Problems for the special course on finite groups and their representations - 2026

(lecturer I.A. Chubarov)

1. Suppose G is a set with a binary operation (multiplication) satisfying the following axioms:
   1) associativity; 2) there exists $e_L \in G$: for any $g \in G$: $e_L \cdot g = g$ (left unit); 3) for any $g \in G$ there exists g_R^{-1}: $g \cdot g_R^{-1} = e_L$. Is G a group? (Prove or construct counterexamples.)
2. Let G be a (non-Abelian) finite group and $H < G$. Prove that there is a common full system of representatives for left and right cosets of G by H.
3. Prove that a finite group G can't be covered with conjugates of it's proper subgroup.
4. Let $H < G$ is a subgroup of index p, where p is a minimal prime divisor of $|G|$ then H is normal in G.
5. Prove that for any natural $n \geq 3$ there is a finite non-Abelian group G with n conjugate classes.
6. Prove that any maximal subgroup of a finite nilpotent group is normal and has index p for some prime number p.
7. Let P be a Sylow p-subgroup of G. Show that for every natural k the numbers of subgroups of order p^k in G and P are congruent modulo p.
8. Find all non-isomorphic groups of order 2026.
9. Denote $r = |\{(x,y) \in G \times G \mid xy = yx\}|$ where G is a group of order n. Count the number of conjugate classes of G.
10. Prove that if G is non-cyclic, then the group Aut(G) is non-cyclic.
11. Prove that a cyclic group of odd order can't be the automorphism group of a (finite) group.
12. Let G is non-Abelian, and G is not isomorphic with S_3. Prove that $|Aut(G)| \geq 8$.
13. Let m be the maximum of the orders of Abelian subgroups of a group G. Prove that $|G|$ divides $m!$.
14. Let $|G| = 2^l m$, m is odd and a Sylow 2-subgroup of G is cyclic. Prove that G has the normal subgroup of order m.
15. a) Find two non-isomorphic groups G with $G/Z(G) \cong D_4$. b) Prove that there is no group G with $G/Z(G) \cong Q_8$.
16. Prove that for a finite group G the following conditions are equivalent: (1) G is solvable ($G^{(m)} = 1$ for some $m \geq 1$); (2) G has a finite ascending (or descending) normal series with Abelian factors, and so all principal factors are elementary Abelian; (3) G has a finite ascending (or descending) subnormal series with Abelian factors, and so all composition factors are of prime orders.
17. Let G is nilpotent of class **r** (the class of a nilpotent group is the least natural r such that $G_r = 1$, where $G_0 = G$ and for any $k \geq 1$, $G_k = [G_{k-1}, G]$). Prove that G is solvable, and its derived length $\mathbf{s} \leq [\log_2 r]+1$ (the derived length of a group G is the least natural number s such that $G^{(s)} = 1$).
18. Prove that for any finite nilpotent group G the converse to Lagrange's theorem is true: if $|G| = n,\ m|n$, then there exists $H < G,\ |H| = m$.
19. Prove that if N is a nilpotent normal subgroup of a finite group G with G/N' nilpotent, then G is nilpotent.
20. Suppose that for every non-nilpotent proper subgroup H of a group G, H is less than $N_G(H)$. Prove that G is solvable.
21. A divisor m of a natural number n is called Hall divisor, if $(m, n/m) = 1$. A subgroup $H < G$ is called Hall subgroup, if $(|H|, |G:H|) = 1$. Prove that in a solvable group G: a) for any any Hall divisor m of $n = |G|$ every subgroup of order dividing m is contained in some Hall subgroup of order m of G; b) any two Hall subgroups of equal orders are conjugate. (Existence was proved on a lecture.)
22. Let all proper subgroups of a finite non-Abelian group G are Abelian. Prove that: a) G is solvable; b) $G'' = \{1\}$; c) $|G|$ is divisible with at most two different prime numbers.
23. Define conjugate classes, centers, commutants and factors by commutant; one-dimensional complex and real representations of groups: (a) S_4 ; (b) Q_8; (c) $D_n, n \geq 3$
24. Construct all irreducible complex representations of groups $D_n, n \geq 3$.
25. Make character tables for groups (a, b) and (c, for n=6, 9) of Problem 23.

26. Construct a non-Abelian group of order: a) 55; b) 27 of exponent 3; c) 27 of exponent 9; d) 75. Make its character table (over ***C***).

27. For each of the groups: a) $G = D_6 \times S_4$; b) $G = A_4 \times D_{10}$; c) $G = D_5 \times Q_8$ define: (1) conjugate classes; (2) derived series; (3) dimensions and numbers of irreducible complex representations.

28. For a finite group G denote **cd(G)** the set {χ(1)| χ ϵ Irr(G)} of degrees of irreducible complex characters of G. a) Let **cd(G)**={1,d} (d>1). Prove that G is solvable, moreover $G^{(,,)}=1$. b) Let **cd(G)**={1,d_1,d_2} (1<d_1<d_2). Prove that G is solvable.

29. Prove that the degrees of all irreducible representations over the field of real numbers of a finite Abelian group equal 1 or 2 (and show that 2 is possible).

30. a) Show that if an element $g \in G$ is conjugate with its inverse then $\chi(g) \in R$ for all complex characters χ of G. b) Let g and g^k are conjugate in a finite group G for any $k \in \mathbb{N}, (k, |g|) = 1$. Prove that $\chi(g) \in Z$ for all complex characters χ of G.

31. Let χ is an irreducible character (over $\mathbb{C}$) of degree $\chi(1) > 1$ of a finite group G. Prove that $\chi(g) = 0$ for some $g \in G$.

32. Let χ is a faithfull (it means that the kernel of the corresponding representation is {1}) character (over $\mathbb{C}$) of degree $\chi(1) > 1$ of a finite group G, which takes exactly k values. Prove that the characters $\chi^0 = 1_G, \chi, \chi^2, ..., \chi^{k-1}$ contain all irreducible characters of G.