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Dedicated to my mother

ROLLING SIMPLEXES AND THEIR COMMENSURABILITY, 2 (FIELD EQUATIONS IN ACCORDANCE WITH TYCHO BRAHE)

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Abstract. Various Cartesian models of central power fields with quadratic dynamics are studied. These examples lead the reader to comprehension of basic aspects of the differential algebraic-geometrical Brahe – Descartes – Wotton theory, which embraces central power fields whose dynamics is composed of flat affine algebraic curves of degree at most N ($N = 1, 2, 3, ...$). When $N = 2$, a **quadratic rolling simplexes law** is proved by purely algebraic means.

Key words: *field, Cartesian plane, affine chart, rolling, incompressibility, quadratic curve, focus, directrix, gravitation, promeasure, prometrics, differential algebra.*

"A pure unlimited mind is a deity itself." Hegel.

The notes below do not talk about foundations of cosmic navigation and the basics of field theory. They are written neither for fans of science fiction, nor for "proud and learned" scientists (see [1]). They are for those who following me will read them again. These are my thoughts on our education. About the ways and reasons explaining one how and why we are all together proved where to be predestined. Hoping for the future, skeptically referring to the past, indulging a wishful thinking, one is aware and knows (or he does think so), the other tries to calculate what is in advance waiting us for. I am not impartial. My sympathy entirely lies on the side of such masters as Democritus, Tycho Brahe, Descartes, Cavendish, Faraday, Maxwell. But I have to be objective.

"I like persons better than principles,
and I like persons with no principles
better than anything." H. Wotton.

1. There are Principles at Stake that One cannot Surrender. When I was a second-year student, my desk-mate Yuli Koshevnik drew a circle on a blank sheet of paper and offered me to plot a point. Feeling some kind of trick I put it, just in case, not at the center of the circle, but a little to the left. Then Yuli pictured a typical child drawing of a house with chimney on the same sheet and asked me to raise smoke from it. I cautiously directed it straight up. He solemnly claimed that Albert Einstein put a point outside of the circle and directed smoke in such a way that the house became looking like a rhinoceros preparing to fight. Somehow it was branded on my memory just as the fact that during examinations at the end of the second semester of our studies Eugene Solomonovich Golod fraid our nerves when he offered to drop a perpendicular to a fixed point from a given line in the well-known model of the Lobachevsky plane with the use of a compasses and a ruler.

I recalled these tests on mediocrity many years later, when I was finishing my manuscript [2]. Looking through a popular series, in which names such as Copernicus, Kepler, Galileo, and Newton were praised up to the skies, I suddenly realized that Tycho Brahe (1542 - 1601), who laid an experimental foundation and initiated the development of theoretical principles of the future celestial mechanics (Kepler worked at his observatory), was aware of the following fact.

Proposal (T. Brahe). *All points of intersection of tangent lines to a circle that pass through the ends of bisectants intersecting at one point inside of the circle lie on a straight line.*

Proof (Descartes). There exists a projective transformation over the affine plane that maps the circle examined onto itself and sends the point of intersection of bisectants into the circle center.

At that moment the ring was circled, the solution to the task suggested by E. S. Golod and the tests of Yu. Koshevnik recurred to me. I was dawned – in the 17th century R. Descartes had already walked through the "wall" of quadratic curve. In his doubts about the location of the center of the world Tycho Brahe was not as judgemental as Kepler, who put that center at its focus. "Directrix" and "focus" were still coexisting as two shoulders of one yoke and were not bound to a particular Euclidian metrics. The way was still open. The barrier was set up later.

"De Omnibus Dubitandum." R. Descartes.

2. Cartesiana: Several Examples of Central Fields, in Each of Which Every Movement Is Realized over Its Cuadratic Curve. In the three-dimensional affine space K^3 ($K = \mathbf{R}, \mathbf{C}$) a "centripetal" movement $\vec{R}(t) = (x(t), y(t), z(t))$ round the point O is characterized in any affine coordinate system with the origin at $O \overset{\text{def}}{=} (0,0,0)$ by the vector equality $[\vec{R}(t), \vec{R}''(t)] = \vec{0}$, which leads to $\frac{d}{dt}([\vec{R}(t), \vec{R}'(t)]) = \vec{0}$ and $[\vec{R}(t), \vec{R}'(t)] = (i_1, i_2, i_3)$, where $i_1, i_2, i_3 \in K$. That is why the space curve $\vec{R}(t)$ lies on the plane given by the equation $i_1 \cdot x + i_2 \cdot y + i_3 \cdot z = 0$. So, our further consideration is reduced to flat models of central fields.

2.1. Galileo fields – uniformly accelerated motion: $x''(t) = \mathrm{g}_x$, $y''(t) = \mathrm{g}_y$ ($\mathrm{g}_x, \mathrm{g}_y \in K$). In this case, the point O lies on the ideal line in the direction of the vector $(\mathrm{g}_x, \mathrm{g}_y)$.

2.2. Harmonic oscillator: $(x''(t), y''(t)) = -h \cdot (x(t), y(t)) \quad (h \in K)$.

2.3. Coulomb fields: $(x''(t), y''(t)) = -\frac{4 \cdot \pi^2 \cdot k}{(r(t))^3} \cdot (x(t), y(t)) \quad (r^2 = x^2 + y^2, k \in K)$.

2.4. Ptolemy fields. Classics of the genre:

$(x''(t), y''(t)) = -2 \cdot \frac{(x'(t)^2 + y'(t)^2)}{x^2(t) + y^2(t) - \delta} \cdot (x(t), y(t))$ $(\delta \in K)$.

When $\delta > 0$ ($\delta \in \mathbf{R}$), the quantity $r_\delta \overset{\text{def}}{=} \delta^{1/2}$ is called the *radius of celestial sphere* (a prototype of the Schwarzschild radius).

2.5. Solar oscillator (volnokhron). There are two equivalent ways to describe fields of that kind in terms of a differential algebra S.

2.5.1. A system of equations in prometrics (O. V. Efimovskaya):

(a) $(x''(t), y''(t)) = -\frac{\tau_S}{(\tau(t))^3} \cdot (x(t), y(t)) \quad (\tau_S \overset{\text{def}}{=} 4 \cdot \pi^2 \cdot k_S / c^3 \in K)$,

(b) $|x^2(t), x(t) \cdot y(t), y^2(t), \tau^2(t)| = 0$ (an equation of prometrics),

where k_S is Tycho Brahe's solar constant and c is the speed of light.

2.5.2. A system of equations in directrices (O. V. Gerasimova):

(a) $(x''(t), y''(t)) = -\frac{\tau_S}{(\tau(t))^3} \cdot (x(t), y(t)) \quad (\tau_S \overset{\text{def}}{=} 4 \cdot \pi^2 \cdot k_S / c^3 \in K)$,

(b) $(\vec{R}'(t), [\vec{R}''(t), \vec{R}'''(t)]) = 0$ (an equation of flat waves).

2.6. Theorem. *Any analytic solution* $x(t), y(t), \tau(t)$ $(x(t) \cdot y'(t) - x'(t) \cdot y(t) \neq 0)$ *of equations* 2.5.1 *with respect to complex variable* t *satisfies equations* 2.5.2, *and any analytic solution of system* 2.5.2 *satisfies equations* 2.5.1. *In particular, every curve* $\vec{R}(t) \overset{\text{def}}{=} (x(t), y(t), \tau(t))$ *is flat and contained in a quadratic one.*

2.7. E.–G.–H.-universe. Equations 2.5.1, 2.5.2, defining the differential algebra S, admit first integrals, by means of which E.-G. algebra S can be realized in a more habitual way, one should say.

2.7.1. s_0-deformation. Let us determine a differential algebra H_S by four generators x, y, s, s_0 and four defining (differential) relations

$s_0' = 0$, $(x'', y'', s'') = -\frac{1}{\tau_S^2} \cdot \frac{1}{s^3(t)} \cdot (x, y, s - s_0)$ $(\tau_S \overset{\text{def}}{=} 4 \cdot \pi^2 \cdot k_S / c^3 \in K)$.

A homomorphism of imbedding $S/\mathrm{Rad}\, S \to H_S$ is given by the mapping $x \to x$, $y \to y$, $\tau \to \tau_S \cdot s$ (thereat $\frac{1}{\tau_S}$ has the same order as frequency). It goes without saying that these four equations generate quadratic dynamics.

2.7.2. A black point (Garin – Hooke accelerator). A differential algebra H_0 and the equations

$(x'', y'', s'') = -\frac{1}{\tau_S^2} \cdot \frac{1}{s^3(t)} \cdot (x, y, s) \quad (\tau_S \overset{\text{def}}{=} 4 \cdot \pi^2 \cdot k_S / c^3 \in K)$,

arising when $s_0 = 0$, are worthy of separate discussion and study. In this case, each quadratic curve passes through its "focus" at the origin of the coordinate system.

2.8. Expressibility by differential relations in two variables (without constants) of two conditions: a condition of field centrality and a condition of quadratic dynamics. Let us point out two models of such universes in the formalism of differential algebras.

2.8.1. A universal differential algebra in the signature $y, t, \frac{d}{dx}$**.** It is given by two differential generators y, t and two differential relations of Descartes:

(a) $t''_x \cdot (x \cdot y'_x - y) = t'_x \cdot x \cdot y''_x$ (a condition of field centrality),

(b) $9 \cdot y'''''_x \cdot (y''_x)^2 - 45 y''''_x \cdot y'''_x \cdot y''_x + 40 \cdot (y'''_x)^3 = 0$ (a condition of quadratic dynamics).

2.8.2. Tycho Brahe quadratic chaos: a model of the universal central field with quadratic dynamics in the natural signature $x, y, \frac{d}{dt}$**.** Let us define a differential algebra B_2 by two generators x, y and two differential relations of Capelli's kind:

(a) $\sigma_{0,2}(x, y) = 0$ (a condition of field centrality),

(b) $b_2(x, y) = 0$ (a condition of quadratic dynamics),

where $\sigma_{i,j}(x, y) \stackrel{\text{def}}{=} x^{(i)} \cdot y^{(j)} - x^{(j)} \cdot y^{(i)}$,

$$b_2(x,y) \stackrel{\text{def}}{=} -9 \cdot \sigma_{1,5} \cdot \sigma_{1,2}^2 - 45 \cdot \sigma_{2,4} \cdot \sigma_{1,2}^2 + 45 \cdot \sigma_{1,4} \cdot \sigma_{1,3} \cdot \sigma_{1,2} + 90 \cdot \sigma_{2,3} \cdot \sigma_{1,3} \cdot \sigma_{1,2} - 40 \cdot \sigma_{1,3}^3 =$$
$$-9 \cdot \sigma'''_{1,2} \cdot \sigma_{1,2}^2 - 27 \cdot \sigma'_{2,3} \cdot \sigma_{1,2}^2 + 45 \cdot \sigma''_{1,2} \cdot \sigma'_{1,2} \cdot \sigma_{1,2} + 45 \cdot \sigma_{2,3} \cdot \sigma'_{1,2} \cdot \sigma_{1,2} - 40 \cdot (\sigma'_{1,2})^3.$$

Let us call it the Tycho Brahe algebra.

Till this moment, we were led in our narration by pure reason. But rumors, from time to time, have brought in tidings to us from the bank of the Thames. It is the time to despise common sense and dive into the depths of their paradigm.

"Entities must not be multiplied beyond necessity."
William Ockham (1290 – 1347)

3. The History and the Time: Jokes and Stratagems of Creators? In 1634, living in Holland, Rene Descartes, officer of the operating reserve of His Eminence Cardinal Richelieu guards, expressed his credo in a letter to one of his correspondents in the following way: "To live well, one should live inconspicuously." Times he lived in were uneasy. The ideas of the hyperboloid of Engineer Garin were most certainly not flying in the air. Circling rumors about women flying on broomsticks were spread from mouth to mouth. One not only had to interact and collaborate with representatives of enforcement authorities and the prototype of the future British "Intelligence Service", but also was forced to make it up with the secret services of the Inquisition: Dominicans, Franciscans, Jesuits, at first spreading the education over the territories, independent of the latest. The fates of Copernicus, Giordano Bruno, Galileo showed that might is right. The formation and the breeding of the manpower of a new kind in science and education was necessary. It became evident that this task could not be solved in nonconformist terms. The institutions were essential. At the first step of realization of the purpose, this part of the problem was taken by the specialized colleges and monasteries.

It was the time of the giants. Philosopher-strategists. The conception of the colossal project, not designed for immediate profits. For centuries. The emphasis was placed not on the studies of the results and truths already gained, but on rediscovery of them in the other's heads, on the adoption of the methods of investigation that had already been elaborated and on the search for new methods of study. By virtue of that, the number of followers for whom these inventions became their own increased multifold. The psychological element of that was delicately counted: "Blood bought in battle is valued more." According to the credo True heroes had no right to show up, they should remain in shadow.

The circumstances favored the next breakthrough, the moment was convenient:

(a) Kepler had already published the results of Tycho Brahe's observatory;

(b) Descartes gave a grounding and developed the foundations of analytical geometry, which turned the art of geometric reasoning about the properties of quadratic curves into the routine of numerical and symbolical calculations;

(c) he was also the one who restored the idea of force as an entity producing acceleration.

It all came together itself.

“There opened an abyss full of stars.
The stars are countless, the abyss is bottomless.”
M.V. Lomonosov.

3.1. The bait: Kepler laws. Peoples say that there is no such thing as a simple solution. Quite so. But the times, as a result of a long, laborious processing of experimental data, do sometimes give birth to visual models, accepting inornate, adequate formalization and its fair interpretation.

The Kepler laws can now be studied from schoolbooks.

3.1.1. 1st law. *The orbit of every planet is an ellipse with the Sun at one of the two foci.*

3.1.2. 2nd law. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

3.1.3. 3rd law. *The ratio of the squares of the orbital periods of planets round the Sun is equal to the ratio of the cubes of semi-major axis of their orbits.*

3.2. The hook: formalization. It is not well spread that Aristotle had already known, how all the projections of a cone of revolution sections on the plane, orthogonal to its axis, are organized, meaning:

(a) any quadratic curve in this plane, a focus of which lies at the top of the cone, can be gained by this construction;

(b) all the projections, which are not lines or pairs of lines, are quadratic curves with common foci.

With the introduction of the elements of analytical geometry into the mathematical custom, it became possible to express such statements in terms of algebraic equations. Let us take as a trial one the cone of revolution, given in the Euclidian coordinate system (x, y, r) by the equation $x^2+y^2-r^2 = 0$. Then for any plane defined by the equality $r = \alpha \cdot x + \beta \cdot y + \delta$ the projection of the section specified by these two equations on the plane (x, y) is a nontrivial quadratic curve when $\delta \neq 0$: $x^2 + y^2 - (\alpha \cdot x + \beta \cdot y + \delta)^2 = 0$.

3.2.1. Interpretation of the 1st law. *The orbit of every planet is a quadratic curve of the form* $x^2 + y^2 - (\alpha \cdot x + \beta \cdot y + \delta)^2 = 0$, $(\delta \neq 0, \alpha, \beta \in K)$. *The Sun is located at the coordinate origin.*

Exercise. Prove that the focal distance of the quadratic curve $x^2 + y^2 - (\alpha \cdot x + \beta \cdot y + \delta)^2 = 0$ is equal to δ.

3.2.2. The formalization of the 2nd law. *At every time moment t the radius-vector $\vec{R}(t) = (x(t), y(t))$ joining the origin and the planet satisfies the equality $[\vec{R}(t), \vec{R}'(t)] = (x(t) \cdot y'(t) - x'(t) \cdot y(t)) = \kappa$, where $\kappa \in K$. In particular, during any interval of time Δt the radius-vector $\vec{R}(t)$ sweeps out an area equal to $\frac{1}{2} \cdot |\kappa| \cdot \Delta t$, and $[\vec{R}(t), \vec{R}''(t)] = (x(t) \cdot y''(t) - x''(t) \cdot y(t)) = 0$.*

The arguments in favor of this treatment of the second law are well known and are intelligible to students familiar with the elements of first-grade analysis. However, a person considering that to understand this interpretation one should know what is differentiation and integration is mistaken. (At least in the common sense of the majority. After all, everything can be credited with a regular genius hunch, there is no dout that more complex statements and formulas were already written in the seventeenth century (see 2.8.1)). Indeed, let us consider a uniformly moving object. Then any observer staying at an arbitrary point of the space out of the movement line will find out that the radius-vector joining him and the object sweeps equal areas during equal time intervals, and as the formula $\frac{1}{2} \cdot |x(t) \cdot y'(t) - x'(t) \cdot y(t)|$ of the area of the triangle generated by vectors $\vec{R}(t), \vec{R}'(t)$ concerns the basics of analytical geometry, he will get the formulas of the statement 3.2.2 and will automatically accept the same results applied to a line as a highly believable conjecture about a uniform, by the second law of Kepler, movement over every quadratic curve. His confidence in the correctness of such an assumption will be reinforced by the fact that when constructing each point of the quadratic curve by any five its points and the tangency at this points in the affine Descartes plane (see [2]), it is sufficient to use only a setsquare and a ruler. As for the rolling, allowing us to practically move the triangles, generated by vectors $\vec{R}(t_1), \vec{R}'(t_1)$ and $\vec{R}(t_2), \vec{R}'(t_2)$, one into another, will finish the business. Indeed, if a pupil at the entrance examination at mechanic-mathematical faculty must be able to prove the formula of a circle sector by elementary means, then one may not be as wise as Solomon to perform the same work for any ellipse with respect to its inner point in the seventeenth century or even earlier.

3.2.3. Tycho Brahe solar constant. Let a and b be the major and the minor semi-axes of an ellipse, and T be the solar orbital period. Then, according to 3.1.3 for any two planets (i and j) the following equality holds: $\frac{a^3(i)}{a^3(j)} = \frac{T^2(i)}{T^2(j)}$. In can be rewritten in the form $\frac{a^3(i)}{T^2(i)} = \frac{a^3(j)}{T^2(j)}$. Consequently, the relation $\frac{a^3}{T^2}$ does not depend on the planet number. We denote this constant by k_S and call it the Tycho Brahe solar constant. Now the third law of Kepler can be formulated in the following way.

3.2.4. *For every planet the ratio of the cube of the major semi-axis of the ellipse and the square of its solar orbital period is equal to the solar constant* k_S.

3.2.5. The mystery of the three laws: $\kappa^2/\delta = 4 \cdot \pi^2 \cdot k_S$**.** Indeed, as the area of an ellipse is equal to $\pi \cdot a \cdot b$, by virtue of 3.2.2 we see that $T = \frac{\pi \cdot a \cdot b}{|\kappa|/2}$. However, $a^3 = k_S \cdot T^2$ (see 3.2.3) and $a^3 = k_S \cdot (\frac{\pi \cdot a \cdot b}{|\kappa|/2})^2$. This leads to $\kappa^2 = 4 \cdot \pi^2 \cdot k_S \cdot (b^2/a)$. But b^2/a is the focal length of the ellipse, and in 3.2.1 (see the exercise) it was noticed that the focal length of the curve $x^2 + y^2 - (\alpha \cdot x + \beta \cdot y + \delta)^2 = 0$ is equal to δ, which proves the proportion between δ, κ and k_S.

3.2.6. Resume. The formula derived above describes important dynamic characteristics of the planet motion such as the velocity of sweeping areas, the orbital periods, the velocity in each point in terms geometric characteristics and the Tycho Brahe solar constant.

3.3. The trap: from the postulates of formalism to the equations of Kepler dynamics. Imagine that at some moment One appears on the time axis, who

(a) extends the customary plane with an ideal (infinitely distant) line;

(b) understands what is meant by a projective mapping of an extended plane, and is able to use it;

(c) introduces an oblique coordinate system into the scientific custom and makes calculations in it together with a common rectangular one;

(d) can formulate in any coordinate system equations of a cone and a plane in three-dimensional space, a line and a tangency to an ellipse in plane;

(e) discovers in the class of rational functions with respect to a variable t the main law of differential-algebraic formalism $(f(t) \cdot g(t))' = f'(t) \cdot g(t) + f(t) \cdot g'(t)$ and starts to interpret the term "force" as an entity causing only velocity change: $v'(t)$;

(f) guesses and can justify in local cases that the second law of Kepler is equivalent to the fact that the vector of acceleration of a planet is directed toward the Sun;

(g) develops a formula for calculation of the area of a triangle in any coordinate system, proves the invariance of the derived expression with respect to the rolling, shows the geometrical way of rolling of the triangles spanned on vectors $R(0), R'(0)$ and $R(t), R'(t)$ one into another, when the ends of the radius-vectors $R(t) \stackrel{\text{def}}{=} (x(t), y(t))$ lie on quadratic curves.

Irrespective of the axis, where Descartes stays, the existing situation suggests to what-not One that it would be fair to visualize the proportionality factor of two collinear vectors $R(t)$ and $R''(t)$, and he does the preliminary step.

3.3.1. The first step: the calculation of the velocity vector $\vec{R}'(t)$**.** This procedure slightly differs from the derivation of the formula of a tangency for a quadratic curve. The curve is given by the equation $x^2 + y^2 - (\alpha \cdot x + \beta \cdot y + \delta)^2 = 0$ and its (Kepler's) parametrization is defined by the equation $x(t) \cdot y'(t) - x'(t) \cdot y(t) = \kappa$ (see 3.2.2). Changing both sides of the first equality with the transformation "prime" (see property (e)) he will get the system of two equations for $x'(t)$ and $y'(t)$

$$2 \cdot x'(t) \cdot x(t) + 2 \cdot y'(t) \cdot y(t) - 2 \cdot (\alpha \cdot x'(t) + \beta \cdot y'(t))(\alpha \cdot x + \beta \cdot y + \delta) = 0,$$

$$x(t) \cdot y'(t) - x'(t) \cdot y(t) = \kappa$$

. What-not One will have no problem solving it:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \frac{-\kappa}{\delta \cdot d(t)} \cdot \begin{pmatrix} y(t) - \beta \cdot d(t) \\ -x(t) - \alpha \cdot d(t)) \end{pmatrix}, \qquad (!)$$

$$d(t) \stackrel{\text{def}}{=} \alpha \cdot x(t) + \beta \cdot y(t) + \delta \quad (d^2(t) = x^2(t) + y^2(t)).$$

Moreover, unlike us, he already knows that the line given by the equation $\alpha \cdot x + \beta \cdot y + \delta = 0$ is the directrix to the curve $x^2 + y^2 - (\alpha \cdot x + \beta \cdot y + \delta)^2 = 0$, and interprets it in the spirit of Tycho Brahe's proposal (see section 1).

3.3.2. The second step: the calculation of the acceleration vector $\vec{R}''(t)$. Finding one's feet at the first step in manipulation of the "prime" One (whatnot) will "derive" both parts of equality (!) and, using it one more time to exclude the newly gained equalities $x'(t), y'(t), d'(t)$ from the right-hand part, will simply get relations:

$$\begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = \frac{-\kappa^2}{\delta} \cdot \frac{1}{d^3(t)} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

$$d(t) \stackrel{\text{def}}{=} \alpha \cdot x(t) + \beta \cdot y(t) + \delta \quad (d^2(t) = x^2(t) + y^2(t)).$$

3.3.3. In the grip of formalism: the equations of Kepler dynamics. By virtue of the third postulate (see 3.2.3 and 3.2.4), the relation κ^2/δ differs from the Solar constant by the numerical factor $4 \cdot \pi^2$. That is why the latest equalities can be rewritten in the form

$$\begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix} = -\frac{4 \cdot \pi^2 \cdot k_S}{r^3(t)} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $r(t) = (x^2(t) + y^2(t))^{1/2}$.

3.3.4. The intelligence and the mind: O. Whatnot relations. A devotee of an oblique coordinate system will try to go beyond the postulates of formalisms of sections 3.2.1–3.2.4 and do the third step. He will try to calculate the centripetal acceleration vector for any quadratic curve defined by the equation

$$a_{11} \cdot x^2 + 2 \cdot a_{12} \cdot x \cdot y + a_{22} \cdot y^2 + 2 \cdot a \cdot x + 2 \cdot b \cdot y + c = 0$$

and will inevitably gain the relations

$$\frac{d^2}{dt^2} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -\kappa^2 \cdot \frac{det \begin{pmatrix} a_{11} & a_{12} & a \\ a_{12} & a_{22} & b \\ a & b & c \end{pmatrix}}{(a \cdot x(t) + b \cdot y(t) + c)^3} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

When $c \neq 0$, the quadratic curve can be defined by the equation

$$\mu_{11} \cdot x^2 + 2\mu_{12} \cdot x \cdot y + \mu_{22} \cdot y^2 - (\alpha \cdot x + \beta \cdot y + \delta)^2 = 0.$$

Then the previous equalities take the form

$$\frac{d^2}{dt^2} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{-\kappa^2}{\delta} \cdot det \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix} \cdot \frac{1}{d^3(t)} \cdot \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $d(t) = \alpha \cdot x(t) + \beta \cdot y(t) + \delta \quad (d^2(t) = \mu_{11} \cdot x^2(t) + 2 \cdot \mu_{12} \cdot x(t) \cdot y(t) + \mu_{22} \cdot y^2(t))$.

3.3.5. Girl dreams: $\pm$ focusing of plane waves. [1] An object throttles down and suddenly "materializes" on the locator screen, then describes an arc, accelerates and disappears, not having reached the border of the display. The picture is familiar, is it not? Minding the truth that ignorance starts before science and barbarism starts after it, I will not give you any physical interpretation of the latest acceleration formulas, but restrict myself to dry the statement of the fact.

Theorem (O. Whatnot). *For any analytical solutions* $x(t), y(t) \quad (dim_K(K \cdot x(t) + K \cdot y(t)) = 2)$ *of the equations (with respect to real or complex variables)* 2.8.2 *generating the quadratic Tycho Brahe chaos, the following equalities are correct:* $\frac{x''(t)}{x(t)} = -\frac{x'(t) \cdot y''(t) - y'(t) \cdot x''(t)}{x(t) \cdot y'(t) - y(t) \cdot x'(t)} = \frac{y''(t)}{y(t)}$, *and when* $(dim_K(K \cdot x'(t) + K \cdot y'(t)) = 2)$ *the linear spaces* $K \cdot 1 + K \cdot x(t) + K \cdot y(t) + K \cdot (x(t)/x''(t))^{1/3}$, $K \cdot x^2(t) + K \cdot x(t) \cdot y(t) + K \cdot y^2(t) + K \cdot (x(t)/x''(t))^{2/3}$ *are not better than three-dimensional.*

[1] Hyper-speeds, UFO, flying saucers, gravityflight, the Paraboloid of Engineer Garin, witch's broom...

3.4. The trick: from the equations of dynamics to the postulates of formalism. The second of the postulates is more than obvious:

$(x(t)\cdot y'(t)-y(t)\cdot x'(t))'=[R(t),R'(t)]'=x(t)\cdot y''(t)-y(t)\cdot x''(t)=-\frac{4\cdot\pi^2\cdot k_S}{r^3(t)}\cdot[R(t),R(t)]=0.$
(I have not had a chance to meet a scientist who would try to differentiate the just gained first integrals at my watch, while solving differential equations. It is hard to tell how I would rate such a situation. Probably, I would consider him an idiot.) Let us scalar multiply both parts of the equality $(x''(t),y''(t))=-\frac{4\cdot\pi^2\cdot k_S}{r^3(t)}\cdot(x(t),y(t))$, $(r(t)=(x^2(t)+y^2(t))^{1/2})$ by the vector $\vec{R}'(t)=(x'(t),y'(t))$. Then

$$\frac{1}{2}\cdot\left(x'(t)^2+y'(t)^2\right)'=\left(\frac{4\cdot\pi^2\cdot k_S}{\sqrt{x^2(t)+y^2(t)}}\right)'$$

and

$$\frac{1}{2}\cdot\left(x'(t)^2+y'(t)^2\right)-\left(\frac{4\cdot\pi^2\cdot k_S}{r(t)}\right)=E\qquad\left(E\in K, r(t)\stackrel{\text{def}}{=}\sqrt{x^2(t)+y^2(t)}\right).$$

(A student, who would start to differentiate first integrals derived by him at an examination in the situation studied through the length and the breath of, would be scarcely listened to. The possibility in the case of gaining an excellent mark seems to me highly problematic.) Let us use the combinatorial relation of Clairaut

$$(x')^2+(y')^2=\left(\frac{x\cdot x'+y\cdot y'}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{x\cdot y'-x'\cdot y}{\sqrt{x^2+y^2}}\right)^2=\left(\left(\sqrt{x^2+y^2}\right)'\right)^2+\frac{\kappa^2}{x^2+y^2}$$

and rewrite the equality with the letter E in the following way:

$$\frac{1}{2}\cdot\left(r'(t)\right)^2+\frac{1}{2}\cdot\frac{\kappa^2}{r^2(t)}-\frac{4\cdot\pi^2\cdot k_S}{r(t)}=E.$$

The Fortune was generous to fly by me twice: independent of one another, with difference in twelve years, two girls tried in front of my very eyes to differentiate both sides of this equation: [2]

$$\left(\frac{1}{2}\cdot\left(r'(t)\right)^2+\frac{1}{2}\cdot\frac{\kappa^2}{r^2(t)}-\frac{4\cdot\pi^2\cdot k_S}{r(t)}\right)'=(E)'.\qquad(!!)$$

3.4.1. Remark. At that time, I was not familiar with the conventional wisdom that a clever woman cannot be distinguished from a foolish one. To tell you the truth, a more radical rating of the ongoing crept into my mind. But I pulled myself together and directed the further discussion of the question into the customary way. An amazing thing is our memory. Nothing is missing there. At some point useful (and useless) things appear in it with the surprising clearness. I memorized what happened several years ago, when I was considering the problem of parameter disclosing and the classification of the fields in the generalized Tycho Brahe chaos. The statement 2.6 and the equations 2.7.1, 2.7.2 were already gained by me and I kept returning to them. Once again something stroked and the equality (!!) seemed to be familiar to me. A seditious thought crept into my head: "What is the role of the equation $x^2+y^2=r^2$ in all these; may be, Euclidian metrics is just a convenient way of calculation of the ratio of two oriented areas $\frac{x'(t)\cdot y''(t)-y'(t)\cdot x''(t)}{x(t)\cdot y'(t)-y(t)\cdot x'(t)}$?"

3.4.2. Drawing the conclusions and harvesting. Equality (!!) can be rewritten as follows: $\left(r(t)-\frac{4\cdot\pi^2\cdot k_S}{\kappa^2}\right)''=-\frac{4\cdot\pi^2\cdot k_S}{r^3(t)}\cdot\left(r(t)-\frac{4\cdot\pi^2\cdot k_S}{\kappa^2}\right)$, and we get into the known situation (see 2.7.1)

$$\left(x(t),y(t),r(t)-\frac{4\cdot\pi^2\cdot k_S}{\kappa^2}\right)''=-\frac{4\cdot\pi^2\cdot k_S}{r^3(t)}\cdot\left(x(t),y(t),r(t)-\frac{4\cdot\pi^2\cdot k_S}{\kappa^2}\right),$$

[2] A bolt from the blue. The most rational way to the aim. All set traps and temptations are walked around: the symbol E, beloved by physics and cherished by mechanics, is annihilated.(There is a short-cut around E. The combinatorial relation of Clairaut should only be differentiated.)

where the acceleration vector $(x''(t), y''(t), r''(t))$ is collinear to the vector $(x(t), y(t), r(t) - \frac{4 \cdot \pi^2 \cdot k_S}{\kappa^2})$ and is directed to the same point $(0, 0, \frac{4 \cdot \pi^2 \cdot k_S}{\kappa^2})$. Consequently, the curve $(x(t), y(t), r(t))$ is flat and the corresponding plane passes through the point $(0, 0, \frac{4 \cdot \pi^2 \cdot k_S}{\kappa^2})$. That is why $r(t) - \delta = \alpha \cdot x + \beta \cdot y$ (for the corresponding $\alpha, \beta \in K$ and $\delta = \frac{4 \cdot \pi^2 \cdot k_S}{\kappa^2}$), that together with the equality $r^2(t) = x^2(t) + y^2(t)$ leads to the relations $x^2(t) + y^2(t) = (\alpha \cdot x + \beta \cdot y + \delta)^2$, $\delta = \frac{4 \cdot \pi^2 \cdot k_S}{\kappa^2}$, which express the first and the third postulates of the formalism, respectively.

3.5. An odd fish: second inspiration. It is well known that Newton was an adherent of geometrical optics and stuck to the corpuscular view on the nature of light. He harshly criticized the theories of quintessence and whirls of Descartes. According to his views, a particle of light should uniformly fly over a natural geodesic in the three-dimensional (affine Descartes!) space with a high, but constant velocity. We can only guess how shocked he must be when he, outpacing his time over three centuries, tried to replace a trial cone $x^2 + y^2 - r^2 = 0$ with an optical $x^2 + y^2 - c^2 \cdot \tau^2 = 0$.

3.5.1. Emotions.

До сих пор мы все, Ньютон,
Чтим тебя, твой сан, твой дом.
Разве мог подумать он,
Что навек займёт свой трон?

Да конечно, все при нём:
Массы, сила и закон.
Дыр не видно, спору нет:
Метрикой прикрыт проект,
Фокус сунули в просвет,
Директрису под запрет...

Но сквозит, сквозит проём –
Меру надо знать во всём!
Ведь ходить одним путём
Всё равно что днем с огнём –
Ну ни капли мысли в том.

Роллинг есть и вихрем он
Сквозь эфир струит в объём
Разум, волю, дух, подъём,
Мирозданий новых сонм,
Превращая билдинг в лом.

* * *

Полем метрику чуть ткни,
Вмеру карту подцепи,
И тогда уж не вернуть,
Мир, что удалось проткнуть...

3.5.2. The great skill is impossible to spend on drink. That is a parish wisdom. But nobody can transmit it to anyone else. When he is gone, it is gone with him. The only things the master leaves here to us are his deeds and thoughts in brains of his followers.

Newton, unlike Plato, destroying the physical samples of Democritus ideas, should be given credit for the contribution to the reprint of the Descartes works (1596–1650). There were published: “Rules for the Direction of the Mind” (publ. 1701), “Tractate about the light” (publ. 1664), “Principles of Philosophy” (1644), “Meditation on First philosophy” (1641), “Geometry” (1637).

Well, about the facts, written in the notes, passed by Descartes to his colleague-correspondents, [3] we will, apparently, never know. [4]

3.6. Improvements and developments: from Tycho Brahe and Descartes through Faraday and Maxwell into the wilderness of Riemannian geometry. The starting point of any other scientific impulsion in mathematics and mechanics is characterized by two statements. The first was made by Boltzman about the electromagnetic nature of electron mass origination. The second is connected to the name of Poincaré, who noticed that the second group of Maxwell equations, showing the absence of magnetic currents and charges, could be interpreted as a cocycle, and pointed out that those equations were expressed in the language of four-generated metabelian Lie algebras, defined by these cocycles, and the Hamiltonian formalism of dynamics of movement of a charged particle in the electromagnetic field was restored by the deformation [5] of the universal enveloping algebra after the split-up (see [3]) of the metalbeian Lie algebra by the usual incremental filtration.

The start of this stage was finished by the introduction of the electromagnetic potential and the development of the elements of relativist mechanics by Minkowski and Poincare, where customary Hamiltonians of the classic parabolic type were replaced by relativist hyperbolic ones.

The further expansion of this course went on in full correspondence with the three principles of successful management in science and education:

(1) when the science feels a lack of arguments, it expands its dictionary;

(2) ugly facts kill beautiful hypotheses;

(3) everything is good in its season.

3.7. Jackpot. In particular, Herman Weyl's formalism, in which he proposed using Capelli relations not only for distingwishing between the different classes of trajectories (see 2.5.1, 2.5.2 and 2.8.1, 2.8.2), but also for searching for the rational Hamiltonian types in the ocean of developing gravitational theories, was left unclaimed.

Following him (see [1]), we leave "the mountain massive of topology" and the wilderness of Riemann geometry together with the theories of strings, superstrings, cosmogonies and return from the sky (see the pioneer work of Kaluza [4]) down to our hero-girls.

(By the way about the witches broom. Further fates of both of them were quite satisfactory. The first of them defended a thesis over a series of works about a top (see [5]). It is to the second that we owe an alternative (more than simple) quantum-mechanic model of the energy levels of the hydrogen atom (see [6], [7]).)

3.8. Va banque: the differential-combinatorial glamour of an affine algebraic quadratic curve. From the theorem of O. Whatnot (see 3.3.5) follows that for any imaginary solutions $x(t), y(t)$ of equations 2.8.2, generating the quadratic chaos of Tycho Brahe, the following equality should hold:

$$det\begin{pmatrix} x'(t) & y'(t) & d'(t) \\ x''(t) & y''(t) & d''(t) \\ x'''(t) & y'''(t) & d'''(t) \end{pmatrix} = 0, \qquad (!!!)$$

where $d(t) \overset{\text{def}}{=} \left(\frac{x'(t)\cdot y''(t)-y'(t)\cdot x''(t)}{x(t)\cdot y'(t)-y(t)\cdot x'(t)}\right)^{-\frac{1}{3}}$. From the Gerasimova equations (see 2.5.2, 2.6, 2.7) it follows that the reverse statement holds too.

[3]Henry Wotton: "There is no such thing as a good influence, Mr. Gray. All influence is immoral. Immoral from the scientific point of view. Because to influence a person is to give him one's own soul. The aim of life is self-development. To realize one's nature perfectly that is what each of us is here for." *The Portrait of Dorian Gray.*

[4]Henry Wotton: "Diplomat is a respectable national, sent to a foreign land to lie to the profit of a monarch and an entrusted to him Motherland." (History of diplomacy. Vol. 1. OGIZ, Moscow (1941)).

[5]It is in this place that the gate was left wide open, leading its way to a new, essentially noncommutative theory – quantum mechanics, which gave the second breath to the classic parabolic Hamiltonians. This possibility was highly ahead of its time, as it contained the Heisenberg relations as a very important, but local case in the class of methabelian Lie algebras.

3.8.1. Gerasimova – Whatnot theorem. *Let analytical functions $x(t)$, $y(t)$ be such that the dimension of the subspace $K \cdot x'(t) + K \cdot y'(t)$ equals two. Then they are solutions of equations* 2.8.2 *(determining the quadratic chaos of Tycho Brahe) if and only if $x(t)$, $y(t)$ satisfy the equalities*

$$x(t) \cdot y''(t) - x''(t) \cdot y(t) = 0,$$

$$\left| x'(t), y'(t), \left(\left(\frac{x'(t) \cdot y''(t) - y'(t) \cdot x''(t)}{x(t) \cdot y'(t) - y(t) \cdot x'(t)} \right)^{-\frac{1}{3}} \right)' \right| = 0.$$

In connection with this theorem, a very capital result, admitting a simple geometrical interpretation, should be noted.

3.8.2 Lemma on a directrix and a focus (O.V. Gerasimova). *For any real numbers α, β, δ, and γ every infinitely differentiable solution $x(t)$, $y(t)$ $(x(t) \cdot y'(t) - x'(t) \cdot y(t) \neq 0)$ of the system of equations*

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -\frac{\gamma}{(\alpha \cdot x + \beta \cdot y + \delta)^3} \cdot \begin{pmatrix} x - a \\ y - b \end{pmatrix} \quad (a, b \in \mathbf{R})$$

lies on its own quadratic curve, for which the "focus" is located at the point (a, b) and the "directrix" in terms of Tycho Brahe's proposal (see section 1) is defined by the equation $\alpha \cdot x + \beta \cdot y + \delta = 0$.

3.8.3. Efimovskaya – Whatnot theorem. *Let analytical functions $x(t)$, $y(t)$ be such that the dimension of the subspace $K \cdot x'(t) + K \cdot y'(t)$ equals two. Then they are solutions of equations* 2.8.2 *(determining quadratic chaos of Tycho Brahe) when and only when for $x(t)$, $y(t)$ the following equalities hold:*

$$x(t) \cdot y''(t) - x''(t) \cdot y(t) = 0,$$

$$\left| x^2(t), x(t) \cdot y(t), y^2(t), \left(\frac{x'(t) \cdot y''(t) - y'(t) \cdot x''(t)}{x(t) \cdot y'(t) - y(t) \cdot x'(t)} \right)^{-\frac{2}{3}} \right| = 0.$$

It is appropriate to note a very curious local case of this theorem, known, apparently, to founders of the theory way before Robert Hooke.

3.8.4 Lemma on a crocodile choking with an apple. *For any real numbers γ, α_1, β_1, α_2, β_2 $(\alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1 \neq 0)$ every infinitely differentiable solution $x(t)$, $y(t)$ $(x(t) \cdot y'(t) - x'(t) \cdot y(t) \neq 0)$ of the system of equations*

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -\frac{\gamma}{\sqrt{(\alpha_1 \cdot x + \beta_1 \cdot y) \cdot (\alpha_2 \cdot x + \beta_2 \cdot y)}^3} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

lies on its quadratic curve, tangent to both lines, defined by the equalities $\alpha_1 \cdot x + \beta_1 \cdot y = 0$, $\alpha_2 \cdot x + \beta_2 \cdot y = 0$.

3.8.5. The key equation of central quadratic dynamics. Each theorem 3.8.1, 3.8.3 implies that the ratio

$$\Delta \stackrel{\text{def}}{=} \frac{x' \cdot y'' - x'' \cdot y'}{x \cdot y' - x' \cdot y}$$

satisfies the ordinary differential equation

$$9 \cdot \Delta''' \cdot \Delta^2 - 45 \cdot \Delta'' \cdot \Delta' \cdot \Delta + 40 \cdot (\Delta')^3 + 9 \cdot (\Delta)' \cdot (\Delta)^3 = 0$$

(see for comparison the law of rolling simplexes in the next section).

"Only one can get there
nobody knows where."
A. Kostrikin.

4. A Restoration: Basics of an Algebraic Brahe – Descartes – Wotton Theory (N = 2). All ways end at the same point. The name of that point is "Disillusion". In its desperate attempt to assert itself, each new generation declares that it is more clever than previous ones. There are all grounds for optimism. Over the last four hundred years the concept of "rolling" (see [2]) has disappeared from all the stages of mathematical education. Working with an affine chart of Cartesian projective plane geometers introduced a "similar" term "Desargues plane". All the textbooks on analytic geometry begin with the formula of distance between two points. Measure and promesure are no longer considered to be something primary. They are understood to be something derived from metrics and prometrics. There are calculated length of curves and areas of surfaces, taking no notice of the fact that solutions of problems of classical mechanics contain the proportion of momentum independent of the choice of Euclidian metrics and systems of affine coordinates. [6] As a result, the ideas of T. Brahe, H. Wotton, R. Descartes, and R. Hooke deflected by Kepler's laws degenerated into the law of gravity, which from the purely mathematical point of view did not have such reserve of generality as theories put forward by these four scientists. And the development of new metrics versions of gravitational theory cannot be stopped. So it goes, one should say.

Let them go forward.

Forward to the future!

4.1. "Mysteries" of the central quadratic chaos. Let us define a differential algebra D_2 by three generators u, v, w and the following relations:

(a) $u'' = -w \cdot u$, $v'' = -w \cdot v$, (b) $9 \cdot w''' \cdot w^2 - 45 \cdot w'' \cdot w' \cdot w + 40 \cdot (w')^3 + 9 \cdot w' \cdot w^3 = 0$.

Let us call it the Descartes – Wotton algebra. It turns out that there is a simple relationship between D_2 and the algebra of Brahe B_2.

Theorem. *The localizations $B_2[\sigma_{0,1}^{-1}(x,y)]$ and $D_2[\sigma_{0,1}^{-1}(u,v)]$ of B_2 and D_2 at the elements $x \cdot y' - x' \cdot y$ and $u \cdot v' - u' \cdot v$, respectively, are differentially isomorphic.*

Proof. Denote by $|f_1, f_2, ..., f_n|$ the Capelli – Wronsky determinant of f_1, f_2, ..., f_n. Let us consider a chain of homomorphisms $E_2 \overset{\varepsilon_{-2}}{\to} G_2 \overset{\varepsilon_{-1}}{\to} B_2 \overset{\varepsilon_0}{\to} D_2 \overset{\varepsilon_1}{\to} B_2[(x \cdot y' - x' \cdot y)^{-1}] \overset{\varepsilon_2}{\to} D_2[(u \cdot v' - u' \cdot v)^{-1}]$, where E_2 is a differential algebra of quadratic dynamics given by generators x, y and a single defining relation $|x^2, x \cdot y, y^2, x, y, 1| = 0$, and G_2 is a reduced algebra of quadratic dynamics $\{x, y \,|\, b_2(x,y) = 0\}$ (see 2.8.2). A routine verification shows that $|x^2, x \cdot y, y^2, x, y, 1| = -4 \cdot (x' \cdot y'' - x'' \cdot y') \cdot b_2(x,y)$.

Since the algebra E_2 is countable-dimensional and the algebraically closed field $\mathbf{C}$ is continuous, for any nonnilpotent element $a \in E_2$ there exists a homomorphism $\psi : E_2 \to \mathbf{C}$ such that $\psi(a) \neq 0$. But for any homomorphism $\psi : E_2 \to \mathbf{C}$ under the Taylor homomorphism $\tilde{\psi} : E_2 \to \mathbf{C}[[t]]$ the power series $\tilde{\psi}(b_2(x,y))$ vanishes; in particular, its free term $\psi(b_2(x,y))$ is equal to zero. Hence, the element $b_2(x,y)$ is nilpotent and lies in the Jacobson radical of E_2. Consequently, the differential equations $|x^2, x \cdot y, y^2, x, y, 1| = 0$ and $b_2(x,y) = 0$ have the same sets of solutions in the classes of: (a) analytical functions, (b) formal power series, (c) infinitely differentiable functions. This proves, in particular, that the Brahe algebra B_2 (its relations) generates (define) a universal model of central field with quadratic dynamics over the fields $\mathbf{R}$ and $\mathbf{C}$.

The homomorphisms $\varepsilon_0 : B_2 \to D_2$, $\varepsilon_1 : D_2 \to B_2[\sigma_{0,1}^{-1}(x,y)]$ of the algebras B_2 and D_2 are defined in a natural way on their generators: $\varepsilon_0(x) \overset{\text{def}}{=} u$, $\varepsilon_0(y) \overset{\text{def}}{=} v$, $\varepsilon_1(u) \overset{\text{def}}{=} x$, $\varepsilon_1(v) \overset{\text{def}}{=} y$, $\varepsilon_1(w) \overset{\text{def}}{=} \sigma_{1,2}(x,y)/\sigma_{0,1}(x,y)$.

The correctness of the definition of ε_0 is verified by means of the equalities $\sigma_{0,2}(u,v) = 0$, $\sigma_{1,2}(u,v) = w \cdot \sigma_{0,1}(u,v)$, $w^2 \cdot \sigma_{0,1}(u,v) = \sigma_{2,3}(u,v)$, which, in turn, directly follow from the defining relation (a) of the algebra D_2. The defining relation $\sigma_{0,2}(x,y) = 0$ of B_2 implies one by one the following equalities:

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[6] Over centuries, people have ignored the fact (lying on the surface!) that *in all central power fields of three-dimensional affine space the dynamics of which is quadratic the very movement realizes* **the following law of quadratic rolling simplexes:**

(a) $\vec{R}'' = -\Delta \cdot \vec{R}$, (b) $9 \cdot \Delta''' \cdot \Delta^2 - 45 \cdot \Delta'' \cdot \Delta' \cdot \Delta + 40 \cdot (\Delta')^3 + 9 \cdot (\Delta)' \cdot (\Delta)^3 = 0$ $(\Delta \overset{\text{def}}{=} (\vec{R}', \vec{R}'')/[\vec{R}, \vec{R}'])$.

$\sigma'_{0,1}(x,y) = 0$, $\sigma_{0,1}(x,y)\cdot\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -\sigma_{1,2}(x,y)\cdot\begin{pmatrix} x \\ y \end{pmatrix}$, $0 = \sigma'_{0,2}(x,y) = \sigma_{0,3}(x,y) + \sigma_{1,2}(x,y)$, $\sigma_{0,3}(x,y) = -\sigma_{1,2}(x,y)$, $\sigma_{0,1}(x,y)\cdot\sigma_{2,3}(x,y) = -\sigma_{1,2}(x,y)\cdot\sigma_{0,3}(x,y) = \sigma^2_{1,2}(x,y)$. With a little help of them, the correctness of ε_1 is checked like a charm.

As $\sigma_{0,1}(u,v)\cdot w = \sigma_{1,2}(u,v) \in \varepsilon_0(B_2) \subset D_2$, localizations of the algebras $\varepsilon_0(B_2)$ and D_2 by the element $\sigma_{0,1}(u,v)$ do coincide. The statement of the theorem is now obvious.

4.1.1. Consequence. *The localization $B_2[(x\cdot y' - x'\cdot y)^{-1}]$ of the Brahe algebra B_2 is characterized by the relations*

$(x'', y'') = -\Delta\cdot(x,y)$, $9\cdot\Delta'''\cdot\Delta^2 - 45\cdot\Delta''\cdot\Delta'\cdot\Delta + 40\cdot(\Delta')^3 + 9\cdot(\Delta)'\cdot(\Delta)^3 = 0$,

where $\Delta \stackrel{\text{def}}{=} (x'\cdot y'' - x''\cdot y')/(x\cdot y' - x'\cdot y)$.

4.1.2. Consequence. *The localizations $B_2[(x'\cdot y'' - x''\cdot y')^{-1}]$, $D_2[(u'\cdot v'' - u''\cdot v')^{-1}]$, $D_2[w^{-1}]$ are integral domains. Moreover, each power series $u(t), v(t), w(t) \in K[[t]]$, being a formal solution of the differential equations*

$u'' = -w\cdot u$, $v'' = -w\cdot v$, $9\cdot w''' = w^{-2}\cdot(45\cdot w''\cdot w'\cdot w - 40\cdot(w')^3 - 9\cdot w'\cdot w^3)$,

converges in some neighborhood of zero of the field K ($K = \mathbf{R}, \mathbf{C}$).

4.2. Prointegrals: a tensor of Descartes – Hooke. Formulas never burn. They tend to rise from ashes at the most unsuitable moment. Let us fill a square 3×3 symmetric matrix H_2 by elements of the integral domain $D_2[w^{-1}]$ setting

$\mathrm{g}_{3,3} \stackrel{\text{def}}{=} -(u\cdot v' - u'\cdot v)^2\cdot(4\cdot(w')^2 - 3\cdot w\cdot w'' + 9\cdot w^3)$,

$\mathrm{g}_{3,2} = \mathrm{g}_{2,3} \stackrel{\text{def}}{=} (u\cdot v' - u'\cdot v)\cdot(-4\cdot(w')^2\cdot u' + 3\cdot w\cdot w''\cdot u' + 3\cdot w'\cdot w^2\cdot u)$,

$\mathrm{g}_{3,1} = \mathrm{g}_{1,3} \stackrel{\text{def}}{=} -(u\cdot v' - u'\cdot v)\cdot(-4\cdot(w')^2\cdot v' + 3\cdot w\cdot w''\cdot v' + 3\cdot w'\cdot w^2\cdot v)$,

$\mathrm{g}_{2,2} \stackrel{\text{def}}{=} 9\cdot w^4\cdot u^2 + 6\cdot w'\cdot w^2\cdot u\cdot u' + (u')^2\cdot(9\cdot w^3 + 3\cdot w''\cdot w - 4\cdot(w')^2)$

$\mathrm{g}_{1,1} \stackrel{\text{def}}{=} 9\cdot w^4\cdot v^2 + 6\cdot w'\cdot w^2\cdot v\cdot v' + (v')^2\cdot(9\cdot w^3 + 3\cdot w''\cdot w - 4\cdot(w')^2)$

$\mathrm{g}_{2,1} = \mathrm{g}_{1,2} \stackrel{\text{def}}{=} -(9\cdot w^4\cdot u\cdot v + 3\cdot w'\cdot w^2\cdot(u\cdot v' + u'\cdot v) + u'\cdot v'\cdot(9\cdot w^3 + 3\cdot w''\cdot w - 4\cdot(w')^2))$.

The following statement shows that each imaginable model of central fields with quadratic dynamics can be derived from the central quadratic chaos by a quotient algebra of the Descartes – Wotton algebra over a prime radical differential ideal.

Theorem. *The integral domain $D_2[w^{-1}]$ and tensor H_2 exhibit the following properties:*

(i) $H'_2 = \frac{10\cdot w'}{3\cdot w}\cdot H_2$ (in particular, $f\cdot g' - f'\cdot g = 0$, $(f/g)' = 0$ for all elements f, g of matrix H_2),

(ii) $g_{1,1}\cdot u^2 + 2\cdot g_{1,2}\cdot u\cdot v + g_{2,2}\cdot v^2 + 2\cdot g_{1,3}\cdot u + 2\cdot g_{2,3}\cdot v + g_{3,3} = 0$,

($\mathrm{g}_{1,3}\cdot u + \cdot\mathrm{g}_{2,3}\cdot v + \mathrm{g}_{3,3} = -9\cdot\sigma^2_{0,1}(u,v)\cdot w^3$, $\det(\mathrm{g}_{i,j} \mid i,j = 1,2,3) = -729\cdot\sigma^4_{0,1}(u,v)\cdot w^{10}$);

(iii) for any proper differential ideal I of $D_2[\sigma^{-1}_{1,2}(u,v)]$ the corresponding quotient algebra contains non-zero elements among $g_{i,j} + I$ ($i,j = 1,2,3$) and properties (i), (ii) are true without a degeneration in any quotient algebra $(D_2[\sigma^{-1}_{1,2}(u,v)])/I$ containing no zero divisor.

4.2.1. Consequence. *Each homogeneous prime ideal of the subalgebra $K[g_{i,j} \mid i,j = 1,2,3]$ can be elevated to a radical prime differential ideal of $D_2[\sigma^{-1}_{1,2}(u,v)]$.*

4.2.2. Consequence. *For any solution in power series $u(t), v(t), w(t) \in K[[t]]$ ($u'\cdot v'' - u''\cdot v' \neq 0$) of the differential equations*

$u'' = -w\cdot u$, $v'' = -w\cdot v$, $9\cdot w''' = w^{-2}\cdot(45\cdot w''\cdot w'\cdot w - 40\cdot(w')^3 - 9\cdot w'\cdot w^3)$

the equalities $g_{i,j}(t)/w^3(t) = c_{i,j}\cdot w^{\frac{1}{3}}(t)$ hold for appropriate $c_{i,j} \in K$ ($i,j = 1,2,3$).

The last statement explains why the adjunction of the irrational element $w^{\frac{1}{3}}$ to the differential algebra $D_2[\sigma^{-1}_{1,2}(u,v)]$ “imbeds” the quadratic chaos of Tycho Brahe into the Hooke universe (see also 2.7.1, 2.7.2).

4.3. Central extensions of localizations of the algebras B_2 and D_2. Let us consider the most primitive ideas and ways of ignoring the law of quadratically rolling simplexes by means of increasing the dimension of the phase space.

4.3.1. A five-dimensional complex model (factorization of homogeneous forms in two variables into linear factors). Any polynomial $f(x,y) \in \mathbf{C}[x,y]$ of degree at most two over an algebraically closed field can be represented as a linear combination of homogeneous polynomials: 1, $f_1(x,y) = z_0$, $f_2(x,y) = z_{-1}\cdot z_1$ ($z_i \stackrel{\text{def}}{=} \alpha_i\cdot x + \beta_i\cdot y$, $i = 0, \pm 1$). That is why every quadratic curve can be given by the equation $\gamma \cdot z_{-1} \cdot z_1 + \alpha \cdot z_0 + \delta = 0$ for appropriate linear forms z_{-1}, z_0, z_1 and our universal centrally quadratic dynamics in the plane Oxy may be realized by means of a differential algebra M_{10} that has six generators z_{-1}, $z_{-\frac{1}{2}} \stackrel{\text{def}}{=} x$, z_0, $z_{\frac{1}{2}} \stackrel{\text{def}}{=} y$, z_1, Δ satisfying the following defining relations:

(a) $z_i'' = -\Delta \cdot z_i$ $(i = 0, \pm\frac{1}{2}, \pm 1)$ (a condition of the field centrality in five-dimensional space),

(b) $|z_{-1} \cdot z_1, z_0, 1| = 0$ (a condition of quadratic dynamics).

Equations (a), as usual, give us ten first integrals $\sigma_{0,1}(z_i, z_j) = \mathit{const}$. (Indeed, $\sigma_{0,1}'(z_i, z_j) = (z_i \cdot z_j' - z_i' \cdot z_j)' = 0$). And the equalities $\sigma_{0,1}(z_i, z_j) \cdot z_k + \sigma_{0,1}(z_j, z_k) \cdot z_i + \sigma_{0,1}(z_k, z_i) \cdot z_j = 0$ imply that $z_i = \frac{\sigma_{0,1}(z_i,y)}{\sigma_{0,1}(x,y)} \cdot x + \frac{\sigma_{0,1}(x,z_i)}{\sigma_{0,1}(x,y)} \cdot y$ $(i = 0, \pm 1)$ in the localization $M_{10}[\sigma_{0,1}^{-1}(x, y)]$ of the algebra M_{10} by the element $x \cdot y' - x' \cdot y$. Hence, $z_i(t) = \alpha_i \cdot x(t) + \beta_i \cdot y(t)$ $(i = 0, \pm 1)$ for any formal solution $z_i(t)$ $(i = 0, \pm\frac{1}{2}, \pm 1)$ of the equations (a) in power series from $\mathbf{C}[[t]]$.

The explicit form of relation (b)

$$0 = |z_{-1} \cdot z_1, z_0, 1| = |(z_{-1} \cdot z_1)', z_0'| = (z_{-1} \cdot z_1)' \cdot (-\Delta \cdot z_0) - (2 \cdot z_1' \cdot z_{-1}' - 2 \cdot \Delta \cdot z_1 \cdot z_{-1}) \cdot z_0'$$

gives us two more first (rational) integrals $(z_{-1} \cdot z_1)'/z_0' = -\beta$, $z_{-1} \cdot z_1 - \frac{(z_{-1} \cdot z_1)'}{z_0'} \cdot z_0 = -\delta$, for which $z_{-1} \cdot z_1 + \beta \cdot z_0 + \delta = 0$, and enables us to rationally express Δ in terms of z_{-1}, z_0, z_1 and their derivatives:

$$\Delta = \frac{2 \cdot z_{-1}' \cdot z_0' \cdot z_1'}{\sigma_{0,1}(z_{-1}, z_0) \cdot z_1 + \sigma_{0,1}(z_1, z_0) \cdot z_{-1}} = \frac{2 \cdot z_{-1}' \cdot z_1'}{z_{-1} \cdot z_1 + (z_{-1} \cdot z_1 - \frac{(z_{-1} \cdot z_1)'}{z_0'} \cdot z_0)} \text{ (compare with 2.4).}$$

Hence, the localization of the algebra $M_{10}[\sigma_{0,1}^{-1}(x, y)]$ by the element

$$2 \cdot z_0' \cdot z_1 \cdot z_{-1} - z_0 \cdot (z_{-1} \cdot z_1)' = \sigma_{0,1}(z_{-1}, z_0) \cdot z_1 + \sigma_{0,1}(z_1, z_0) \cdot z_{-1}$$

is described by one rational vector differential equation

$$(z_{-1}, x, z_0, y, z_1)'' = -\frac{2 \cdot z_{-1}' \cdot z_0' \cdot z_1'}{2 \cdot z_0' \cdot z_1 \cdot z_{-1} - z_0 \cdot (z_{-1} \cdot z_1)'} \cdot (z_{-1}, x, z_0, y, z_1),$$

which combines both the condition of field centrality in five-dimensional affine space and the requirement that all movements are realized by flat essentially quadratic curves.

The homomorphism $\varepsilon_{10} : D_2 \to M_{10}[(2 \cdot z_0' \cdot z_1 \cdot z_{-1} - z_0 \cdot (z_{-1} \cdot z_1)')^{-1}]$ is uniquely determined on the generators u, v, w: $\varepsilon_{10}(u) \stackrel{\text{def}}{=} x$, $\varepsilon_{10}(v) \stackrel{\text{def}}{=} y$, $\varepsilon_{10}(w) \stackrel{\text{def}}{=} \Delta$. We leave it as an exercise for the reader to verify the correctness of such a definition of the mapping ε_{10}.

4.3.2. A four-dimensional "real" model (conjugate directions). Let a quadratic curve in the affine coordinate system Oxy be given by the equation

$$\mu_{11} \cdot x^2 + 2\mu_{12} \cdot x \cdot y + \mu_{22} \cdot y^2 + 2\alpha \cdot x + 2\beta \cdot y + \delta = 0,$$

where $\alpha \cdot x + \beta \cdot y \not\equiv 0$. Then there exist non-zero homogeneous linear forms $u \stackrel{\text{def}}{=} \alpha_u \cdot x + \beta_u \cdot y$, $v \stackrel{\text{def}}{=} \alpha_v \cdot x + \beta_v \cdot y$ such that in the coordinate system Ouv a new equation of the quadratic curve has the following form: $(u^2 + v) + \gamma v^2 + \delta_{u,v} = 0$. (In this case the line $u = 0$ passes through the origin of the coordinate system and the center of the curve, and v coincides, up to a non-zero factor, with $\alpha \cdot x + \beta \cdot y$.)

That is why, as in the previous case, our universal quadratic dynamics of central kind can be determined by means of a differential algebra M_8, which has five generators x, y, u, v, w satisfying the following defining relations:

(a) $z'' = -w \cdot z$ $(z \stackrel{\text{def}}{=} x, y, u, v)$, (a condition of the field centrality in four-dimensional space),

(b) $|u^2 + v, v^2, 1| = 0$ (a condition of quadratic dynamic).

Equations (a), as usual, give us six first integrals $\sigma_{0,1}(z_1, z_2) = \mathit{const}$ $(z_1, z_2 \stackrel{\text{def}}{=} x, y, u, v)$, and the equalities $u = \frac{\sigma_{0,1}(u,y)}{\sigma_{0,1}(x,y)} \cdot x + \frac{\sigma_{0,1}(x,u)}{\sigma_{0,1}(x,y)} \cdot y$, $v = \frac{\sigma_{0,1}(v,y)}{\sigma_{0,1}(x,y)} \cdot x + \frac{\sigma_{0,1}(x,v)}{\sigma_{0,1}(x,y)} \cdot y$ in the localization $M_8[\sigma_{0,1}^{-1}(x, y)]$ of algebra M_8 by the element $x \cdot y' - x' \cdot y$. That is why $u(t) = \alpha_u \cdot x(t) + \beta_u \cdot y(t)$, $v(t) = \alpha_v \cdot x(t) + \beta_v \cdot y(t)$ for any formal solution $x(t), y(t), u(t), v(t)$ of equations (a) in power series from $K[[t]]$ $(K = \mathbf{R}, \mathbf{C})$.

The explicit form of relation (b)

$$0 = |u^2 + v, v^2, 1| = |(u^2 + v)', (v^2)'| = \begin{vmatrix} 2 \cdot u \cdot u' + v' & 2 \cdot v \cdot v' \\ 2 \cdot (u')^2 & 2 \cdot (v')^2 \end{vmatrix} - w \cdot \begin{vmatrix} 2 \cdot u \cdot u' + v' & 2 \cdot v \cdot v' \\ 2 \cdot u^2 + v & 2 \cdot v^2 \end{vmatrix} =$$

$$\begin{vmatrix} 2 \cdot u \cdot u' + v' & 2 \cdot v \cdot v' \\ 2 \cdot (u')^2 & 2 \cdot (v')^2 \end{vmatrix} - w \cdot \begin{vmatrix} 2 \cdot u \cdot u' & 2 \cdot v \cdot v' \\ 2 \cdot u^2 & 2 \cdot v^2 \end{vmatrix} = (4u' \cdot v' \cdot (u \cdot v' - u' \cdot v) + 2(v')^3) - 4u \cdot v \cdot (u' \cdot v - u \cdot v') \cdot w$$

gives us two more first (rational) integrals $\frac{2\cdot u\cdot u'+v'}{2\cdot v\cdot v'} = -\gamma$, $v\cdot\frac{v\cdot v'+2\cdot u\cdot(u\cdot v'-u'\cdot v)}{2\cdot v\cdot v'} = -\delta_{u,v}$, for which $(u^2+v)+\gamma\cdot v^2+\delta_{u,v}=0$, and allows us to rationally express w in terms of u, v and their derivatives:

$$w=\frac{u'\cdot v'\cdot(u\cdot v'-u'\cdot v)+\frac{1}{2}\cdot(v')^3}{u\cdot v\cdot(u'\cdot v-u\cdot v')}$$

Hence, the localization of the algebra $M_8[\sigma_{0,1}^{-1}(x,y)]$ by the element $u\cdot v\cdot(u\cdot v'-u'\cdot v)$ is described by one rational vector differential relation

$$(u,v,x,y)''=-\frac{u'\cdot v'\cdot(u\cdot v'-u'\cdot v)+\frac{1}{2}\cdot(v')^3}{u\cdot v\cdot(u'\cdot v-u\cdot v')}\cdot(u,v,x,y),$$

which combines both the condition of field centrality in the four-dimensional affine space and the requirement that all movements proceed along flat essentially quadratic curves (with the case of harmonic oscillator left out of consideration).

The homomorphism $\varepsilon_8 : D_2\to M_8[(u\cdot v\cdot(u\cdot v'-u'\cdot v))^{-1}]$ is uniquely determined on the generators u, v, w: $\varepsilon_8(u)\stackrel{\text{def}}{=}x$, $\varepsilon_8(v)\stackrel{\text{def}}{=}y$, $\varepsilon_8(w)\stackrel{\text{def}}{=}\frac{u'\cdot v'\cdot(u\cdot v'-u'\cdot v)+\frac{1}{2}\cdot(v')^3}{u\cdot v\cdot(u'\cdot v-u\cdot v')}$. We leave it as an exercise for the reader to check the correctness of such a definition of the mapping ε_8.

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P. S.

"Как ручеёк
Струится луч
Сквозь горы туч
Нетленных истин.

Его поток,
Смывая муть,
Находит путь
В свободе жизни."

$\hbar\cdot w$