,\dots,\delta_d) = d!\operatorname{vol}(\mathcal{N}(\delta_1),\dots,\mathcal{N}(\delta_d))$, with the mixed volume of $\mathcal{N}(\delta_1),\dots,\mathcal{N}(\delta_d)$ on the r.h.s., giving the number of solutions for a system of d equations in general position on a d-dimensional torus (Bernstein–Kouchnirenko [Kou]).

20. More generally, consider $G = (G\times G)/\operatorname{diag} G$ as a homogeneous space under the doubled group, cf. 3.3–3.4. Suppose $\delta = \operatorname{div} f$, $f \in \mathbb{C}[G]$. (There is no essential loss of generality, because a finite cover of G is a factorial variety.) From Example 18 and results of 3.3 we see that $\mathcal{N}(\delta) = (\operatorname{conv}\{\lambda_1,\dots,\lambda_m\} - C^\vee)\cap C = (\operatorname{conv} W\{\lambda_1,\dots,\lambda_m\})\cap C$ if f is expressed as the sum of matrix entries of $G : V_{\lambda_i}$, $i = 1,\dots,m$, and $\pi(\delta) = 0$. We have $\Lambda = \{(-\lambda,\lambda) \mid \lambda \in \Lambda(B)\}$, the positive roots of $G\times G$ are $(-\alpha,0)$, $(0,\alpha)$, where α is a positive root of G, and $(-\rho,\rho)$ is half the sum of positive roots for $G\times G$. Now Theorem 30 yields Kazarnovskii's "Bézout theorem" on any reductive group [Kaz]:

$$(\delta^d) = d! \int_{\mathcal{N}(\delta)} \prod_{\alpha} \frac{(\lambda,\alpha)^2}{(\rho,\alpha)^2}\, d\lambda$$

21. Consider the Grassmannian $\mathrm{Gr}_k(\mathbb{P}^n)$ acted on by $G = \mathrm{GL}_{n+1}$. Let δ be a hyperplane section of its Plücker embedding into $\mathbb{P}(\bigwedge^{k+1}\mathbb{C}^{n+1})$. We have $\delta \sim D$, where D is the unique colour, which generates $\operatorname{Pic}\mathrm{Gr}_k(\mathbb{P}^n)$. Here $\Lambda = 0$, whence $\mathcal{N}(\delta) = \{0\}$, and $\pi(\delta) = \pi(D) = -\epsilon_{k+2} - \dots - \epsilon_{n+1}$. Positive roots are of the form $\alpha = \epsilon_i - \epsilon_j$, $i < j$, and $\rho = (n/2)\epsilon_1 + (n/2-1)\epsilon_2 + \dots + (-n/2)\epsilon_{n+1}$. The degree of the Plücker embedding equals

$$\begin{aligned}(\delta^d) = d! \prod_{\alpha \not\perp \pi(\delta)} \frac{(\pi(\delta),\alpha)}{(\rho,\alpha)} &= [(k+1)(n-k)]! \prod_{i\le k+1<j} \frac{1}{j-i} \\ &= [(k+1)(n-k)]! \frac{0!\dots k!}{n!\dots(n-k)!}\end{aligned}$$

This is a classical result of Schubert.

22. Now we come back to the classical enumerative problem mentioned in the introduction. In the notation of Example 11, all conics tangent to a given one fill the divisor δ given by the equation $f(q) = \operatorname{Dis}\det(sq - tq_0) = 0$, where q_0 is the matrix of the given conic, s, t are indeterminates, and Dis denotes the discriminant of a binary form. Note that $f \in \mathbb{C}[Q_2]$, whence $\delta = \operatorname{div} f$ is principal.

From the expression for the discriminant of a binary cubic form and from Example 11, it is easy to see that $f = f_{(4\omega_1+4\omega_2)} + f_{(6\omega_1)} + f_{(6\omega_2)} + f_{(2\omega_1+2\omega_2)} + f_{(0)}$, where $f_{(\lambda)}$ is the projection to $\mathbb{C}[Q_2]_{(\lambda)}$. It follows by Examples 18, 11 that $\mathcal{N}(\delta) = \operatorname{conv}\{4\omega_1 + 4\omega_2, 6\omega_1, 6\omega_2, 0\}$ and $\pi(\delta) = 0$. (Actually, it suffices to know the highest weight $4\omega_1 + 4\omega_2$ occurring in f.) We subdivide $\mathcal{N}(\delta)$ into 2 triangles $\mathcal{N}_i = \operatorname{conv}\{4\omega_1 + 4\omega_2, 6\omega_i, 0\}$ $(i = 1, 2)$.

The positive roots are α_1, α_2, $\rho = \alpha_1 + \alpha_2$. Write $\lambda = 2x_1\alpha_1 + 2x_2\alpha_2$, $\forall \lambda \in \Lambda \otimes \mathbb{Q}$. The number of plane conics tangent to 5 given conics in general position equals

$$\begin{aligned}(\delta^5) &= 5! \int_{\mathcal{N}(\delta)} \frac{(\lambda,\alpha_1)(\lambda,\alpha_2)(\lambda,\rho)}{(\rho,\alpha_1)(\rho,\alpha_2)(\rho,\rho)}\, d\lambda \\ &= 5! \int_{\mathcal{N}_1} (4x_1 - 2x_2)(4x_2 - 2x_1)(2x_1 + 2x_2)\, dx_1\, dx_2 \\ &= 5! \int_0^2 dx_1 \int_{x_1/2}^{x_1} dx_2\, (4x_1 - 2x_2)(4x_2 - 2x_1)(2x_1 + 2x_2) = 3264\end{aligned}$$

(Chasles, 1864)

4. Spaces of complexity one

The embedding theory of homogeneous spaces of complexity one is developed in [Tim1] from the general Luna–Vust theory of embeddings in a way similar to the theory of spherical varieties. In this survey, we will only give a brief exposition of this theory, skipping most proofs and attracting reader's attention to common points and distinctions from the spherical case.

4.1. **Coloured data.** In contrast with the spherical case, a B-semiinvariant rational function on a homogeneous space G/H of complexity 1 is not uniquely determined (up to a constant) by its weight. Observe by the Lüroth theorem that $\mathbb{C}(G/H)^B \simeq \mathbb{C}(\mathbb{P}^1)$ is the field of rational functions in one variable, and a B-eigenfunction f_λ is determined by its weight $\lambda \in \Lambda$ only up to a multiple in $\mathbb{C}(\mathbb{P}^1)^\times$. We have a short exact sequence

$$0 \longrightarrow \mathbb{C}(\mathbb{P}^1)^\times/\mathbb{C}^\times \longrightarrow \mathcal{A} \longrightarrow \Lambda \longrightarrow 0$$

recalling $\mathcal{A} = \mathbb{C}(G/H)^{(B)}/\mathbb{C}^\times$ from 2.2. It is convenient to fix a (non-canonical) splitting $\mathcal{A} \simeq \Lambda \times (\mathbb{C}(\mathbb{P}^1)^\times/\mathbb{C}^\times)$, so that each B-semiinvariant function is represented as $f = f_\lambda q$, where f_λ is a fixed function of weight λ, and $q \in \mathbb{C}(\mathbb{P}^1)$.

Geometrically, the identification $\mathbb{C}(G/H)^B \simeq \mathbb{C}(\mathbb{P}^1)$ gives rise to a surjective rational map $\pi : G/H \dashrightarrow \mathbb{P}^1$, whose generic fibers are (the closures of) generic B-orbits in G/H. Thus the set of colours depends on one continuous parameter. We may fix a cofinite subset $\mathring{\mathcal{D}} \subseteq \mathcal{D}$ consisting of $D_z = \pi^{-1}(z)$, $z \in \mathring{\mathbb{P}}^1$, a cofinite subset of $\mathbb{P}^1$.

To any colour $D \in \mathcal{D}$ we associate a vector $v_D \in \Lambda^*$ by restriction of ord_D to $\{f_\lambda \mid \lambda \in \Lambda\}$. The restriction of ord_D to $\mathbb{C}(G/H)^B$ yields a valuation of $\mathbb{C}(\mathbb{P}^1)$ with center $z_D \in \mathbb{P}^1$ and the order $h_D \in \mathbb{Z}_+$ of a local coordinate at z_D. We have $\mathrm{ord}_D f = \langle v_D, \lambda\rangle + h_D(\mathrm{ord}_{z_D} q)$. (If ord_D vanishes on $\mathbb{C}(\mathbb{P}^1)$, then we put $h_D = 0$ and take any point of $\mathbb{P}^1$ for z_D.) Similarly, G-valuations are determined by triples (v, h, z), where $v \in \Lambda^*_{\mathbb{Q}}$, $h \in \mathbb{Q}_+$, $z \in \mathbb{P}^1$.

Consider the union $\Lambda^+_{\mathbb{Q}} = \bigcup_{z\in\mathbb{P}^1} \Lambda^+_{\mathbb{Q}}(z)$, where $\Lambda^+_{\mathbb{Q}}(z) = \Lambda^*_{\mathbb{Q}} \times \mathbb{Q}_+$ are half-spaces naturally attached together along their common boundary hyperplane $\Lambda^*_{\mathbb{Q}}$. We say that $\Lambda^+_{\mathbb{Q}}$ is the *hyperspace* associated with G/H. By the above, colours and G-valuations are represented by points of the hyperspace. Reducing $\mathring{\mathcal{D}}$ if necessary, we may assume that $\mathrm{ord}_D f_\lambda = 0$, $\forall D \in \mathring{\mathcal{D}}$, $\lambda \in \Lambda$. Hence D_z is represented by the vector $(0, 1) \in \Lambda^+_{\mathbb{Q}}(z)$, $\forall z \in \mathring{\mathbb{P}}^1$. The following result generalizes Theorem 19:

Theorem 31 ([Kn3]). *G-valuations form a subset $\mathcal{V} \subseteq \Lambda^+_{\mathbb{Q}}$, called the* valuation hypercone, *such that the $\mathcal{V}(z) = \mathcal{V} \cap \Lambda^+_{\mathbb{Q}}(z)$ are solid convex polyhedral (in fact, cosimplicial) cones.*

4.2. **Equivariant embeddings.** Now we reorganize coloured data of B-charts and G-germs in a way similar to the spherical case.

The class **CD** consists of the pairs $(\mathcal{W}, \mathcal{F})$ such that $\mathcal{W}$ is finite and $\mathcal{F}$ differs from $\mathring{\mathcal{D}}$ by finitely many elements. Take $(\mathcal{W}, \mathcal{F}) \in \mathbf{CD}$.

Condition (F) is always satisfied, but in this case it is non-trivial, see [Tim1, 3.1].

Let $\mathcal{C}(z)$ be the cone generated by those elements of $\mathcal{W}$ and $\mathcal{F}$ which map to $\Lambda^+_{\mathbb{Q}}(z)$ and by

$$(16)\quad \mathcal{Z} = \sum_{z\in\mathbb{P}^1} \mathcal{Z}(z) \subseteq \Lambda^*_{\mathbb{Q}} \qquad \text{(Minkowski sum), where}$$

$$\mathcal{Z}(z) = \operatorname{conv}\left\{ v/h,\ v_D/h_D \;\middle|\; \begin{array}{l} (v,h) \in \mathcal{W}\cap\Lambda^+_{\mathbb{Q}}(z) \\ (v_D,h_D) \in \mathcal{F}\cap\Lambda^+_{\mathbb{Q}}(z) \\ h, h_D \neq 0 \end{array} \right\}$$

Put $\mathcal{C} = \bigcup_{z\in\mathbb{P}^1} \mathcal{C}(z)$. Condition (C) means that $(\mathcal{C},\mathcal{F})$ is a coloured hypercone in the sense of the following

Definition 10. A *coloured hypercone* is a pair $(\mathcal{C},\mathcal{F})$, where $\mathcal{C} \subseteq \Lambda^+_{\mathbb{Q}}$, $\mathcal{F} \subseteq \mathcal{D}$, and there exists a finite subset $\mathcal{W} \subset \mathcal{V}$ such that:

- $\mathcal{F}$ differs from $\mathring{\mathcal{D}}$ by finitely many elements, and $\mathcal{F} \not\ni 0$.
- $\mathcal{Z} \not\ni 0$, where $\mathcal{Z}$ is defined by Formula (16).
- $\mathcal{C}(z) = \mathcal{C}\cap\Lambda^+_{\mathbb{Q}}(z)$ are strictly convex cones generated by $\mathcal{W}\cap\Lambda^+_{\mathbb{Q}}(z)$, $\mathcal{F}\cap\Lambda^+_{\mathbb{Q}}(z)$, and by $\mathcal{Z}$.

The *interior* of $(\mathcal{C},\mathcal{F})$ is $\operatorname{int}\mathcal{C} = \left(\bigcup_{z\in\mathbb{P}^1}\operatorname{int}\mathcal{C}(z)\right) \cup \operatorname{int}(\mathcal{C}\cap\Lambda^*_{\mathbb{Q}})$ whenever $\mathcal{C}(z) \not\subseteq \Lambda^*_{\mathbb{Q}}$, $\forall z \in \mathbb{P}^1$, and $\emptyset$, otherwise. The coloured hypercone is said to be *supported* if $(\operatorname{int}\mathcal{C}) \cap \mathcal{V} \neq \emptyset$.

A *face* of $(\mathcal{C},\mathcal{F})$ is either a coloured cone $(\mathcal{C}',\mathcal{F}')$ in some $\Lambda^+_{\mathbb{Q}}(z)$ such that $\mathcal{C}'$ is a face of $\mathcal{C}(z)$ and $\mathcal{C}'\cap\mathcal{Z} = \emptyset$, or a coloured hypercone $(\mathcal{C}',\mathcal{F}')$ such that $\mathcal{C}'(z)$ are faces of $\mathcal{C}(z)$ and $\mathcal{C}'\cap\mathcal{Z} \neq \emptyset$, and $\mathcal{F}' = \mathcal{F}\cap\mathcal{C}'$ in both cases.

A *coloured hyperfan* is a collection of supported coloured cones and hypercones which is obtained from finitely many coloured hypercones by taking all the supported faces, and has the property that different cones and hypercones intersect along faces inside $\mathcal{V}$.

Condition (W) says that $\mathcal{W}$ is recovered from $(\mathcal{C},\mathcal{F})$ as the set of generators of those edges of $\mathcal{C}$ which do not intersect $\mathcal{F}$ and $\mathcal{Z}$.

Conditions (V), (V′), (D′), (S) are reformulated verbatim alike the spherical case.

The following theorem is a counterpart of Theorem 20.

Theorem 32. *B-charts are in bijection with coloured hypercones, G-germs with supported coloured cones and hypercones, and embeddings of G/H are in bijection with coloured hyperfans.*

Theorem 21 transfers verbatim to the case of complexity 1 if we only replace "G-orbits" by "closed G-subvarieties", "cones" by "cones and hypercones", and "fan" by "hyperfan".

4.3. **Divisors and intersection theory.** Results of 3.5–3.7 are generalized in [Tim2] to the complexity one case (and even, to some extent, to arbitrary complexity).

Theorem 26 generalizes together with the proof if we take $\mathring{X} = U_P \times A \times C$ from Theorem 2 and observe that C is a smooth rational curve, hence $\mathring{X}$ is factorial. There is a description of B-stable Cartier, base point free, and ample divisors similar to Theorem 28, see [Tim2, §4].

However, the G-module structure of global sections for a B-stable Cartier divisor $\delta = \sum m_i D_i$ on an embedding $X \hookleftarrow G/H$ is more complicated. We may assume that the sum ranges over all B-stable prime divisors $D_i \subset X$ (with only finitely many $m_i \neq 0$), and let $(v_i, h_i) \in \Lambda^+_{\mathbb{Q}}(z_i)$ be the respective vectors of the hyperspace. Put

$$\mathcal{P}(\delta) = \{\lambda \in \Lambda_{\mathbb{Q}} \mid \langle v_i, \lambda\rangle \geq -m_i \text{ whenever } h_i = 0\}$$

$$m_z = \min_{\substack{z_i = z \\ h_i \neq 0}} \frac{\langle v_i, \lambda\rangle + m_i}{h_i}, \qquad \forall z \in \mathbb{P}^1$$

$$m(\delta, \lambda) = \max\Bigl(1 + \sum_{z \in \mathbb{P}^1} m_z,\ 0\Bigr)$$

Theorem 33. *Let $\pi(\delta)$ be the B-weight of the canonical section s_δ of $\mathcal{O}(\delta)$ with $\operatorname{div} s_\delta = \delta$. Then the multiplicity of $V_{\lambda+\pi(\delta)}$ in $H^0(X, \mathcal{O}(\delta))$ equals $m(\delta, \lambda)$ if $\lambda \in \mathcal{P}(\delta)$, and 0, otherwise.*

Remark. Note that the multiplicity function $m(\delta, \lambda)$ is a piecewise affine concave function of λ on its support.

Proof. It suffices to examine the space of highest weight vectors of a given weight in $H^0(X, \mathcal{O}(\delta))$. A section $s = f_\lambda q s_\delta$ ($\lambda \in \Lambda$, $q \in \mathbb{C}(\mathbb{P}^1)$) is a highest weight vector iff $\operatorname{div} f_\lambda q \geq -\delta$ iff $\langle v_i, \lambda\rangle + h_i(\operatorname{ord}_{z_i} q) \geq -m_i$, $\forall i$. The latter condition is equivalent to $\lambda \in \mathcal{P}(\delta)$ and $\operatorname{ord}_z q \geq -m_z$, $\forall z \in \mathbb{P}^1$. Hence the dimension of the space of highest weight vectors equals $\dim H^0\bigl(\mathbb{P}^1, \mathcal{O}\bigl(\sum_z m_z z\bigr)\bigr) = m(\delta, \lambda)$. □

Unfortunately, the intersection theory on homogeneous spaces of complexity one is not as nice as for spherical spaces. The reason is that embeddings of G/H generally have infinitely many G-orbits, and there might exist no compactification $X \hookleftarrow G/H$ with finitely many orbits such that the closures $\overline{Z_i}$ of given subvarieties $Z_1, \dots, Z_s \subset G/H$ intersect $X \setminus (G/H)$ properly. Then $\overline{Z_1} \cap \dots \cap \overline{Z_s}$ may have points "at infinity", and the intersection product of $[\overline{Z_i}]$ in $H^*(X)$ has no relation with $|Z_1 \cap \dots \cap Z_s|$. In particular, there is generally no "Bézout theorem" for the intersection number of hypersurfaces in G/H. However, there is a weaker version of Theorem 30:

Theorem 34 ([Tim2]). *Let δ be a base point free divisor on a projective embedding $X \hookleftarrow G/H$, $\dim G/H = d$. Then*

$$(\delta^d) = d! \int\limits_{\mathcal{P}(\delta)} m(\delta, \lambda) \prod_{\alpha \not\perp \Lambda + \langle \pi(\delta)\rangle} \frac{(\lambda + \pi(\delta), \alpha)}{(\rho, \alpha)}\, d\lambda \tag{17}$$

The proof is essentially the same as for Theorem 30 using Theorem 33 instead of Theorem 29. Details are left to the reader.

Consider the problem of finding the intersection number of divisors on G/H. Suppose we managed to construct a compactification $X \supset G/H$ with finitely many orbits such that all divisors, whose intersection number we are looking for, intersect each orbit properly. Then Theorem 34 leads to a "Bézout theorem" on G/H. Another application of Theorem 34 is the computation of the degree of any orbit in any $\mathrm{SL}_2(\mathbb{C})$-module or projective representation [Tim2]. (For irreducible representations this degree was computed in [MJ] using the description of Chow rings for smooth embeddings of $\mathrm{SL}_2/\{e\}$.)

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