Group characters

1. Definitions and basic properties

Let $\varphi: G \to GL(V)$ be a linear representation of a group G in a finite-dimensional vector space over a field K. For any element $g \in G$ let $A_\varphi(g)$ will be the matrix of the operator $\varphi(g)$ with respect to some basis $e = \{e_1, e_2, ..., e_n\}$ *of* V. As is well known, the trace of $A_\varphi(g)$ does not depend upon a basis, so the function $\chi = \chi_\varphi : G \to K, \chi_\varphi(g) := tr_V \varphi(g) = trA_\varphi(g)$ is defined; χ_φ is called the character of the representation φ.

Let's establish basic properties of characters.

Proposition 1. 1) $\chi_\varphi(1) = \dim V$ - degree (dimension) of the representation φ.

2) $\forall g, h \in G,\ \chi_\varphi(h^{-1}gh) = \chi_\varphi(g)$, so χ_φ is constant on conjugate classes. (A function on G which is constant on its conjugate classes is called central, or class function.)

3) If $g \in G$ has finite order and $K = \mathbb{C}$ the field of complex numbers, then $\chi_\varphi(g^{-1}) = \overline{\chi_\varphi(g)}$ (complex conjugate).

4) The character of the direct sum of representations equals the sum of characters of the summands: $\chi_{\varphi\oplus\psi} = \chi_\varphi + \chi_\psi$ (and similarly for any finite number of summands).

5) Characters of equivalent representations are equal.

Proofs. 1) $\chi_\varphi(1) = trE = n = \dim V$.

2),5) follow from the property that $tr(AB) = tr(BA)$ (recall that two representations are equivalent iff there is a matrix C such that $A_\psi(g) = C^{-1}A_\psi(g)C$).

3) If $g^m = 1$ and λ is an eigenvalue of $\varphi(g)$, then $\lambda^m = 1 \Rightarrow \lambda^{-1} = \overline{\lambda}$. Now $trA_\varphi(g) = \sum_{k=1}^{n} \lambda_k, trA_\varphi(g^{-1}) = \sum_{k=1}^{n} \lambda^{-1}{}_k = \sum_{k=1}^{n} \overline{\lambda_k}$, q.e.d.

4) In the basis of the direct sum $V \oplus W$ which is combined of bases of spaces V, W

we have $A_{\varphi\oplus\psi}(g) = \begin{pmatrix} A_\varphi(g) & 0 \\ 0 & A_\psi(g) \end{pmatrix}$ hence $trA_{\varphi\oplus\psi}(g) = trA_\varphi(g) + trA_\psi(g)$, qed.

Note that the set of central functions ZF_G on G is the linear space of dimension r, the number of conjugate classes $K_1, ..., K_r$ *of* G (its basis consists of functions $\Gamma_i : \Gamma_i(K_j) = \delta_{i,j}, 1 \le j \le r,\ i = 1, ..., r$), while the set F_G of all the functions on G is the linear space of dimension $n = |G|$ with basis of "delta-functions" $\delta_g : \delta_g(x) = 1, x = g,\ 0, x \ne g \in G$.

The character of any *irreducible representation* cannot be decomposed into sum of characters, by 4, that' s why it is named ***irreducible character***.

2. Schur's Lemma and its consequence for finite groups

Let me remind

Schur's Lemma. If $\varphi: G \to GL(V), \psi: G \to GL(W)$ are irreducible representations of a group G and $f: V \to W$ is a homomorphism of representations (i.e. $f\varphi = \psi f$), then f=0 or f is isomorpyism of representations.

Consequence. If the field of definition of V, W is algebraically closed and V, W are finite-dimensional then f=0 or (when the representations are isomorphic and the spaces and representations are identified) $f = \lambda E, \lambda \in K$.

In what follows the ground field will be the field of complex numders $\mathbb{C}$.

Lemma. Let $\varphi: G \to GL(V), \psi: G \to GL(W)$ are finite-dimensional irreducible representations of a finite group G and $f: V \to W$ over $\mathbb{C}$ and $f: V \to W$ is some linear mapping. Then the average mapping equals

$$\tilde{f} = \frac{1}{|G|}\sum_{g\in G}\psi(g) f \varphi(g)^{-1} = \begin{cases}\lambda E,\ if\ V = W, \varphi = \psi,\ where\ \lambda = \frac{trf}{\dim V}, \\ 0, otherwise\end{cases}.$$

Proof. It's evident that $\tilde{f}$ is a homomorphism of representations, so, by Schur Lemma, f=0 or (in the first case) $\tilde{f} = \lambda E, \lambda \in \mathbb{C} \Rightarrow tr\tilde{f} = trf = \lambda trE = \lambda \dim V$.

■

Matrix version of lemma.

Fix bases in V, W: $V = \langle v_i | i \in I\rangle, W = \langle w_j | j \in J\rangle$ V, W, and the mappings get matrices: $\varphi(g) = \left(\varphi_{i,i'}(g)\right), \psi(g) = \left(\psi_{i,i'}(g)\right), f = \left(f_{ji}\right), \tilde{f} = \left(\tilde{f}_{ji}\right)$. By definition,

$$\tilde{f}_{ji} = \frac{1}{|G|}\sum_{g,i',j'}\psi_{jj'}(g) f_{j'i'}\varphi_{i'i}(g^{-1}). \quad (1)$$

In particular, taking $f = E_{j_0 i_0}$, $f_{j_0 i_0} = 1, f_{ji} = 0$ $((j,i) \neq (j_0, i_0))$ - the matrix unit – we get

1) If φ, ψ are non-isomorphic, then from (1)

$$\frac{1}{|G|}\sum_{g\in G}\psi_{jj_0}(g)\varphi_{i_0 i}(g^{-1}) = 0, \forall i, i_0, j, j_0. \quad (2)$$

2) If $V = W, \varphi = \psi$ then $\tilde{f} = \frac{trf}{\dim V}E, trf = \sum_i f_{ii} = \sum_{j',i'}\delta_{j'i'}f_{j'i'} \Rightarrow \tilde{f}_{ji} = \delta_{ji}\frac{trf}{\dim V} = \frac{\delta_{ji}}{\dim V}\sum_{j',i'}\delta_{j'i'}f_{j'i'}$

In view of (1) we get $\frac{1}{|G|}\sum_{g,i',j'}\psi_{jj'}(g) f_{j'i'}\varphi_{i'i}(g^{-1}) = \frac{\delta_{ji}}{\dim V}\sum_{j',i'}\delta_{j'i'}f_{j'i'}$.

Taking again $f = E_{j_0 i_0}$, we finally have

$$\frac{1}{|G|}\sum_{g\in G}\psi_{jj_0}(g)\varphi_{i_0 i}(g^{-1}) = \begin{cases}\frac{\delta_{ji}}{\dim V} if\ j_0 = i_0, \\ 0,\ otherwise\end{cases} \quad (3)$$

3. Orthogonality relations for characters

Introduce in the space F_G of all complex-valued functions on a finite group G the Hermitian form $(\chi, \eta)_G = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\eta(g)} = 0$ $(\chi, \eta \in F_G)$,

that can be taken as a (Hermitian) scalar product in F_G. Taking into account the fact that characters are constant on conjugate classes, we can write the ***scalar product of two characters*** χ, η in form

$$(\chi, \eta)_G = \frac{1}{|G|}\sum_{j=1}^{r}|K_j|\chi(g_j)\overline{\eta(g_j)}, g_j \in K_j,$$

where $K_j, j = 1, ..., r$ are conjugate classes of G.

Theorem 1. (The first orthogonality relation).

Let φ, ψ are (finite-dimensional) irreducible representations of a finite group G. Then

$$(\chi_\varphi, \chi_\psi)_G = \delta_{\varphi,\psi} = \begin{cases}1 if\ \varphi \cong \psi, \\ 0 otherwise\end{cases}. \quad (I)$$

Proof. By definition, $\chi_\varphi(g) = \sum_i \varphi_{ii}(g), \chi_\psi(g) = \sum_j \psi_{jj}(g)$. Putting $i = i_0, j = j_0$ in (2) and (3) and sum on i, j we receive in the case 1) of Lemma $\frac{1}{|G|}\sum_{g\in G}\psi_{jj}(g)\varphi_{ii}(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\psi_{jj}(g)\overline{\varphi_{ii}(g)} = (\chi_\psi, \chi_\varphi) = 0$.

In the case 2), $1=\frac{\sum_{j,i}\delta_{ji}}{\dim V}=\frac{1}{|G|}\sum_{g,i,j}\varphi_{jj}(g)\varphi_{ii}(g^{-1})=\frac{1}{|G|}\sum_{g\in G}(\sum_{j}\varphi_{jj}(g))(\sum_{i}\varphi_{ii}(g^{-1}))=(\chi_{\varphi},\chi_{\psi})_G$. ■

Consequence 1. The number ***s*** of equivalence classes of irreducible complex representations of a finite group G is less or equal the number of conjugate classes of G: $s\le r$.

Really, the characters $\chi_1,...,\chi_s$ of all nonequivalent irreducible complex representations are pairwise orthogonal, hence, linear independent, in the space ZF_G of central functions on G, consequently, $s\le\dim ZF_G=r$. ■

Consequence 2. If two representations have equal characters, then they are equivalent .

Proof. Let $\varphi:G\to GL(V),\psi:G\to GL(W)$ two representations and $\chi_{\varphi}=\chi_{\psi}$. Consider_the canonical decomposition of V into direct sum of non-equivalent irreducible representations: $V=n_1V_1\oplus...\oplus n_sV_s$ (n_iV_i means that there are n_i invariant subspaces isomorphic with V_i as representations. Denote $\chi_{\varphi_i}=\chi_i$, $i=1,...,s$, φ_i is the full system of irreducible representations.

It follows, that $\chi_{\varphi}=n_1\chi_1\oplus...\oplus n_s\chi_s$, and $(\chi_{\varphi},\chi_i)=n_1(\chi_1,\chi_i)+...+n_s(\chi_s,\chi_i)=n_i(\chi_i,\chi_i)=n_i$. Now, if $W=m_1W_1\oplus...\oplus m_sW_s$ is canonical decomposition, then $m_i=(\chi_{\psi},\chi_i)=(\chi_{\varphi},\chi_i)=n_i, i=1,...,s$. It shows that that V and W are isomorphic. ■

Theorem 2. s = r.

Proof. It is sufficient to prove that the orthogonal system $\chi_1,...,\chi_s$ is complete in the space ZF_G of central functions on G, that is equivalent to the condition: $\forall f\in ZF_G$ from $(\chi_i,f)=0, i=1,...,s$ it follows that $f\equiv 0$.

For arbitrary representation (φ,G,V) construct the linear operator $\varphi^*(f)=\sum_{g\in G}\bar{f}(g)\varphi(g): V\to V$.

Note, that $(\varphi\oplus\psi)^*(f)=\varphi^*(f)\oplus\psi^*(f)$.

This is evident in matrix form:

$$(\varphi\oplus\psi)(g)=\begin{Vmatrix}\varphi(g) & 0\\ 0 & \psi(g)\end{Vmatrix}\Rightarrow(\varphi\oplus\psi)^*(f)=\begin{Vmatrix}\varphi^*(f) & 0\\ 0 & \psi^*(f)\end{Vmatrix}.$$

For any irreducible representation φ_i the operator $\varphi_i^*(f)$ is the endomorphism of φ_i because f is central function: $\varphi(h)\varphi^*(f)\varphi(h^{-1})=\sum_{g\in G}\bar{f}(g)\varphi(hgh^{-1})=\sum_{g\in G}\bar{f}(hgh^{-1})\varphi(hgh^{-1})=\varphi^*(f)$.

So by Schur's Lemma $\varphi_i^*(f)=\lambda_iE$. Calculating traces, we find $\lambda_i\chi_i(1)=\sum_{g\in G}\bar{f}(g)tr\varphi_i(g)=\sum_{g\in G}\bar{f}(g)\chi_i(g)=|G|(\chi_i,f)=0$.

But for every representation φ, $\chi_{\varphi}=n_1\chi_1+...+n_s\chi_s\Rightarrow\varphi^*(f)=0$. Apply this for the regular representation $\Lambda:G\to GL(\mathbb{C}G),\Lambda(g)(x)=gx$ $for\, x\in G\, and$ $\mathbb{C}G$.

Namely, $0=\Lambda^*(f)(1)=(\sum_{g\in G}\bar{f}(g)\Lambda(g))(1)=\sum_{g\in G}\bar{f}(g)g$ (1 is the unit element of G).

By definition of $\mathbb{C}G$,the elements $g\in G$ linearly independent $\Rightarrow f(g)=0, \forall g\in G$, q.e.d.

Consequence. Any irreducible representation of G enters in the regular representation with multiplicity equal to its degree.

Proof. Recall, that the value of the character $\chi(g), g\in G$ of any permutation representation of G considered as a linear representation equals the number of fixed points of g in this action. In the regular representation $\Lambda:G\to S_{|G|}$(action of G on itself by left multiplications) non-identity elements have no fixed points. Therefore $\chi_\Lambda(g)=\begin{cases}|G|, g=1,\\ 0, g\neq 1\end{cases}$, and by the proof of consequence 2, the multiplicity $n_i=(\chi_i,\chi_\Lambda)=\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_\Lambda(g)}=\frac{\chi_i(1)|G|}{|G|}=\chi_i(1)$.

Burnside's equality. $\sum_{g\in G}\chi_i(1)^2=|G|$.

4. Character tables.

Every character $\chi=n_1\chi_1\oplus...\oplus n_r\chi_r$ of the group G is determined on the values of irreducible characters. ***Character table*** X is $r\times r$-matrix the rows of which are labeled with irreducible characters (in some order) and columns with conjugate classes (in some order).

So $X=(x_{ij})$, $x_{ij}=\chi_i(g_j), g_j\in K_j\ (i,j=1,...,r)$. (Usually, $\chi_1=1_G$, $K_1=\{1\}$.)

Illustrate this by a simple example.

Example. Character table of $G=S_3$.

	K_1	K_2	K_3
1	1	1	1
ε	1	-1	1
χ	2	0	1

The group $G=S_3$ has three conjugate classes: $K_1=\{1\},K_2=\{(1,2),(1,3),(2,3)\},K_3=\{(1,2,3),(3,2,1)\}$ and hence three irreducible characters. Two of them are one-dimensional, because $S_3'=A_3$ has index two, namely, unit $\chi_1=1_G,\chi_2=\varepsilon=sign$ and sign characters. The third one is two-dimensional, as $1^2+1^2+d_3^2=6$. We could calculate it from the corresponding representation, as the group of triangle, but let's invoke orthogonality relation. Let $\chi(1)=a,\ \chi(K_2)=b,\chi(K_3)=c$.

Calculate scalar products:
$$(\chi,1)=\frac{1}{6}(a+3b+2c)=0,\ (\chi,\varepsilon)=\frac{1}{6}(a-3b+2c)=0\Rightarrow b=0;$$
$$6(\chi,\chi)=a^2+3|b|^2+2|c|^2=a^2+2|c|^2=6\Rightarrow a=2,c=\pm1.$$

From $a+3b+2c=0,a=2\Rightarrow c=-1$. So we can finish the table.

Now rewrite the first orthogonality relation $(\chi_i,\chi_k)_G=\sum_{j=1}^{r}\frac{|K_j|}{|G|}\chi_i(g_j)\overline{\chi_k(g_j)}=\delta_{ik}$

in the form $\sum_{j=1}^{r}\frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}\cdot\frac{\overline{\chi_k(g_j)}}{\sqrt{|C_G(g_j)|}}=\delta_{ik}, g_j\in K_j$, because of $|K_j|=\frac{|G|}{|C_G(g_j)|}$.

It means that the matrix $M=\left(\frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}\right)$ is unitary on rows, namely $M\cdot\overline{M}^T=E$.

Then it is unitary on columns: $M^T\cdot\overline{M}=E\Rightarrow\sum_{i=1}^{r}\frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}\cdot\frac{\overline{\chi_i(g_k)}}{\sqrt{|C_G(g_j)|}}=\delta_{jk}$.

We derived

Theorem 3. (the ***second orthogonality relation***):

$$\sum_{i=1}^{r}\chi_i(g)\overline{\chi_i(h)}=\begin{cases}|C_G(g)|, if\ g,h\ are\ conjugate,\\ 0\ otherwise\end{cases}\quad \text{(II)}.$$

5. Theorem on the dimension of an irreducible representation.

Theorem 4. The degree of any irreducible complex representation $\varphi: G \to GL(V)$ of a finite group G divides the order $|G|$ of G.

The proof uses characters and is based on the theory of algebraic numbers.

Let χ be the character of φ.

Note that the values of all complex characters of the group G are contained in the cyclotomic field $\mathbb{Q}(\sqrt[|G|]{1})$ because $\chi(g) = \sum_{k=1}^{n} \lambda_k$, but λ_k are the roots $\sqrt[|g|]{1}$ *and* $|g| \mid |G|$, therefore $\chi(g) = \sum_{k=1}^{n} \lambda_k \in \mathbb{Q}(\sqrt[|G|]{1})$.

A complex number z is called algebraic if it is a root of a polynomial $p(x) = a_0 x^n + \ldots + a_n \ (n \geq 1)$ with (rational) integer coefficients. It is ***algebraic integer*** if $a_0 = 1$.

We need some lemmas.

Lemma 1. The set of all algebraic integer numbers is the ring (denoted O).

Proof. First prove that if $\omega_1, \ldots, \omega_m \in \mathbb{C}, \omega_j \neq 0, j = 1, \ldots, m$ and $M = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_m$ is a ring, then all numbers of M are algebraic integers. For any $\alpha \in M, \alpha \neq 0$, the products $\alpha\omega_i$ are integer linear combinations of $\omega_1, \ldots, \omega_m$: $\alpha\omega_j = \sum_{i=1}^{m} a_{ij}\omega_i, j = 1, \ldots, m, a_{ij} \in \mathbb{Z}$ (1)

This is a homogeneous linear system:

$$(\alpha E - A)\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0, A = \left(a_{ij}\right)$$ and $\omega_1, \ldots, \omega_m$ is its nonzero solution. It follows that $\det(\alpha E - A) = 0$, but $\det(\alpha E - A)$ is the polynomial with integer coefficients and leading coefficient equal 1. So α is algebraic integer.

For any $\alpha, \beta \in \mathrm{O}, \alpha^n + a_1\alpha^{n-1} + \ldots + a_n = 0, \beta^k + b_1\alpha^{k-1} + \ldots + b_k = 0, a_i, b_j \in \mathbb{Z}$ (2), the set $M = \{\sum_{i,j \geq 0} c_{ij}\alpha^i\beta^j : c_{ij} \in \mathbb{Z}, 0 \leq i < n, 0 \leq j < k\}$ is the ring, because relations (2) enable to express $\alpha^p\beta^q$, $p \geq n$ *or* $q \geq k$ through $\alpha^i\beta^j$ $(0 \leq i < n, 0 \leq j < k)$. Especially, $\alpha \pm \beta, \alpha\beta \in M$, hence, they are algebraic integers. Q.e.d.

Lemma 2. If the number $\alpha \in \mathbb{Q}$ is algebraic integer, then $\alpha \in \mathbb{Z}$.

Proof. Let $\alpha = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}, (p,q) = 1, \alpha^n + a_1\alpha^{n-1} + \ldots + a_n = 0, a_i \in \mathbb{Z}$

$p^n + a_1 p^{n-1}q + \ldots + a_n q^n = 0 \Rightarrow -p^n = a_1 p^{n-1}q + \ldots + a_n q^n \ \vdots q \Rightarrow q = 1$, q.e.d.

Return to characters.

Lemma 3. Let χ be the character of the representation of $\varphi: G \to GL(V)$ over C, then $\forall g \in G: \chi(g)$ is algebraic integer and $|\chi(g)| \leq \chi(1)$.

Proof. We have $\chi(g) = \sum_{k=1}^{n} \lambda_k$, where λ_k are the roots the polynomial $x^{|g|} - 1$, hence are algebraic integers, therefore $\chi(g) = \sum_{k=1}^{n} \lambda_k$, is algebraic integer by Lemma 1. Moreover, $|\chi(g)| = \left|\sum_{k=1}^{n} \lambda_k\right| \leq \sum_{k=1}^{n} |\lambda_k| = n = \chi(1)$.

Now let's introduce the ***group algebra*** $\mathbb{C}G$ ***of the group*** G ***over the field C*** of complex numbers: $\mathbb{C}G = \{\sum_{g \in G} \alpha_g g \mid \alpha_g \in \mathbb{C}\}$ is the linear space of formal linear combinations of elements of G which are considered as a basis, and with multiplication

$$(\sum_{g\in G}\alpha_g g)\cdot(\sum_{h\in G}\beta_h h)=\sum_{g,h\in G}\alpha_g\beta_h gh=(gh=x)=\sum_{x\in G}\gamma_x x,\ \gamma_x=\sum_{g\in G}\alpha_g\beta_{g^{-1}x}\ .$$

We may consider that $G\subset\mathbb{C}G$. Let $K_1,...,K_r$ are all conjugate classes of G. Construct the following elements of $\mathbb{C}G$: $\bar{K}_i=\sum_{g\in K_i}g, i=1,...,r$.

Lemma 4. $\{\bar{K}_1,...,\bar{K}_r\}$ is the basis of the center of the group algebra $Z(\mathbb{C}G)$. Moreover, $\forall i,j:\bar{K}_i\cdot\bar{K}_j=\sum_{l=1}^{r}a_{ij}^l\bar{K}_l$ (1) where a_{ij}^l are non-negative integers.

Proof. Find by what condition an element $z=\sum_{g\in G}\alpha_g g$ belongs to the center of $\mathbb{C}G$: for any $h\in G$ calculate $hzh^{-1}=\sum_{g\in G}\alpha_g hg\,h^{-1}$. Denote $hg\,h^{-1}=a$; when g runs over G then a runs over G. So $hzh^{-1}=\sum_{g\in G}\alpha_g hg\,h^{-1}=\sum_{a\in G}\alpha_{h^{-1}ah}a=z\Leftrightarrow\alpha_{h^{-1}ah}=\alpha_g,\forall h\in G$, hence α is a central function on G. Therefore, $z=\sum_{g\in G}\alpha_g g=\sum_{i=1}^{r}\sum_{g\in K_i}\alpha_g g=\sum_{i=1}^{r}\alpha_{g_i}\sum_{g\in K_i}g=\sum_{i=1}^{r}\alpha_{g_i}\bar{K}_i\,(g_i\in K_i)$.

Note that K_iK_j is the union of some conjugate classes: namely, if $x\in K_l$, $x=x_ix_j\in K_iK_j\Rightarrow\forall g\in G, gxg^{-1}=(gx_ig^{-1})(gx_jg^{-1})\in K_iK_j\Rightarrow K_l\subseteq K_iK_j$. Consequently, every expression $x=x_ix_j, x\in K_l$ contributes one into decomposition of $\bar{K}_i\bar{K}_j$, and the total coefficient a_{ij}^l equals the number of ways to present $x\in K_l$ in the form $x=x_j,x_i\in K_i,x_j\in K_j$. Q.e.d.

Lemma 5. Let χ is an irreducible complex character of the group G, then for any $g\in K_g\subset G$ the number $\omega(\chi,g)=|K_g|\frac{\chi(g)}{\chi(1)}$ (K_g is the conjugate class containing g) is algebraic integer.

Proof. First note: if χ is the character of the representation $\varphi:G\to GL(V)$, we may extend it to representation $\Phi:\mathbb{C}G\to L(V)$ of the group algebra by the rule $\Phi(\sum_{g\in G}\alpha_g g)=\sum_{g\in G}\alpha_g\varphi(g)$ We receive linear operators $\Phi_j=\Phi(\bar{K}_j)=\sum_{g\in K_j}\varphi(g), j=1,...,r$. It follows from lemma 4 that Φ_j commute with all $\varphi(g), g\in G$, hence, by Schur's Lemma, $\Phi_j=\lambda_j E$. Calculating trace we get

$$tr\Phi_j=\sum_{g\in K_j}tr\varphi(g)=|K_j|\chi(g_j)=\lambda_j\chi(1)\Rightarrow\lambda_j=\frac{|K_j|\chi(g_j)}{\chi(1)}=\omega(\chi,g_j),\ g_j\in K_j,\quad\text{and}\quad\Phi_j=\frac{|K_j|\chi(g_j)}{\chi(1)}E.$$

On the other hand, applying Φ to the decomposition (1) from Lemma 4, we have $\Phi_i\Phi_j=\sum_{l=1}^{r}a_{ij}^l\Phi_l$ whence $\frac{|K_i|\chi(g_i)}{\chi(1)}\cdot\frac{|K_j|\chi(g_j)}{\chi(1)}=\sum_{l=1}^{r}a_{ij}^l\frac{|K_l|\chi(g_l)}{\chi(1)}$ or $\omega(\chi,g_i)\omega(\chi,g_j)=\sum_{l=1}^{r}a_{ij}^l\omega(\chi,g_l)$. Consequently (confer with the proof of Lemma 1) $\omega(\chi,g_i)$ is algebraic integer. Q.e.d.

Proof of Theorem 4. As χ is irreducible,

$$|G|(\chi,\chi)_G=\sum_{g\in G}\chi(g)\overline{\chi(g)}=\sum_{i=1}^{r}|K_i|\chi(g_i)\overline{\chi(g_i)}=|G|\Rightarrow$$

$$\sum_{i=1}^{r}\frac{|K_i|\chi(g_i)}{\chi(1)}\overline{\chi(g_i)}=\frac{|G|}{\chi(1)}\in\mathrm{O}\cap\mathbb{Q}=\mathbb{Z}$$

by lemmas 1,2 and 5. The theorem is proved.