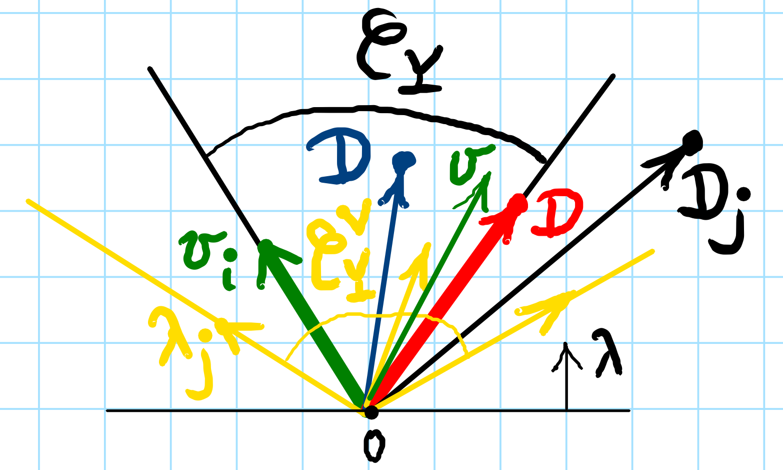
Spherical varieties: Lecture 16

Aim: colored cone $(\mathcal{C},\mathcal{B}) \rightsquigarrow$ simple sph. var. $X = X_Y$

s.t. $(\mathcal{C}_Y, \mathcal{D}^Y) = (\mathcal{C},\mathcal{B})$



Construction:

$\mathcal{D}\setminus\mathcal{B} = \{D_1,\dots,D_t\}$

$v_1,\dots,v_r$ generators of extr. rays of $\mathcal{C}$ not containing v_D, $D\in\mathcal{B}$

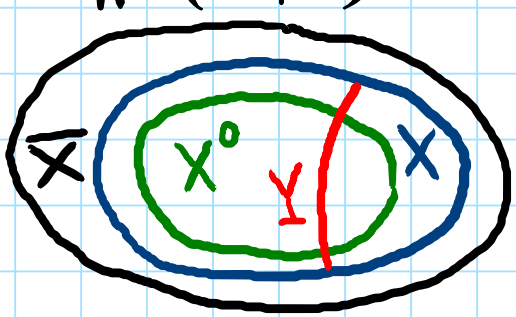
$\lambda_1,\dots,\lambda_n$ generators of $\mathcal{C}^\vee\cap\Lambda$

$\delta = m\cdot(D_1+\dots+D_t)$, $m \gg 0$, $\mathcal{L} = \mathcal{O}(\delta)$

$s_0 \in H^0(Y_0,\mathcal{L})^{(B)}$, $\operatorname{div}(s_0) = \delta$, $s_j = f_{\lambda_j}\cdot s_0 \in H^0(Y_0,\mathcal{L})^{(B)}$

$M_j = \langle G\cdot s_j\rangle_k$, $j = 0,\dots,n$, $M = M_0\oplus\dots\oplus M_n \subset H^0(Y_0,\mathcal{L})$

$\varphi : Y_0 \longrightarrow \mathbb{P}(M^*)$ $\Rightarrow$ $\overline{X} = \overline{\varphi(Y_0)}$



$X^0 = \overline{X}\setminus\{s_0 = 0\}$

$X = G\cdot X^0$

Proved:

1) $k[X^\circ] = \{ f \in k[Y^\circ] \mid v_i(f) \geq 0, \forall i = 1, \dots, r;\ v_D(f) \geq 0, \forall D \in \mathcal{B} \}$

2) X° and X are normal

Proceed w. proof:

3) $\varphi : Y_0 \underset{\text{open}}{\hookrightarrow} X$

Suffices to prove: $k(X^\circ) \supset k[Y^\circ] \subset k(Y_0)$

$$\left.\begin{array}{l} \mathcal{C} \text{ pointed} \\ \overline{v_D} \neq 0,\ \forall D \in \mathcal{B} \end{array}\right\} \Longrightarrow \begin{array}{r} \exists \lambda \in \mathcal{C}^\vee \cap \Lambda\ \ \forall i = 1, \dots, r: \langle v_i, \lambda \rangle > 0 \\ \forall D \in \mathcal{B}: \langle \overline{v_D}, \lambda \rangle > 0 \end{array}$$

$$f \in k[Y^\circ] \Longrightarrow v_i(f \cdot f_\lambda^k),\ v_D(f \cdot f_\lambda^k) \geq 0 \text{ for } k \gg 0$$

$$f_\lambda \in k(Y_0)^{(\mathcal{B})}$$

$$\Longrightarrow f \cdot f_\lambda^k \in k[X^\circ] \ni f_\lambda$$

$$\Longrightarrow f \in k(X^\circ)$$

4) X simple

Choose $v \in \mathcal{C}^\circ \cap \mathcal{V} \neq \emptyset$

$\Longrightarrow$ $v \geq 0$ on M/s_0, as in proof of 1)

$\Longrightarrow$ $v \geq v(s_0)$ on M

$\Longrightarrow$ $v \geq d \cdot v(s_0)$ on $\underbrace{M \cdot \ldots \cdot M}_{d} = k[\hat{X}]_d$

$$\mathcal{I} = \bigoplus_{d=1}^{\infty} \mathcal{I}_d \subset k[\hat{X}]$$

where $\mathcal{I}_d = \{F \in k[\hat{X}]_d \mid v(F) > d \cdot v(s_0)\}$

G-stable prime ideal

Exercise 1: Prove it.

$\Longrightarrow$ $\mathcal{I} = \mathcal{I}(Y)$, $Y \subset \overline{X}$ G-orbit
$\not\ni s_0$ $\Longrightarrow$ $Y \cap X^\circ \neq \emptyset$

Prove that $Y \subset \overline{Y'}$, $\forall$ G-orbit $Y' \subset X$,

In particular, Y unique closed orbit in X

Blow-up:
$$\begin{array}{ccc} \overline{Y'} & \subset & X \\ \uparrow & & \uparrow \\ D' & \subset & X' \end{array}$$

$\Rightarrow$ $v' = v_{D'} \in \mathcal{V}$

$v' \geq 0$ on $k[X^\circ]$ because $Y' \cap X^\circ \neq \emptyset$

$\hookrightarrow$ $\langle v', \lambda_j \rangle \geq 0$, $\forall j = 1, \dots, n$ $\Rightarrow$ $v' \in \mathcal{E}$

If $Y \not\subset \overline{Y'}$, then $\exists f \in k[X^\circ] : f \neq 0|_Y, f = 0|_{Y'}$

$\Rightarrow$ $v'(f) > 0$, $v(f) = 0$

2nd appox. lemma $\Rightarrow$ may assume $f = f_\lambda$, $\lambda \in \mathcal{E}^\vee$

$\Rightarrow$ $\left.\begin{array}{l} \langle v', \lambda \rangle > 0 \\ \langle v, \lambda \rangle = 0 \end{array}\right\}$ contradicts $v \in \mathcal{E}^\circ$

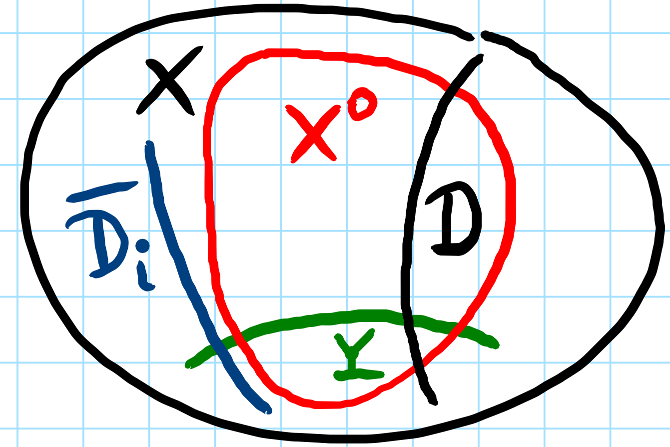
because then $\langle \mathcal{E}, \lambda \rangle = 0$

5) $X^\circ = X^\circ_Y$, $\quad \mathcal{D}^Y = \mathcal{B}$

$X^\circ = X \setminus \{s_0 = 0\} = X \setminus (\overline{D}_1 \cup \ldots \cup \overline{D}_t)$

$D \in \mathcal{D} \setminus \mathcal{B} \Rightarrow D = \overline{D}_i \Rightarrow D \not\supset Y$

$D \in \mathcal{B} \Rightarrow D \cap X^\circ \neq \emptyset$



If $D \not\supset Y$, then

$\exists f \in k[X^\circ]$, $f = 0|_D$,

$f \neq 0|_Y$ $\quad \Rightarrow \; v_D(f) > 0$

$v(f) = 0$

2nd approx. lemma $\Rightarrow$ may assume $f = f_\lambda$

$\Rightarrow \lambda \in \mathcal{C}^\vee$, $\langle v_D, \lambda \rangle > 0$, $\langle v, \lambda \rangle = 0$

Contradicts $v \in \mathcal{C}^\circ$

Hence $D \supset Y$

Therefore $\mathcal{D}^Y = \mathcal{B}$ and $X^\circ = X \setminus \bigcup\limits_{D \in \mathcal{D} \setminus \mathcal{D}^Y} D = X^\circ_Y$

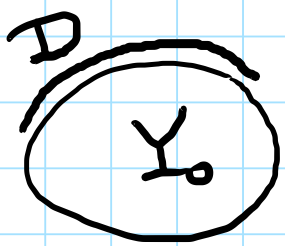
6) $\mathcal{C}_Y = \mathcal{C}$

$$\mathcal{C}_Y^\vee \cap \Lambda = \{\lambda \mid f_\lambda \in k[X_Y^\circ]\} = \mathcal{C}^\vee \cap \Lambda$$

$$\Longrightarrow \mathcal{C}_Y^\vee = \mathcal{C}^\vee \Longrightarrow \mathcal{C}_Y = \mathcal{C}$$

Proof complete.

Examples: colored cone $(0, \emptyset) \rightsquigarrow X = Y_0$



$(\mathbb{Q}_{\geq 0} \cdot v, \emptyset) \rightsquigarrow X = Y_0 \cup D$

$v \in \mathcal{V}$, $\neq 0$

$G \curvearrowright D$ transitive

$v = p \cdot v_D$, $p \in \mathbb{Q}_{>0}$

If $X \supset$ G-orbits other than Y_0 and open orbit in D, then removing other orbits $\rightsquigarrow$ simple sph. var. w. same colored cone

$X =: X_v$ elementary embedding of Y_0, $D =: D_v$