Spherical varieties: Lecture 12

$G \curvearrowright X$ sph. variety, $\mathcal{E}_X = \mathrm{Hom}(\Lambda_X, \mathbb{Q})$

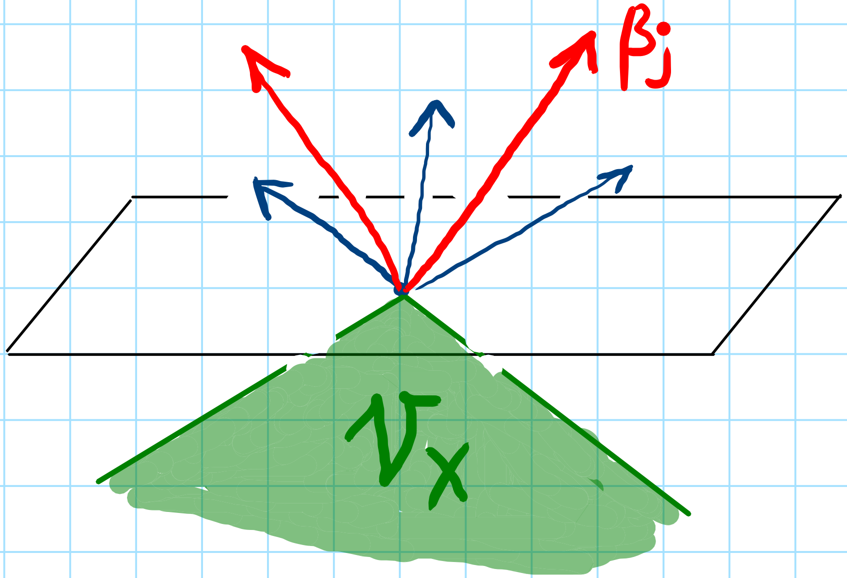
Lemma 4. G-inv. vals. of $k(X)/k$ form a solid convex cone $\mathcal{V}_X \subset \mathcal{E}_X$

Quasiaff. case:

$$k[X]_{(\lambda)} \cdot k[X]_{(\mu)} = k[X]_{(\lambda+\mu)} \oplus \underbrace{k[X]_{(\lambda+\mu-\beta_1)} \oplus \ldots \oplus k[X]_{(\lambda+\mu-\beta_r)}}_{\text{tails}}$$

β_i = sums of $\alpha \in \Delta^+$

$$\mathcal{V}_X = \{ v \in \mathcal{E}_X \mid \langle v, \beta_i \rangle \leq 0,\ \forall \beta_i,\ \forall \lambda, \mu \}$$



$\mathcal{V}_X$ solid

$\Uparrow$

Cone(Δ^+) pointed

Cone(tails), too

Claim: $v \in \mathcal{V}_X \iff v$ defines a G-inv. val.

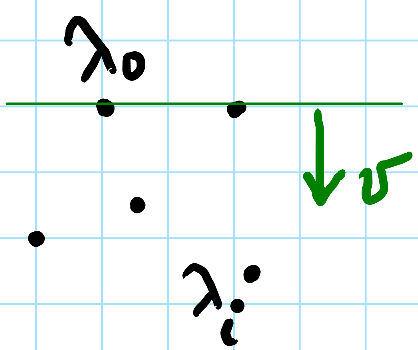
$\Leftarrow$ proved at Lecture 11

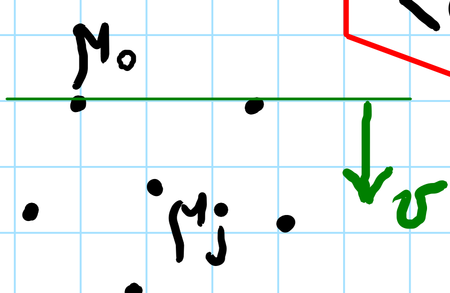
$\Rightarrow$ $f \in k[X]$, $f = \sum_i f_i$, $f_i \in k[X]_{(\lambda_i)} \setminus 0$

$$\Rightarrow \quad v(f) := \min_i \langle v, \lambda_i \rangle$$

Check (1): $g \in k[X]$, $g = \sum_j g_j$, $g_j \in k[X]_{(\mu_j)} \setminus 0$

$$\Rightarrow f \cdot g = \sum_{i,j} f_i \cdot g_j = h_0 + h_1 + \ldots + h_r$$



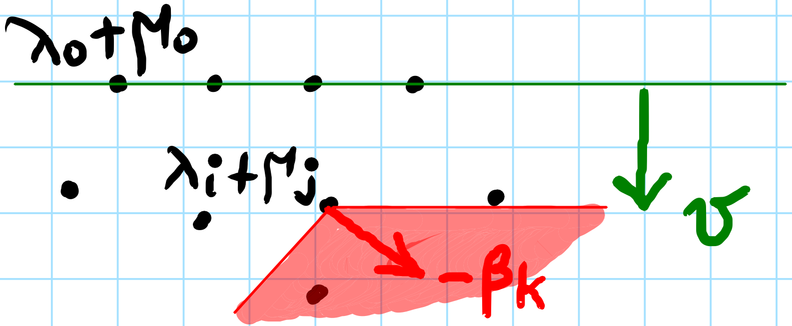


Choose λ_0, μ_0 such that

$\langle v, \lambda_0 \rangle = \min_i \langle v, \lambda_i \rangle$, $\langle v, \mu_0 \rangle = \min_j \langle v, \mu_j \rangle$

$(\lambda_i + \mu_j) - (\lambda_0 + \mu_0) \notin \operatorname{Cone}(\text{tails})$

$$v(f \cdot g) = \langle v, \lambda_0 + \mu_0 \rangle = \langle v, \lambda_0 \rangle + \langle v, \mu_0 \rangle = v(f) + v(g)$$



$h_0 \in k[X]_{(\lambda_0+\mu_0)}$, $h_0 \neq 0$

$h_k \in k[X]_{(\lambda_i+\mu_j-\beta_k)}$

Gen. case:

$$\begin{tikzcd}
\{G\text{-inv. vals. of } k(X)/k\} \arrow[r, leftrightarrow, "1:1"] \arrow[d, squiggly, "\text{ext.}"'] & \mathcal{V} \subset \mathcal{E}_X \\
\{\hat{G}\text{-inv. vals. of } k(\hat{Y}_0)/k\} \arrow[r, leftrightarrow, "1:1"'] \arrow[u, bend right, "\text{res.}"', two heads] & \hat{\mathcal{V}} \subset \mathcal{E}_{\hat{Y}_0} \arrow[u, two heads]
\end{tikzcd}$$

$$k(X)^{(B)} \subset k(\hat{Y}_0)^{(\hat{B})} \Longrightarrow \Lambda_X \subset \Lambda_{\hat{Y}_0}$$

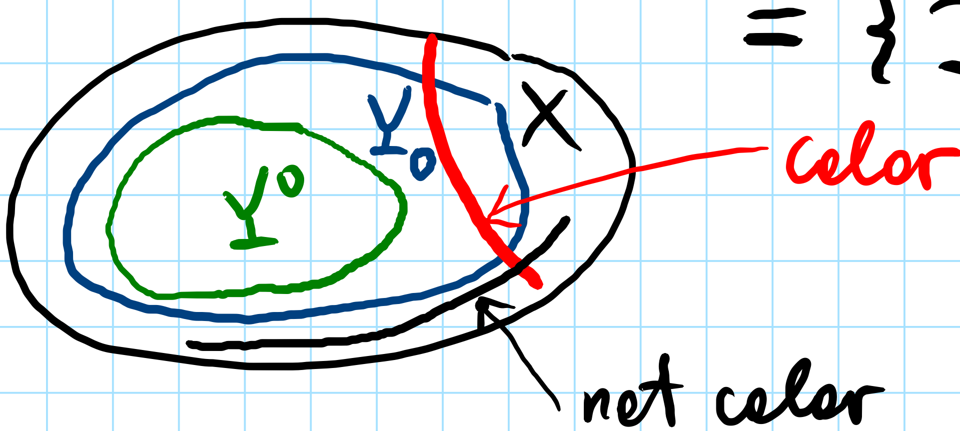
$\blacksquare$

Rmk. $\mathcal{V}_X$ is polyhedral, even cosimplicial

3) Colors.

$$\underbrace{Y_0}_{\text{open } G\text{-}} \setminus \underbrace{Y^0}_{/\ B\text{-orbit in } X} = \underbrace{D_1 \cup \dots \cup D_s}_{B\text{-stable prime divisors}}$$

Set of colors $\mathcal{D}_X = \{\overline{D_1}, \dots, \overline{D_s}\}$
$= \{D \subset X \text{ B-stable, not G-stable}\}$



Map $\mathcal{D}_X \longrightarrow \Lambda_X^* \subset \mathcal{E}_X$

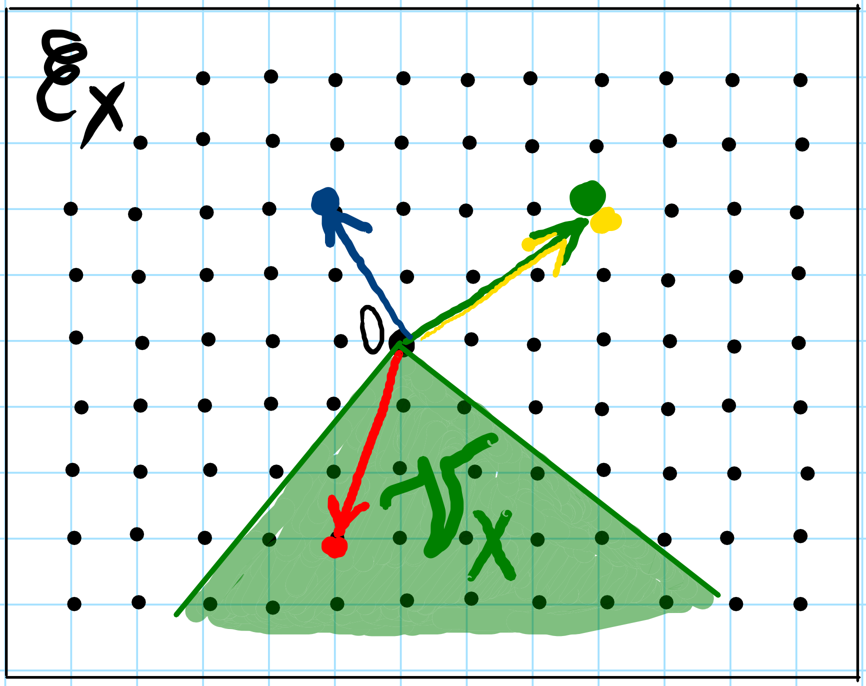
$D \longmapsto \overline{v_D}$ May be not injective!

$$\begin{tikzcd}
1 \arrow[r] & k^\times \arrow[r] \arrow[d] & k(X)^{(B)} \arrow[r] \arrow[d, "v_D"] & \Lambda_X \arrow[r] \arrow[dl, "\overline{v_D}"] & 0 \\
& 0 \quad \in & \mathbb{Z} & &
\end{tikzcd}$$

Summarize:

$$X \rightsquigarrow \left(\Lambda_X \subset \Lambda(T),\ \mathcal{V}_X \subset \mathcal{E}_X = \mathrm{Hom}(\Lambda_X, \mathbb{Q}),\ \mathcal{D}_X \to \Lambda_X^* \subset \mathcal{E}_X\right)$$

colored data



Example. $X = G \circlearrowleft G \times G \supset B^- \times B$ Borel subgrp.

$Y^\circ = B^- \cdot B = U^- \cdot T \cdot U$

$f \in k(G)^{(B^- \times B)} \Rightarrow f(u^- \cdot t \cdot u) = f(t) = t^\lambda \cdot f(e)$

$\lambda \in \Lambda(T)$

$(t_1, t_2) \cdot f\ (t) = f(t_1^{-1} \cdot t \cdot t_2)$

$= t_1^{-\lambda} \cdot f(t) \cdot t_2^\lambda$

eigen wt. $(-\lambda, \lambda)$

Weight lattice $\Lambda_G = \{(-\lambda, \lambda) \mid \lambda \in \Lambda(T)\} \simeq \Lambda(T)$

Irr. reps. : $G \times G \circlearrowright V(\mu) \otimes V(\lambda) \hookrightarrow k[G]$

$\Updownarrow$ Frob. reciprocity

$\mathrm{Hom}_G(V(\mu), V(\lambda^*)) \simeq [V(\mu^*) \otimes V(\lambda^*)]^{\mathrm{diag}(G)} \neq 0$

$\Updownarrow$

$\mu = \lambda^*$

$$V(\lambda^*)\otimes V(\lambda) \xrightarrow{\ \sim\ } \mathrm{Mat}\,V(\lambda) \subset k[G]$$

$$v^*\otimes v \longmapsto f_{v^*,v} \in \left\langle f_{v^*,v}(g) = \langle v^*, g\cdot v\rangle,\ \forall v\in V(\lambda),\ v^*\in V(\lambda^*)\right\rangle_k$$

lowest wt. vector for B in $V(\lambda^*)$

space of matrix entries for $G \curvearrowright V(\lambda)$

$$v^*_{-\lambda}\otimes v_\lambda \longmapsto f_{(-\lambda,\lambda)}$$

ht. wt. vector for $B^-\times B$

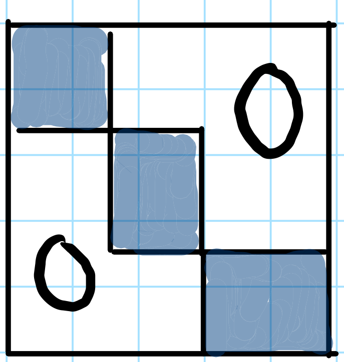
$$\mathrm{Mat}\,V(\lambda) = k[G]_{(-\lambda,\lambda)}$$

$$V(\lambda)\otimes V(\mu) \simeq V(\lambda+\mu)\oplus V(\lambda+\mu-\beta_1)\oplus\cdots\oplus V(\lambda+\mu-\beta_s)$$

$$\mathrm{Mat}\,V(\lambda)\cdot \mathrm{Mat}\,V(\mu) = \mathrm{Mat}\,V(\lambda)\otimes V(\mu)$$

$$= \mathrm{Mat}\,V(\lambda+\mu)\oplus \bigoplus \mathrm{Mat}\,V(\lambda+\mu-\beta_j)$$

β_j distinct



Simple roots $\alpha_1, \dots, \alpha_\ell \in \Delta^+$ lin. indep., $\forall$ pos. root = sum of α_i's

[Onishchik – Vinberg, LG & AG, chap. 4, §§ 1, 2]

$\lambda, \mu \in (C^+)^\circ \Rightarrow$ all $\alpha_1, \dots, \alpha_\ell$ occur among β_j's

$\Downarrow$

$e_{-\alpha_i} v_\lambda, \; e_{-\alpha_i} v_\mu \neq 0$, $\quad e_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i} \subset \mathfrak{g}$

$c_\lambda \cdot e_{-\alpha_i} v_\lambda \otimes v_\mu - c_\mu \cdot v_\lambda \otimes e_{-\alpha_i} v_\mu \in V(\lambda) \otimes V(\mu)$

ht. wt. vector of wt. $\lambda + \mu - \alpha_i$ (for some c_λ, c_μ)

Valuation cone:

$$\mathcal{V}_G = \{ v \in \Lambda(T)^* \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle v, \alpha_i \rangle \leq 0, \; \forall i = 1, \dots, \ell \}$$

negative Weyl chamber