Spherical varieties: Lecture 14

$X = X_Y$ simple sph. G-var.

$\mathcal{V}^Y = \{ v \in \mathcal{V} \mid v = v_D,\ D \subset X$ G-stable, $D \supset Y \}$

$\mathcal{D}^Y = \{ D \in \mathcal{D} \mid D \supset Y \}$

$$X_Y^\circ = X_Y \setminus \bigcup_{D \in \mathcal{D} \setminus \mathcal{D}^Y} D \qquad \text{B-chart}$$

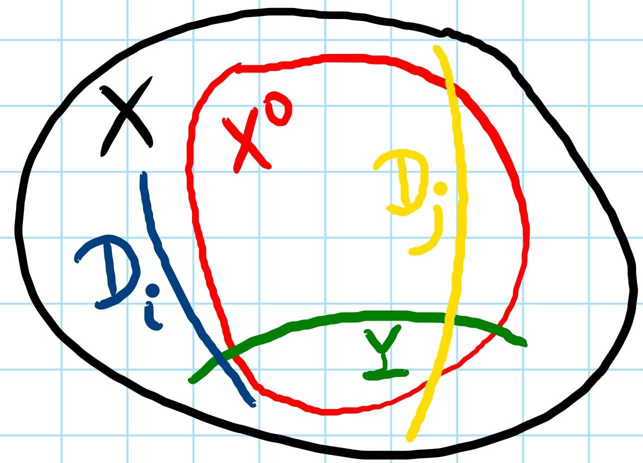
Lemma 1. X_Y° = smallest B-stable affine open subset intersecting Y

$$k[X_Y^\circ] = \{ f \in k[Y^\circ] \mid v(f) \geq 0,\ v_D(f) \geq 0,\ \forall v \in \mathcal{V}^Y,\ D \in \mathcal{D}^Y \}$$

Proved: $\exists$ B-stable affine open $X^\circ \subset X$, $X^\circ \cap Y \neq \emptyset$

$$\mathcal{D} \setminus \mathcal{D}^Y = \{D_1, \dots, D_t\}$$

$$X \setminus X^\circ = D_1 \cup \dots \cup D_s, \quad s \le t$$



Choose $f \in k[X^\circ]$, $f = 0|_{D_j}$, $\forall j > s$, $f \neq 0|_Y$

Blow-up: [Hartshorne, AG, chap. 2, §7]

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \cup & & \cup \\ Y & \longleftarrow & D_0 \\ X \setminus Y & \xleftarrow{\sim} & X' \setminus D_0 \end{array}$$

D_0: G-stable prime divisor

$v_0 = v_{D_0} \in \mathcal{V}$, $v_0(f) = 0$

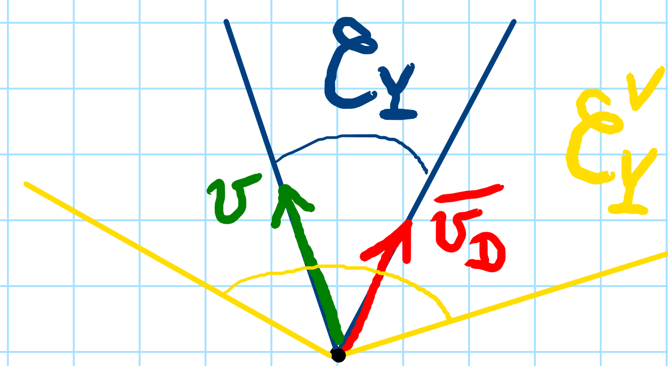
2nd approx. lemma $\Longrightarrow \exists f_\lambda \in k(X)^{(B)} : \quad v_0(f_\lambda) = 0$

$$\forall j > s : \quad v_{D_j}(f_\lambda) > 0$$

$$\Longrightarrow X_Y^\circ = X^\circ \setminus \{D_{s+1}, \dots, D_t\}$$
$$= \{x \in X^\circ \mid f_\lambda(x) \neq 0\} \quad \underline{\text{affine}}$$
$$= X \setminus (D_1 \cup \dots \cup D_t)$$

$\Longrightarrow$ reg. functions on X_Y° may have poles on X only along $D_1, \dots, D_t$

$\Longrightarrow$ <u>description</u> of $k[X_Y^\circ]$



$\blacksquare$

$$\mathcal{C}_Y := \operatorname{cone}(v, \overline{v_D} \mid v \in \mathcal{V}^Y, D \in \mathcal{D}^Y) \subset \mathcal{E} = \Lambda^* \otimes_{\mathbb{Z}} \mathbb{Q}$$

Rmk. $f_\lambda \in k[X_Y^\circ]^{(B)} \Longleftrightarrow \lambda \in \mathcal{C}_Y^\vee \cap \Lambda$

dual cone given by: $\langle v, \lambda \rangle \geq 0, \ \forall v \in \mathcal{C}_Y$

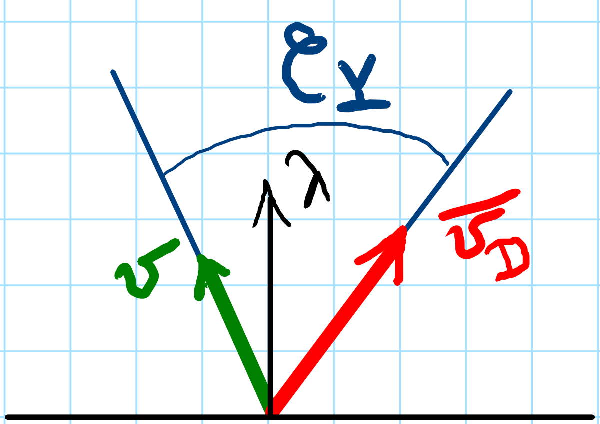
Lemma 2. $\mathcal{C}_Y$ is pointed; $\forall D \in \mathcal{D}^Y$: $\overline{v_D} \neq 0$

Proof: Choose $\underset{\neq 0}{f} \in k[X_Y^\circ]$, $f = 0|_D$, $\forall$ B-stable $D \subset X_Y^\circ$

$$\Longrightarrow v(f) > 0, \quad \forall v \in \mathcal{V}^Y$$
$$v_D(f) > 0, \quad \forall D \in \mathcal{D}^Y$$

2nd approx. lemma $\Longrightarrow$ may assume $f = f_\lambda$

$$\Longrightarrow \langle v, \lambda \rangle > 0, \quad \forall v \in \mathcal{V}^Y$$
$$\langle \overline{v_D}, \lambda \rangle > 0, \quad \forall D \in \mathcal{D}^Y$$
$$\Longrightarrow \langle w, \lambda \rangle > 0, \quad \forall w \in \mathcal{C}_Y \setminus 0$$
$$\Longrightarrow \mathcal{C}_Y \text{ pointed}$$



$\square$

Lemma 3. $\mathcal{C}_Y^\circ \cap \mathcal{V} \neq \emptyset$

rel. interior = cone $\setminus$ proper faces

Proof: Blow-up of $\begin{array}{ccc} Y & \subset & X \\ \uparrow & & \uparrow \\ D_0 & \subset & X' \end{array}$, $\quad v_0 = v_{D_0} \in \mathcal{V}$

$$\lambda \in \Lambda \cap \mathcal{C}_Y^\vee \implies f_\lambda \in k[X_Y^\circ]$$

Suppose $\exists\, v \in \mathcal{V}^Y$ or $D \in \mathcal{D}^Y$ s.t. $\langle v,\lambda\rangle > 0$ or $\langle v_D,\lambda\rangle > 0$

$\implies f_\lambda = 0|_D$ for some B-stable $D \subset X_Y^\circ$

$\implies f_\lambda = 0|_Y \implies f_\lambda = 0|_{D_0} \implies \langle v_0,\lambda\rangle > 0$

Hence $v_0 \in \mathcal{C}_Y^\circ$.

$\blacksquare$

Lemma 4.

$$\mathcal{V}^Y = \{\text{indivisible } v \in \Lambda^* \text{ on } \overbrace{\text{extr. rays}}^{\text{edges}} \text{ of } \mathcal{C}_Y \not\ni \overline{v_D},\ D \in \mathcal{D}^Y\}$$

Proof: $v \in \mathcal{V}^Y,\quad v = v_D,\quad D \subset X \quad G\text{-stable}$

$$\Rightarrow\ \exists f \in k[X_Y^\circ],\quad f \neq 0|_D,\quad f = 0|_{D'},$$

$$\forall\, B\text{-stable } D' \subset X,\ D' \supset Y,\ D' \neq D$$

$$\Rightarrow\ v(f) = 0$$

$$v'(f) > 0,\quad \forall\, v' \in \mathcal{V}^Y \setminus \{v\}$$

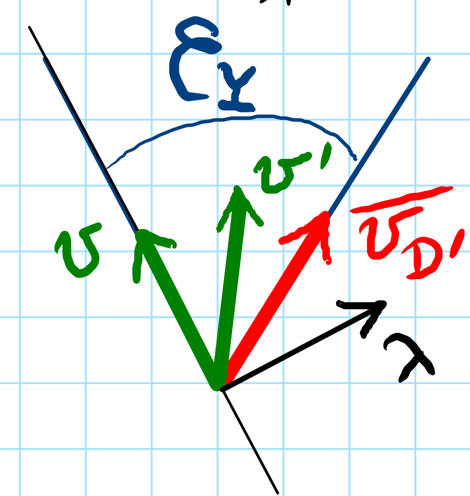
$$v_{D'}(f) > 0,\quad \forall\, D' \in \mathcal{D}^Y$$

2nd approx. lemma $\Rightarrow$ may assume $f = f_\lambda$

$$\langle v, \lambda\rangle = 0$$

$v \in$ extr. ray of $\mathcal{C}_Y \not\ni \overline{v_{D'}}$ $\Leftarrow$ $\langle v', \lambda\rangle > 0$

$$\langle \overline{v_{D'}}, \lambda\rangle > 0$$



Conversely: $\mathcal{C}_Y = \mathrm{cone}(v, \overline{v_D} \mid v \in \mathcal{V}^Y, D \in \mathcal{D}^Y)$

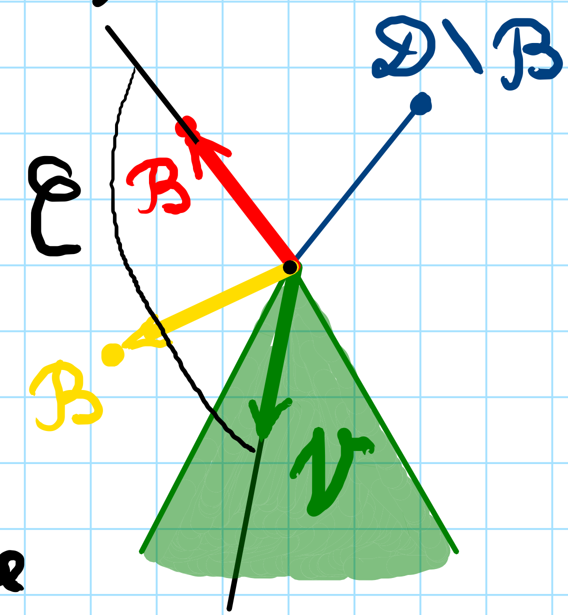
$\Longrightarrow \forall$ extr. ray $\not\ni \overline{v_D}$ is spanned by some $v \in \mathcal{V}^Y$

$\mathrm{Im}(v) = \mathbb{Z} \iff v$ indivisible

$\blacksquare$

Def. Colored cone: $(\mathcal{C}, \mathcal{B})$, where:

- $\mathcal{B} \subset \mathcal{D}$
- $\mathcal{C} \subset \mathcal{E}$ convex cone generated by $\overline{v_D}$, $D \in \mathcal{B}$, and fin. many $v_i \in \mathcal{V}$
- $\mathcal{C}$ pointed
- $\overline{v_D} \neq 0$, $\forall D \in \mathcal{B}$
- $\mathcal{C}^\circ \cap \mathcal{V} \neq \emptyset$



Lemmas 2,3: $(\mathcal{C}_Y, \mathcal{D}^Y)$ is a colored cone

Thm. $\left\{ \text{simple sph. vars.} \underset{\text{open}}{\supseteq} Y_o \right\} \overset{1:1}{\longleftrightarrow} \left\{ \text{colored cones in } \mathcal{E} \right\}$

$$X = X_Y \rightsquigarrow (\mathcal{C}_Y, \mathcal{D}^Y)$$

Proof:

Injectivity. Suppose $X' = X_{Y'}$ another simple sph. var.

$$(\mathcal{C}_{Y'}, \mathcal{D}^{Y'}) = (\mathcal{C}_Y, \mathcal{D}^Y) \underset{\text{Lemma 4}}{\Longrightarrow} \mathcal{V}^{Y'} = \mathcal{V}^Y$$

$$\begin{tikzcd}
Y_o \arrow[r, equal] \arrow[d, phantom, "\cap"] & Y_o \arrow[d, phantom, "\cap"] \\
X \arrow[r, dashed, "\varphi"] & X' \\
X^\circ_Y \arrow[u, phantom, "\cup"] \arrow[r, "\varphi", "\sim"'] & X^\circ_{Y'} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$k[X^\circ_Y] = \left\{ f \in k[Y^\circ] \;\middle|\; v(f) \geq 0,\ v_D(f) \geq 0 \ \forall v \in \mathcal{V}^Y,\ \forall D \in \mathcal{D}^Y \right\} = k[X^\circ_{Y'}]$$

$$\forall g \in G: \quad g \cdot X^\circ_Y \xrightarrow[\sim]{\varphi} g \cdot X^\circ_{Y'}$$

$$\Longrightarrow X = G \cdot X^\circ_Y \xrightarrow[\sim]{\varphi} G \cdot X^\circ_{Y'} = X'$$