Spherical varieties: Lecture 10

Classification of sph. varieties

$Y_o = G/H$ sph. hom. space

Problem: Classify sph. vars. $X \supset Y_o$ open G-orbit

Luna–Vust' 1983, Knop' 1991

Birational invariants

1) Weight lattice

$$B \circlearrowright k(X) = k(Y_o), \quad (b\cdot f)(x) = f(b^{-1}\cdot x)$$

Group of B-semi-inv. rat. functions:

$$k(X)^{(B)} = \{f \in k(X)\setminus 0 \mid b\cdot f = \lambda(b)\cdot f,\ \forall b \in B,\ \text{for some } \lambda \in \Lambda(B)\}$$
$$= k(Y_o)^{(B)}$$

Weight lattice $\Lambda_X = \Lambda_{Y_0} = \{\lambda \in \Lambda(B) \mid \exists f_\lambda \in k(X)^{(B)}$ of wt. $\lambda\}$

unique up to scalar mult.:

$f_\lambda' / f_\lambda \in k(X)^B = k$

$= \text{const}$

Exact Sequence:

$$1 \longrightarrow k^\times \longrightarrow k(X)^{(B)} \longrightarrow \Lambda_X \longrightarrow 0$$

$$f_\lambda \longmapsto \lambda$$

$Y_0 \supset Y^0$ open B-orbit

$\Downarrow$

y_0 base pt.

May choose f_λ s.t. $f_\lambda(y_0) = 1$

Then: $f_{\lambda+\mu} = f_\lambda \cdot f_\mu, \quad \forall \lambda, \mu \in \Lambda_X$

Rank $r(X) := \operatorname{rk} \Lambda_X$

Example. $X = G/P$, $B \subset P \subset G$

Stabilizer $G_{y_0} = P \implies U^- \cdot y_0$ open in X dense

$G \supset B^- \cdot B = U^- \cdot B$ open subset

$B^- \cdot P = U^- \cdot P$ bigger open subset

$$\forall f \in k(X)^{(B^-)} : f = \mathrm{const}\big|_{U^- \cdot y_0}$$

$$\implies r(X) = 0$$

2) Invariant valuations

Def. **Valuation** of $k(X)/k$: a map $v : k(X)^\times \longrightarrow \mathbb{Q}$

s.t. (1) $v(f \cdot g) = v(f) + v(g)$

(2) $v(f+g) \geq \min\{v(f), v(g)\}$ if $f+g \neq 0$

$v(0) := +\infty$

(3) $v(k^\times) = 0$

(4) $\mathrm{Im}(v) \underset{\text{or } 0}{\simeq} \mathbb{Z}$

Exercise 1: $=$ holds in (2) if $v(f) \neq v(g)$

Basic example: $X \supset D$ prime divisor

$\rightsquigarrow$ valuation v_D, $\quad v_D(f) := \operatorname{ord}_D f$

Example. $X = \mathbb{P}^1 \ni \infty$, $\quad k(\mathbb{P}^1) = k(\mathbb{A}^1) = k(t)$

$f \in k(t)$, $f = \frac{p}{q}$, $p, q \in k[t]$

$$\Rightarrow v_\infty(f) = v_\infty(p) - v_\infty(q)$$
$$= \deg(q) - \deg(p)$$

Exercise 2: Suppose X quasiaff.

Prove: every map $v : k[X] \setminus 0 \longrightarrow \mathbb{Q}$ satisfying (1), (2), (3) and s.t. $\operatorname{Im}(v) \subset$ cyclic subgroup of $\mathbb{Q}$ uniquely extends to a val. of $k(X)/k$

X spherical $\Rightarrow$
$$\begin{tikzcd} 1 \arrow[r] & k^\times \arrow[r] \arrow[d] & k(X)^{(B)} \arrow[r] \arrow[d, "v"] & \Lambda_X \arrow[r] \arrow[dl, "\text{lin. function } \bar{v}"] & 0 \\ & 0 \in & \mathbb{Q} & & \end{tikzcd}$$

v val. of $k(X)/k$

Lemma 1. G-inv. valuation v uniquely determined by $\bar{v}$

Proof: Quasiaff. case. X quasiaff. $\underset{\text{Exc. 2}}{\Rightarrow}$ v uniquely determined by $v|_{k[X]}$

$$k[X] = \bigoplus_\lambda k[X]_{(\lambda)},$$

$$k[X]_{(\lambda)} \simeq V(\lambda)$$
$$\rotatebox{90}{$\in$}$$
f_λ ht. wt. vector

G-inv. filtration (decreasing): $k[X] = \bigcup_{c \in \mathbb{Q}} k[X]_{\geq c}$

$$k[X]_{\geq c} = \{ f \in k[X] \mid v(f) \geq c \}$$

$\forall\lambda\ \forall c$: either $k[X]_{\geq c} \supset k[X]_{(\lambda)}$
or $k[X]_{\geq c} \cap k[X]_{(\lambda)} = 0$

Hence : $v = \text{const}$ on $k[X]_{(\lambda)} \setminus 0$

$$\Longrightarrow\ k[X]_{\geq c} = \bigoplus_{\langle \bar{v}, \lambda\rangle \geq c} k[X]_{(\lambda)}$$

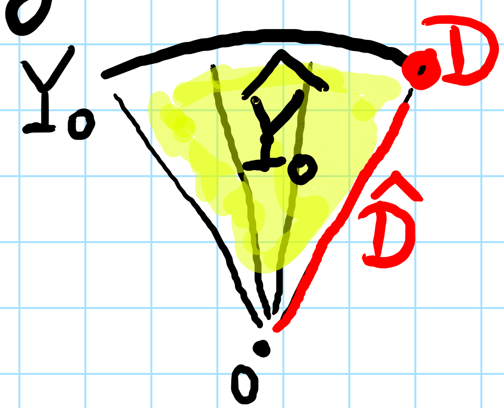
$\forall f \in k[X]\setminus 0$: $f = \sum_i f_i$, $f_i \in k[X]_{(\lambda_i)}$

$$\Longrightarrow\ v(f) = \max\{c \in \mathbb{Q} \mid f \in k[X]_{\geq c}\} = \min_i \langle \bar{v}, \lambda_i\rangle$$

General case. May assume

$$\begin{tikzcd}
X = Y_0 \arrow[d, squiggly] \simeq & G\cdot[v] \subset \mathbb{P}(V) \\
\widehat{Y}_0 = & \widehat{G}\cdot v \subset V \arrow[u, "\text{proj.}"']
\end{tikzcd}$$

$\widehat{G} = G \times k^{\times}$



$k(Y_0) \subset k(\hat{Y}_0) = k(Y_0)(t)$, $\quad t \in V^{*(B)}\big|_{\hat{Y}_0}$

$v \rightsquigarrow \hat{v}$ val. of $k(\hat{Y}_0)/k$

$\hat{v}(t) = 0$

$f = \sum_k f_k \cdot t^k \in k(Y_0)[t] \Rightarrow \hat{v}(f) := \min_k v(f_k)$

$f = p/q$, $\quad p,q \in k(Y_0)[t] \Rightarrow \hat{v}(f) = \hat{v}(p) - \hat{v}(q)$

Exercise 3: Check that $\hat{v}$ is indeed a valuation

Geom. meaning: if v corresponds to a prime divisor D then $\hat{v}$ corresponds to $\hat{D}$

Lemma 2. (approximation lemma) $\forall$ val. v of $k(X)/k$ $\exists$ G-inv. val $\tilde{v}$ s.t. $\forall f \in k(X)$ $\exists$ open $U \subset G$

$\forall g \in U: \ v(g \cdot f) = \tilde{v}(f)$

Apply Lemma 2 to $\hat{v}$ $\rightsquigarrow$ $\hat{G}$-inv. val $\tilde{v}$ of $k(\hat{Y}_0)/k$

$\tilde{v} = v$ on $k(Y_0)$