Spherical varieties: Lecture 13

Example (continued): $\quad G = (G\times G)/\mathrm{diag}(G)$

$$\Lambda_G \simeq \Lambda(T) \hookrightarrow \Lambda(T\times T)$$
$$\lambda \mapsto (-\lambda, \lambda)$$

$$\mathcal{V}_G = \{ v \in \Lambda(T)^* \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \mid \langle v, \alpha_i \rangle \leq 0, \ \forall i = 1, \dots, \ell \}$$

Colors $\leftarrow\!\!\sim\!\sim$ Bruhat decomposition

[Humphreys, LAG, §29]

[Springer, LAG, §8.3]

$$D_i = \overline{B^- \cdot r_i \cdot B}, \qquad i = 1, \dots, \ell$$

simple reflections

$$\mathfrak{g} \supset \mathfrak{s}_i = \langle \underset{\begin{pmatrix}0&1\\0&0\end{pmatrix}}{\underset{\parallel}{e_{\alpha_i}}},\ \underset{\begin{pmatrix}0&0\\1&0\end{pmatrix}}{\underset{\parallel}{e_{-\alpha_i}}},\ \underset{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}{\underset{\parallel}{[e_{\alpha_i}, e_{-\alpha_i}]}} \rangle_k \simeq \mathfrak{sl}_2$$

$$G \supset \underset{\underset{r_i}{}}{\underset{\Psi}{S_i}} \simeq (P)SL_2$$

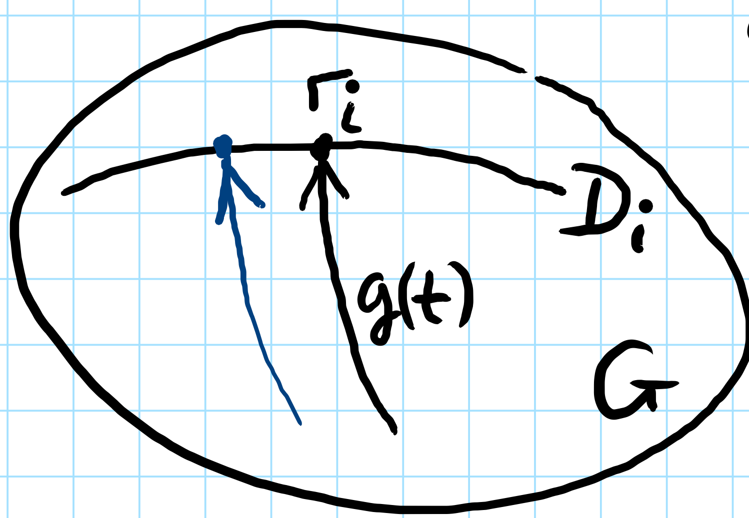
$$r_i = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$f = f_{(-\lambda,\lambda)}$$

$$\operatorname{ord}_{D_i}(f) = \operatorname{ord}_{t=0} f(g(t)), \quad g(t) \in B^- \cdot B \text{ for } t \neq 0$$

transversal to D_i

at $g(0) = r_i$



Bruhat decomp. $\Rightarrow G \supset N_i^- \cdot S_i \cdot T_i \cdot N_i^+$ open

$$N_i^\pm = \exp\Big(\bigoplus_{\substack{\alpha\in\Delta^+\\ \alpha\neq\alpha_i}} \mathfrak{g}_{\pm\alpha}\Big) \subset U^\pm, \quad \underset{\text{codim}=1}{T_i \subset T}$$

$$D_i \supset N_i^- \cdot \underbrace{(D_i \cap S_i)}_{\begin{pmatrix} 0 & * \\ * & * \end{pmatrix} = (B^-\cap S_i)\cdot r_i \cdot (B\cap S_i)} \cdot T_i \cdot N_i^+$$

Take $g(t) = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \in S_i$

$$= \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}$$

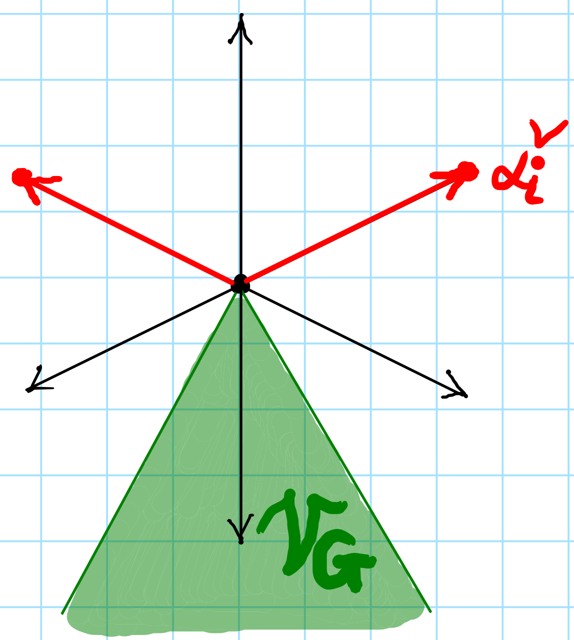
$$= \exp(-t^{-1} \cdot e_{-\alpha_i}) \cdot t^{\alpha_i^\vee} \cdot \exp(t^{-1} \cdot e_{\alpha_i}) \in U^- \cdot T \cdot U$$

[Springer, LAG, § 7.4] $\quad \alpha_i^\vee \in \Lambda(T)^*$ simple coroot

[Onishchik–Vinberg, LG & AG, Chap. 4, § 2]

$$f(g(t)) = f(t^{\alpha_i^\vee}) = t^{\langle \alpha_i^\vee, \lambda \rangle} \cdot f(e)$$

Hence: $\overline{v_{D_i}} = \alpha_i^\vee$



Exercise 1: Compute colored data for:

(a) $X = G/B$

(b) $X = G/U$

Problem: Classify (normal) sph. vars. $X \overset{\text{open}}{\supset} Y_0$ given sph. hom. space

$\Rightarrow$ fixed colored data $(\Lambda, \mathcal{V}, \mathcal{D} \to \Lambda^*)$

Recall: $X \supset$ fin. many G-orbits Y_i

$\forall$ G-orbit $Y \subset X$:

$$X_Y = \{x \in X \mid \overline{Gx} \supset Y\} = X \setminus \bigcup_{\overline{Y_i} \not\supset Y} \overline{Y_i} \quad \text{open in } X$$

$\supset Y$ unique G-orbit closed in X_Y

$$X = \bigcup_{Y \subset X} X_Y \quad \text{finite open cover}$$

Def. A sph. variety X is *simple* if $X \supset$ unique closed G-orbit

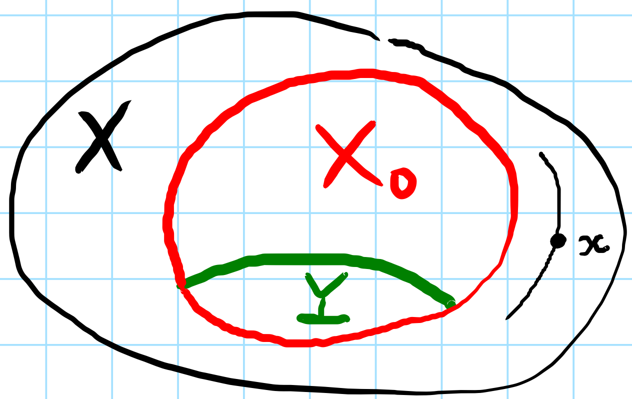
Problem 1: Classify simple sph. vars.

Problem 2: How to glue them together?

Suppose: $\begin{array}{ll} X & \text{simple} \\ \cup & \\ Y & \text{unique closed } G\text{-orbit} \end{array} \Rightarrow X = X_Y$

Prop. Simple sph. vars. are quasiproj.

Proof: Sumihiro thm $\Rightarrow$ $\exists$ G-stable quasiproj. open $X_0 \subset X$, $\supset Y$



$X \setminus X_0 \ni x$ closed $\Rightarrow \overline{Gx} \cap Y = \emptyset$

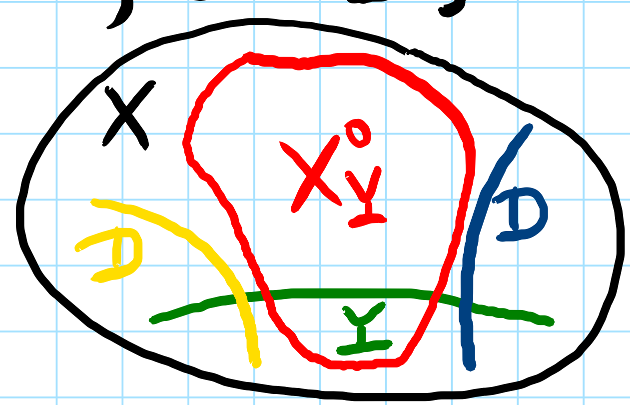
X simple $\Rightarrow X = X_0$

$\blacksquare$

$\mathcal{V}^Y := \{ v = v_D \mid D \subset X \ G\text{-stable}, D \supset Y \} \subset \mathcal{V}$

$\mathcal{D}^Y := \{ D \in \mathcal{D} \mid D \supset Y \}$

$X_Y^\circ := X_Y \setminus \bigcup_{D \notin \mathcal{D}^Y} D$

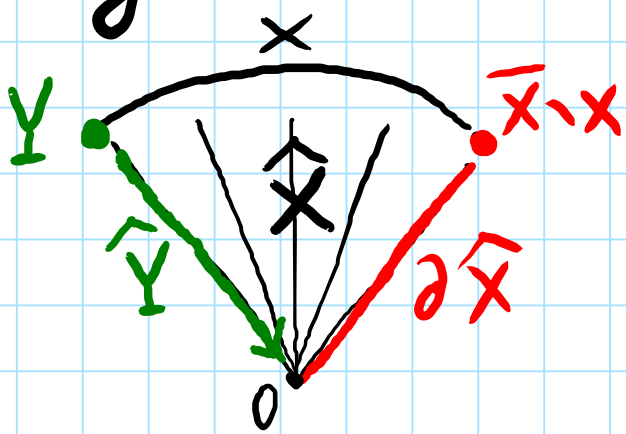


Lemma 1. X°_{Y} = smallest B-stable affine open subset intersecting Y

open **B-chart** of Y

$$k[X^{\circ}_{Y}] = \{ f \in k[Y^{\circ}] \mid v(f) \geq 0,\ v_D(f) \geq 0,\ \forall v \in \mathcal{V}^Y,\ \forall D \in \mathcal{D}^Y \}$$

Proof: May assume $X \subset \mathbb{P}(V) \Rightarrow \overline{X}$ proj. closure



$V \supset \hat{X}$ = aff. cone over $\overline{X}$ $\ni 0$

$\cup$

$\partial \hat{X}$ = cone over $\overline{X} \setminus X$ $\ni 0$

$$Y \not\subset \overline{X} \setminus X \Rightarrow k[\hat{X}] \rhd \mathcal{I}(\hat{Y}) \not\supset \mathcal{I}(\partial \hat{X}) = \mathcal{I}(\hat{Y} \cup \partial \hat{X}) \oplus \underset{\hat{G}\text{-stable}}{M}$$

Choose $F \in M^{(\hat{B})}$, $F \neq 0$ $\Rightarrow X^{\circ} := \{ x \in \overline{X} \mid F(x) \neq 0 \} \subset X$

B-stable, open, affine, $X^{\circ} \cap Y \neq \emptyset$

$\forall$ B-stable aff. open $X^\circ \subset X$, $X^\circ \cap Y \neq \emptyset$:

$$X \setminus X^\circ = D_1 \cup \ldots \cup D_s$$

$$D_i \in \mathcal{D} \setminus \mathcal{D}^Y = \{D_1, \ldots, D_t\}, \quad t \geq s$$