Theorem. If two representations φ, ψ have equal characters: $\chi_\varphi = \chi_\psi$, then $\varphi \cong \psi$.

Proof. Use the direct decompositions:
$\varphi = \bigoplus_{i=1}^{r} k_i \varphi_i$, where $\{\varphi_i\}$ are all non-isomorphic irreducible representations of G, k_i are multiplicities.
$\psi = \bigoplus_{i=1}^{r} k_i' \varphi_i \Rightarrow \chi_\varphi = \sum_{i=1}^{r} k_i \chi_i$, $\chi_\psi = \sum_{i=1}^{r} k_i' \chi_i$.
$\{\chi_i\}$ are all irreducible characters.
$k_i = (\chi_\varphi, \chi_i)_G = (\chi_\psi, \chi_i) = k_i'$, all $i = 1, \dots, r \Rightarrow \psi \cong \varphi$. q.e.d.

Character table of the group G (the table of irreducible characters)

$$X(G) = \begin{array}{c} \\ \chi_1 \\ \vdots \\ \chi_i \\ \vdots \\ \chi_r \end{array} \begin{array}{c} K_1 \dots K_j \dots K_r \\ \left[\begin{array}{ccc} & \vdots & \\ \dots & \chi_i(g_j) & \\ & & \end{array}\right] \end{array} \quad (r \times r)$$

$X_{ij} = \chi_i(g_j)$

(g_j is some representative of the conjugate class K_j)

$G = \bigsqcup_{j=1}^{r} K_j$

1.st orthogonality relation:

$$(\chi_i, \chi_k) = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_k(g)} = \frac{1}{|G|} \sum_{j=1}^{r} \left(\sum_{g_j \in K_j} \chi_i(g_j) \overline{\chi_k(g_j)} \right)$$

$$= \frac{1}{|G|} \sum_{j=1}^{r} |K_j| \chi_i(g_j) \overline{\chi_k(g_j)} = \delta_{ij}$$

Example. Let $G = A_4$ of even permutations of degree 4. $G = \{(i,j,k);\ (i,j)(k,l)\}$

There are conjugate classes: $\{e\}$, $\{(i,j)(k,l)\}$, $\{\underbrace{(1,2,3), \dots}_{4 \text{ elements}}\}$, $\{\underbrace{(1,3,2), \dots}_{4 \text{ elements}}\}$

We know, that the order of $cl(g) = \frac{|G|}{|C_G(g)|}$, $C_G(g) = \{h \mid hg = gh\}$.

$C_G((1,2,3)) = \langle (1,2,3) \rangle$ of order 3 $\Rightarrow K((1,2,3)) = \frac{12}{3} = 4$

	$K_1 = \{e\}$	$K_2 = \{(i,j)(k,l)\}$	$K_3 = \{(1,2,3)\}$	$K_4 = \{(1,3,2) \dots\}$
χ_1	1	1	1	1
χ_2	1	1	ε	ε^{-1}
χ_3	1	1	ε^{-1}	ε
χ_4	3	$\cancel{a}$ -1	$\cancel{b}$ 0	$\cancel{c}$ 0

$A_4' = V_4$, $|G : G'| = 3 \Rightarrow \exists$ three one-dimensional characters

Every one-dimensional representation contains G' in its kernel $\Rightarrow \chi_i((i,j)(k,l)) = 1$, $1 \le i \le 3$

$\chi_4(e) = ?$ $|G| = 12 = 1^2 + 1^2 + 1^2 + \chi_4(e)^2 \Rightarrow \chi_4(e) = 3$

$(\chi_4, 1) = \frac{1}{12}(3 + 3a + 4b + 4c) = 0$

$(\chi_4, \chi_2) = \frac{1}{12}(3 + 3a + 4\bar{\varepsilon} b + 4\varepsilon c) = 0$

$(\chi_4, \chi_3) = \frac{1}{12}(3 + 3a + 4\varepsilon b + 4\bar{\varepsilon} c) = 0$

$(\chi_4, \chi_4) = \frac{1}{12}(9 + 3|a|^2 + 4|b|^2 + 4|c|^2) = 1$

$$\begin{cases} 3a + 4b + 4c = -3 \\ 3a + 4\bar{\varepsilon} b + 4\varepsilon c = -3 \\ 3|a|^2 + 4|b|^2 + 4|c|^2 = 3 \end{cases}$$

$a = -1$, $b = c = 0$

The second orthogonality relation

Rewrite the first relation $(\chi_i, \chi_k) = \sum_{j=1}^{r} \frac{|K_j|}{|G|} \chi_i(g_j) \overline{\chi_k(g_j)} =$

$= \sum_{j=1}^{r} \frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}} \cdot \frac{\overline{\chi_k(g_j)}}{\sqrt{|C_G(g_j)|}}$, because $|K_j| = \frac{|G|}{|C_G(g_j)|}$.

This means that the matrix $M = \left(\frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}} \right)$ is unitary by rows: $M \cdot \overline{M}^T = E$

It means that it is also unitary by columns:

$$M^T \cdot \overline{M} = E \Leftrightarrow \sum_{i=1}^{r} \frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}} \cdot \frac{\overline{\chi_i(g_k)}}{\sqrt{|C_G(g_j)|}} = \delta_{jk}.$$

We derived the second orthogonality relation:

$$\sum_{i=1}^{r} \chi_i(g) \overline{\chi_i(h)} = \begin{cases} 0, & \text{if } g, h \text{ are not conjugate} \\ |C_G(g)|, & \text{if } h \text{ and } g \text{ lie in the same conjugate class.} \end{cases} \quad (*)$$

The theorem on the dimension of an irreducible representation

Th. The dimension of any complex irreducible representation $\varphi : G \to GL(V)$ of a finite group G divides $|G|$.

Let χ be the character of φ. $\forall g \in G$, $\chi(g) \in \mathbb{Q}(\sqrt[|G|]{1})$
($\sqrt[|G|]{1}$ is some primitive complex root of 1 of degree $|G|$)

A complex number $x \in \mathbb{C}$ is algebraic if $\exists\, p(x) = a_0 x^n + \dots + a_n$ ($n \geq 1$) with integer (rational) coeff. It is called algebraic integer if $a_0 = 1$, $p(x) = 0$.

Lemma 1. The set of all algebraic integers (over $\mathbb{Q}$) is the ring (often $\mathbb{O}$)

Lemma 2. Есш $\alpha \in \mathbb{Q} \cap \mathbb{O} \Rightarrow \alpha \in \mathbb{Z}$.