3×6 window lattice

code block	weight	No. of codes with block	No. of trees = weight × number
1	1	6	6
101	3	4	12
1001	4	3	12
10001	6	2	12
10101	9	2	18
100001	8	1	8
101001	12	2	24
			92

TABLE 1

The combinatorial problem in the general case is that of having to determine all admissible distributions of the known number of vertices of the W-lattice vertex matrix. For example, FIGURE 7(a) shows a tree T_H with the all-zeros code 00000 in the 5×5 window lattice where the eight vertices of W not spanned by T_H are distributed in accord with the composition 12131 of 8 among the columns of W, and FIGURE 7(b) shows another code-00000 tree T_H where the 8 nonspanned vertices are distributed as the composition 11312.

References

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Minimum Counterexamples in Group Theory

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In the theory of finite groups, many major theorems have been proved by the minimum counterexample technique which works as follows. If there is a counterexample to a given theorem, then there is a counterexample G of smallest possible order. The assumption that G exists is then used to force a contradiction and the theorem in question is thus established. In practice, the contradiction frequently arises from the existence of a counterexample of order smaller than that of the presumed minimum counterexample. This technique was used by G. A. Miller [1], as early as 1916, though it may have been used by earlier writers. Of course, the minimum counterexample technique is merely a disguised form of mathematical induction and in fact bears the same kind of relation to induction as does proof by the method of infinite descent used in number theory.

However, even when a conjecture about finite groups turns out to be false, the question naturally arises as to which group furnishes us with a counterexample of smallest possible order. The search for counterexamples is an important aspect of the teaching of modern algebra, and the finding of minimum counterexamples, or "least criminals" as they are sometimes called, gives the student an excellent opportunity of becoming familiar with the groups of "small" order. Indeed, it is remarkable just how many conjectures can be refuted with a knowledge of the structures of the groups of order 12 or less.

In this paper we give a minimum counterexample for each of a number of not implausible conjectures about finite groups. Proofs can be found in standard references on group theory, but we have included a few arguments to illustrate the search technique. We also suggest many further problems, solved and perhaps unsolved, in this area. We work with groups given by generators and defining relations, or with groups given by their faithful permutation representations. We note that a minimum counterexample need not be unique since nonisomorphic minimum counterexamples (of the same order) may possibly exist for a given conjecture (as in Conjecture 1 below).

For handy reference throughout this note we list in TABLE 1 the notation which we shall use, and in TABLE 2 the groups of order less than 12.

G	a finite group
$Z(G)$	the center of G
G'	the commutator subgroup of G
$\lvert G\rvert$	the order of G
C_n	the cyclic group of order n
D_n	the dihedral group of order $2n$
A_n	the alternating group of order $n!/2$
Q	the quaternion group of order 8
$A\times B$	the direct product of A and B
F	the set $\{x^2 \mid x\in G\}$ of all squares in G
$G\simeq H$	the groups G and H are isomorphic
$H\lhd G$	H is a normal subgroup of G
Aut G	the group of automorphisms of G
$\Phi(G)$	the Frattini subgroup of G (see [2])

TABLE 1

Order	Distinct nonisomorphic groups
1	C_1
2	C_2
3	C_3
4	$C_4, C_2\times C_2$
5	C_5
6	$C_6\simeq C_2\times C_3, S_3\simeq D_3$
7	C_7
8	$C_8, C_4\times C_2, C_2\times C_2\times C_2, D_4, Q$
9	$C_9, C_3\times C_3$
10	$C_{10}\simeq C_2\times C_5, D_5$
11	C_{11}

TABLE 2

We will use standard notation for generators and relations for a group. For example, $S_3=\langle a, b \mid a^3=b^2=1,\ bab=a^2\rangle$ denotes the fact that the symmetric group S_3 can be described as the group whose elements are products of two elements a, b subject to the conditions that a is of order 3, b is of order 2, and the triple product bab collapses to a^2. This "generators and relations" notation allows us to avoid the tedious use of multiplication tables.

CONJECTURE 1. *In any group G, the set F of all squares of elements of G is a subgroup of G.*

If G is Abelian and x^2, y^2 are elements of F, then $x^2y^2=(xy)^2$, so F is closed and clearly nonempty and thus a subgroup of G. This fact rules out as counterexamples all groups of order less than 12 with the possible exceptions of S_3, D_4, Q and D_5.

Now $S_3\simeq D_3=\langle a, b \mid a^3=b^2=1,\ bab=a^2\rangle$ and direct calculation shows that $F=\{1, a, a^2\}$ is a subgroup of S_3. Next, for $D_4=\langle a, b \mid a^4=b^2=1,\ bab=a^3\rangle$, $F=\{1, a^2\}$, which is a subgroup of D_4. Similarly, $Q=\langle a, b \mid a^4=1,\ a^2=b^2,\ b^{-1}ab=a^3\rangle$ giving $F=\{1, a^2\}$, which is a subgroup of Q. Finally, $D_5=\langle a, b \mid a^5=b^2=1,\ bab=a^4\rangle$ giving $F=\{1, a, a^2, a^3, a^4\}$, a subgroup of D_5.

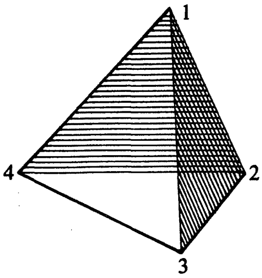

FIGURE 1. **The rotation group of the regular tetrahedron permutes the vertices, and hence can be identified with A_4. This group is a minimum counterexample to several conjectures.**

Thus a minimum counterexample has order at least 12, and we will now show that A_4, of order 12, is a minimum counterexample. Representing A_4 as a permutation group, we have

$$A_4 = \{e,(12)(34),(13)(24),(14)(23),(123),(132),(124),(142),(134),(143),(234),(243)\}$$

(see FIGURE 1) and

$$F = \{e,(123),(132),(124),(142),(134),(143),(234),(243)\}.$$

So, since $|F| = 9$, and 9 is not a divisor of $|A_4|$, F is not a subgroup of A_4. Note that the dicyclic group of order 12 given by $\langle a,b \mid a^6 = 1,\ b^2 = a^3,\ b^{-1}ab = a^{-1}\rangle$ is also a minimum counterexample.

CONJECTURE 2. *The converse of Lagrange's theorem is true; i.e., if n divides $|G|$, then G has a subgroup of order n.*

Since it is known that the converse of Lagrange's theorem is true for all finite Abelian groups, we can rule out each group of order less than 12 as a minimum counterexample except possibly S_3, D_4, Q and D_5. For each of these groups we can produce subgroups of appropriate orders, in terms of the generators already given:

$$S_3\colon \{1\},\quad \{1,b\},\quad \{1,a,a^2\},\quad S_3.$$
$$D_4\colon \{1\},\quad \{1,a^2\},\quad \{1,a,a^2,a^3\},\quad D_4.$$
$$Q\colon \{1\},\quad \{1,a^2\},\quad \{1,a,a^2,a^3\},\quad Q.$$
$$D_5\colon \{1\},\quad \{1,b\},\quad \{1,a,a^2,a^3,a^4\},\quad D_5.$$

Hence none of these groups is a counterexample. However, A_4 is a minimum counterexample, since we can show that it has no subgroup of order 6. Direct calculation shows that the conjugacy classes in A_4 are

$$\{e\},\{(12)(34),(13)(24),(14)(23)\},\ \{(123),(134),(243),(142)\}$$

and

$$\{(132),(143),(234),(124)\}.$$

If H were a subgroup of order 6 in A_4, then, since $[A_4\colon H]=2$, $H \lhd A_4$. Therefore H must consist of complete conjugacy classes of A_4 and of course $e \in H$. The five nonidentity elements of H must be made up by taking complete classes with either 3 or 4 elements, an impossibility. Thus H cannot exist and so A_4 is a minimum counterexample.

CONJECTURE 3. *If A and B are subgroups of G such that $B \lhd A$ and $A \lhd G$, then $B \lhd G$.*

Since every subgroup of an Abelian group is normal, the only group of order less than 8 to be examined is S_3. Now $\{1\} \lhd \{1, a, a^2\} \lhd S_3$ is the only relevant normal chain in S_3, and since $\{1\} \lhd S_3$, S_3 does not produce a counterexample. However, D_4 of order 8 is a minimum counterexample, as we now prove. Since $D_4 = \langle a, b \mid a^4 = b^2 = 1,\ bab = a^3\rangle$, we take $B = \{1, b\}$,

$A=\{1, a^2, b, a^2b\}$. Then $B \lhd A$, since A is Abelian, and $A \lhd D_4$ since $[D_4 : A]=2$, but clearly B is not normal in D_4 since $a^{-1}ba \notin B$.

CONJECTURE 4. *All groups of odd order are Abelian.*

This conjecture might be optimistically made on the strength of Feit and Thompson's remarkable result that all groups of odd order are soluble. We have the following well-known results:

(a) *Groups of order p or p^2 are Abelian, where p is a prime.*

(b) *If p and q are distinct primes with $p>q$ and if q does not divide $p-1$, then there is a unique group of order pq and this group is the Abelian group $C_{pq} \simeq C_p \times C_q$.*

These two results eliminate all groups of odd order less than 21 as counterexamples. However, there is a group G of order 21, given by $G=\langle a, b \mid a^7=b^3=1, b^{-1}ab=a^2\rangle$, and G is clearly nonAbelian, so G is the required minimum counterexample.

CONJECTURE 5. *$Z(G)$ is a fully invariant subgroup of G, i.e., $Z(G)$ is mapped into $Z(G)$ by every endomorphism of G.*

If G is Abelian or $Z(G)=\{1\}$, then $Z(G)$ is fully invariant, so all groups of order less than 12 are ruled out as counterexamples, except possibly D_4 and Q. In both of these groups it is easy to show that $Z(G)=G'$, so $Z(G)$ is fully invariant because it is known that G' is fully invariant. However, $D_6=\langle a, b \mid a^6=1=b^2, bab=a^5\rangle$ of order 12 is a minimum counterexample. In D_6 the mapping $a \to b, b \to b$, induces an endomorphism θ of D_6 such that $(a^3)\theta=b$, where $a^3 \in Z(D_6)$ and $b \notin Z(D_6)$.

CONJECTURE 6. *If $N \lhd G$, then G contains a subgroup isomorphic to the factor group G/N.*

This very natural conjecture is in fact true for finite Abelian groups, so S_3 is the only group of order less than 8 which needs to be considered. Now $\{1\}$, $\{1, a, a^2\}$, and S_3 are the normal subgroups of S_3, and the corresponding factor groups are isomorphic to S_3, C_2, and C_1. However, S_3 has subgroups isomorphic to each of these groups. But the quaternion group Q is a minimum counterexample. Let $Q=\langle a, b \mid a^4=1, a^2=b^2, b^{-1}ab=a^3\rangle$. Then $\langle a^2\rangle=A \lhd Q$ and $Q/A \simeq C_2 \times C_2$. However, Q has no subgroup isomorphic to $C_2 \times C_2$, since Q has only one element of order 2.

We close with a number of conjectures, all of which are **false**. The reader is challenged to produce a minimum counterexample in each case. At the time of writing, minimum counterexamples to those conjectures marked with an asterisk were unknown to the author. Some of these are likely to present considerable difficulty, and the author would welcome comments or solutions.

CONJECTURE 7. *In any group G, the set of all commutators forms a subgroup.*

CONJECTURE 8. *If every subgroup of G is normal, then G is Abelian.*

CONJECTURE 9. *If every proper subgroup of G is cyclic, then G is cyclic.*

CONJECTURE 10. *If every proper subgroup of G is cyclic, then G is Abelian.*

CONJECTURE 11. *If every proper subgroup of G is Abelian, then G is Abelian.*

CONJECTURE 12. *Every normal subgroup of G is characteristic in G.*

CONJECTURE 13. *Every characteristic subgroup is fully invariant in G.*

CONJECTURE 14. *Given a group G, there exists a group H such that $G \simeq H'$.*

CONJECTURE 15. *Given a group G, there exists a group H such that $G \simeq$* Aut *H.*

CONJECTURE 16. *Given a nonAbelian group G, there exists a finite group H such that $G \simeq \operatorname{Aut} H$.*

CONJECTURE 17. *Given a nonAbelian group G, there exists a finite group H such that $G \simeq H/Z(H)$.*

CONJECTURE 18. *Given a group G, there exists a finite group H such that $G \simeq \Phi(H)$.*

CONJECTURE 19. *Every group G has a subgroup of prime index.*

CONJECTURE 20. *If G is a simple group, then G is Abelian.*

CONJECTURE 21. *Every insoluble group G is simple.*

CONJECTURE 22. *If G is a group with trivial center, then $G = G'$.*

CONJECTURE 23. *If G is a group with $G = G'$, then G has trivial center.* (See **[3]**, page 56.)

CONJECTURE 24. *If every group of order $|G|$ is cyclic, then $|G|$ must be a prime number (ignore $|G| = 1$).*

CONJECTURE 25*. *If an automorphism α of G sends every conjugacy class of G onto itself, then α must be inner.* (See **[2]**, page 23.)

CONJECTURE 26*. *If G has a fixed-point-free automorphism, then G is nilpotent.* (See **[3]**, page 336.)

CONJECTURE 27*. *If G is nonAbelian, then* Aut G *is nonAbelian.*

CONJECTURE 28*. *If G is nonAbelian, then* $|\operatorname{Aut} G|$ *is even, i.e., every nonAbelian group has an automorphism of order* 2.

CONJECTURE 29. *If G is nonAbelian, then G is a 2-generator group.*

CONJECTURE 30. *If G is a nonAbelian p-group, then* Aut G *cannot also be a p-group.*

CONJECTURE 31*. *If G is nonAbelian and $|G|$ is odd, then G has an outer automorphism.*

CONJECTURE 32. *If H is a proper subgroup of G, then* $|\operatorname{Aut} G| > |\operatorname{Aut} H|$. (See **[2]**, page 24.)

CONJECTURE 33. *If $|G| = n > 1$, then there are less than n distinct isomorphism classes of groups of order n.*

CONJECTURE 34. *If G and H are groups such that G and H have exactly the same number of elements of each order, then $G \simeq H$.*

CONJECTURE 35. *If H is a normal nilpotent subgroup of G such that G/H is nilpotent, then G is nilpotent.*

CONJECTURE 36*. *If G is a nonAbelian p-group, where p is odd, then* Aut G *cannot also be a p-group.*

CONJECTURE 37. *The kernel of a Frobenius group G is Abelian.*

CONJECTURE 38. *If $|G| = |H|$ and* $\operatorname{Aut} G \simeq \operatorname{Aut} H$, *then $G \simeq H$.*

CONJECTURE 39. *If H is a subgroup of G, then $\Phi(H) \subseteq \Phi(G)$.* (See **[2]**, page 50.)

CONJECTURE 40. *If $|G| = |H|$, and G and H have the same character table, then $G \simeq H$.*

CONJECTURE 41. *In any permutation group G, the product of transpositions, no two of which are equal, cannot be the identity element of G.*

CONJECTURE 42. *If A and B are subgroups of G such that $A \subset B \subset G$, where A is characteristic in G, then A is characteristic in B.*

Conjecture 43. *If G is a noncyclic group with* $|G| = n$, *then G can be faithfully embedded in* S_m *for some* $m < n$.

Conjecture 44. *If G is a nonAbelian group with* $|G| = n$, *then G can be faithfully embedded in* S_m *for some* $m < n$.

Conjecture 45. *Any element of* G' *is the product of at most two elements of* F.

Conjecture 46. *If A and B are normal subgroups of a group G such that* $A \simeq B$, *then* $G/A \simeq G/B$.

Conjecture 47. *If A and B are normal subgroups of a group G such that* $G/A \simeq G/B$, *then* $A \simeq B$.

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Regions of Convergence for a Generalized Lambert Series

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In this note, we discuss infinite series of the form

$$G(z) = \Sigma \frac{a_n z^n}{1 + c_n z^n}$$

where the coefficients a_n and c_n are complex numbers and $c_n z^n \neq -1$. We will call this type of infinite series a **G-series**. A G-series is a power series if, for all $n, c_n = 0$ and a **Lambert series** if, for all $n, c_n = -1$.

In the literature a G-series is usually considered as a generalized Lambert series. For our investigation it might be better to think of a Lambert series as a generalized power series since the questions we will consider can be readily understood in terms of what is known about power series. For example, we inquire—does a G-series have a radius of convergence? Are there convergence criteria for G-series similar to well-known criteria of power series?

We first consider Lambert series. In general, a Lambert series is analytic at the origin and therefore has a power series expansion at the origin. Some of these power series expansions are very interesting. For example, J. H. Lambert found that for $|z| < 1$

$$\sum_{n=1}^{\infty} \frac{z^n}{1 - z^n} = \sum_{n=1}^{\infty} \tau_n z^n = z + 2z^2 + 2z^3 + 3z^4 + 2z^5 + 4z^6 + \cdots,$$

where τ_n is the number of divisors of n. More generally, provided r is a real number and $|z| < 1$,