Spherical varieties: Lecture 11

$$\underset{\text{spherical}}{G \curvearrowright X} \supset \underset{\substack{\text{open}\\ G\text{-orbit}}}{Y_0} \supset \underset{\substack{\text{open}\\ B\text{-orbit}}}{Y^0}$$

Lemma 1. A G-inv. val. v of $k(X)/k$ is uniquely determined by $v|_{k(X)^{(B)}}$, $v(f_\lambda) = \langle \bar{v}, \lambda \rangle$

Done:

- quasiaff. case
  Here: $f = \sum_i f_i \in k[X] \Rightarrow v(f) = \min_i \langle \bar{v}, \lambda_i \rangle$
  $f_i \in k[X]_{(\lambda_i)} \setminus 0$

- $Y_0 \xleftarrow{\text{proj.}} \hat{Y}_0 \curvearrowleft \hat{G} = G \times k^\times$
  $v \overset{\text{ext.}}{\rightsquigarrow} \hat{v}$ $\hat{G}$-inv. val. of $k(\hat{Y}_0)/k$

Lemma 3. (2nd approximation lemma)

$f \in k(X)$ w. B-stable poles $\Rightarrow \exists \tilde{f} \in k(X)^{(B)}$:

- $v(\tilde{f}) = v(f)$
- $v'(\tilde{f}) \geq v'(f)$, $\forall G$-inv. val. v'
- $v_D(\tilde{f}) \geq v_D(f)$, $\forall B$-stable $D \subset X$

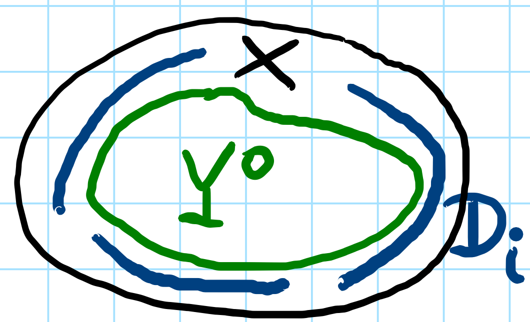
Proof of Lemma 1:

Y^0 affine $\Longrightarrow k(X) = k(Y^0) = \operatorname{Frac} k[Y^0]$

Let $v' \neq v$ another G-inv. val. of $k(X)/k$

$\Longrightarrow$ may assume $\exists f \in k[Y^0] : v(f) < v'(f)$

$\underset{\text{Lemma 3}}{\Longrightarrow} \quad \exists f_\lambda \in k(X)^{(B)} : v(f_\lambda) = v(f) < v'(f) \leq v'(f_\lambda)$



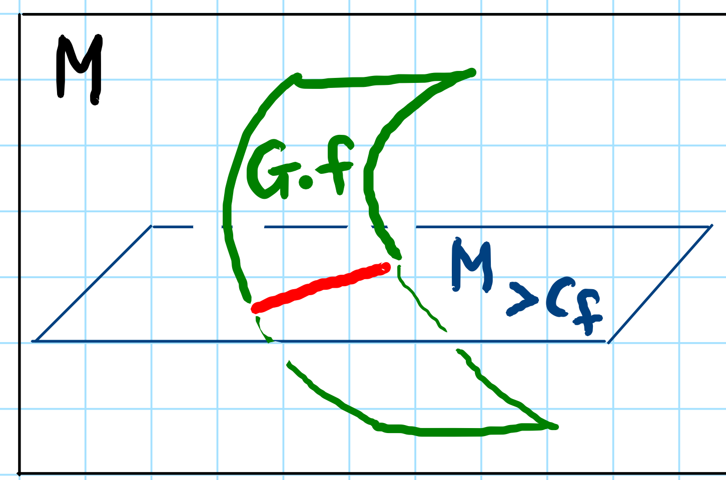
$\blacksquare$

Lemma 2. $\forall$ val. v of $k(X)/k$ $\exists$ G-inv. val. $\tilde{v}$ s.t.
$\forall f \in k(X)$ $\exists$ open $U \subset G$ $\forall g \in U$: $v(g\cdot f) = \tilde{v}(f)$

Proof: Quasiaff. case:

$$f \in k[X] \Rightarrow \underbrace{\langle G\cdot f\rangle_k}_{\text{fin.-dim. } G\text{-submodule}} =: M \subset k[X]$$

$$\boxed{\begin{array}{l}\textbf{Fact.}\quad G \circlearrowright X \longleftarrow \mathcal{L} \circlearrowleft G \rightsquigarrow G \circlearrowright H^0(X,\mathcal{L}) \\ \forall s \in H^0(X,\mathcal{L})\ \exists \text{ fin.-dim. } G\text{-stable } M \underset{\ni s}{\subset} H^0(X,\mathcal{L}) \\ \text{[Knop, Kraft, Luna, Vust]}\end{array}}$$



$v \rightsquigarrow$ filtration: $M = \bigcup_{c \in \mathbb{Q}} M_{\geq c}$

$c_f := \max\{c \mid M_{\geq c} = M\}$

$$\underbrace{G\cdot f \cap M_{>c_f}}_{\text{closed}} \subsetneq G\cdot f$$

$U_f := \{g \in G \mid g\cdot f \notin M_{>c_f}\}$ open in G $\Rightarrow v(g\cdot f) = c_f$

Put $\tilde{v}(f) := c_f$

$\Rightarrow \quad \tilde{v} : k[X] \setminus 0 \longrightarrow \mathbb{Q}$ satisfies (1) — (3)

Exercise 1: Check it

$\Downarrow$

extends to a G-inv. val. of $k(X)/k$

$f \in k(X)$, $f = p/q$, $p, q \in k[X]$,

$U := U_p \cap U_q \quad \Rightarrow \quad \forall g \in U: \quad v(g \cdot f) =$

$= v(g \cdot p) - v(g \cdot q) = \tilde{v}(p) - \tilde{v}(q)$

$= \tilde{v}(f)$

Gen. case: $v \rightsquigarrow \hat{v}$ val. of $k(\hat{Y}_0)/k$

$\rightsquigarrow \hat{\tilde{v}}$ $\hat{G}$-inv. val. of $k(\hat{Y}_0)/k$

$\rightsquigarrow \tilde{v} = \hat{\tilde{v}}\big|_{k(X)^\times}$

$\blacksquare$

Proof of Lemma 3: May assume X smooth.

$$X \setminus Y^0 = D_1 \cup \ldots \cup D_m$$

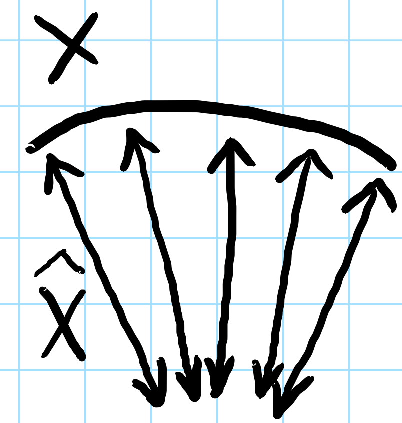
B-stable prime div.

$$\delta = - \sum_i \mathrm{ord}_{D_i}(f) \cdot D_i \quad \text{Cartier, B-stable}$$

$\rightsquigarrow$

B′-semi-inv.

s_δ canonical rat. section of $\mathcal{O}(\delta)$



$$s := f \cdot s_\delta \in H^0(X, \mathcal{O}(\delta))$$

$$\mathrm{ord}_{D_i}(s) = 0$$

$X \twoheadleftarrow \hat{X}$:= total space of $\mathcal{O}(-\delta) \setminus$ zero section

$$k(X) \subset k(\hat{X}) \ni s_\delta, s$$

$v, v', \ldots \underset{\text{Lemma 2}}{\rightsquigarrow}$ G'-inv. vals. of $k(\hat{X})/k$

$$\langle G' \cdot s \rangle_k =: M \subset H^0(X, \mathcal{O}(\delta))$$
$$= \bigoplus_j M_{(\lambda_j)}$$

$$s = \sum_j s_j \,, \quad s_j \in M_{(\lambda_j)} \setminus 0$$

$$\Longrightarrow v(s) = \min_j v(s_j) = v(s_1)$$
w.l.o.g.

Take $\tilde{s} \in M_{\lambda_1}^{(B')} \Longrightarrow v(\tilde{s}) = v(s_1) = v(s)$

$$v'(\tilde{s}) = v'(s_1) \geq v'(s)$$
$$\mathrm{ord}_{D_i}(\tilde{s}) \geq 0 = \mathrm{ord}_{D_i}(s)$$

Put $\tilde{f} := \tilde{s}/s_\delta$. Recall: $f = s/s_\delta$.

$\blacksquare$

May identify G-inv. val. v w. $\overline{v}$

$\Rightarrow$ write $v \in \mathrm{Hom}(\Lambda_X, \mathbb{Q}) = \Lambda_X^* \underset{\mathbb{Z}}{\otimes} \mathbb{Q} =: \mathcal{E}_X$

Lemma 4. G-inv. vals. of $k(X)/k$ form a solid of full dim. convex cone $\mathcal{V}_X \subset \mathcal{E}_X$

Valuation cone

Proof: Quasiaff. case: v any G-inv. val.

$$k[X]_{(\lambda)} \cdot k[X]_{(\mu)} = \underset{\substack{\Downarrow \\ f_{\lambda+\mu} = f_\lambda \cdot f_\mu \neq 0}}{k[X]_{(\lambda+\mu)}} \oplus k[X]_{(\lambda+\mu-\beta_1)} \oplus \ldots \oplus k[X]_{(\lambda+\mu-\beta_r)} \quad (r \leq s)$$

$$\Uparrow$$

$$V(\lambda) \otimes V(\mu) \simeq V(\lambda+\mu) \oplus V(\lambda+\mu-\beta_1) \oplus \ldots \oplus V(\lambda+\mu-\beta_s)$$

all T-eigenwts. are: $\lambda+\mu-\alpha_1-\ldots-\alpha_k$, $\alpha_i \in \Delta^+$

β_j = sums of $\alpha \in \Delta^+$

$$V(\lambda) = H^0(G/B, \mathcal{L}_{-\lambda^*}) \ni s \neq 0$$

$$\otimes$$

$$V(\mu) = H^0(G/B, \mathcal{L}_{-\mu^*}) \ni t \neq 0 \quad \Longrightarrow \quad s \otimes t \neq 0$$

$\otimes$ of sections

$$\to V(\lambda+\mu) = H^0(G/B, \mathcal{L}_{-\lambda^*-\mu^*})$$

$$\mathcal{L}_{-\lambda^*-\mu^*} = \mathcal{L}_{-\lambda^*} \otimes \mathcal{L}_{-\mu^*}$$

Take $f \in k[X]_{(\lambda)}$, $g \in k[X]_{(\mu)}$ general

$$\Rightarrow f \cdot g = h_0 + h_1 + \ldots + h_r, \quad h_j \in k[X]_{(\lambda+\mu-\beta_j)}, \quad h_j \neq 0$$

$$\Rightarrow v(f \cdot g) = v(f) + v(g) = \langle v, \lambda \rangle + \langle v, \mu \rangle$$

$$= \min_j v(h_j)$$

$$= \min \{ \langle v, \lambda+\mu \rangle, \langle v, \lambda+\mu-\beta_1 \rangle, \ldots, \langle v, \lambda+\mu-\beta_r \rangle \}$$

$$\Longrightarrow \quad \langle v, \beta_j \rangle \leq 0 \quad , \quad \forall j = 1, \ldots, r \quad , \quad \forall \lambda, \mu$$

tails

$$\mathcal{V}_X := \{ v \in \mathcal{E}_X \mid \langle v, \beta_j \rangle \leq 0 \text{ for all tails } \beta_j \ \forall \lambda, \mu \}$$