Spherical varieties: Lecture 18

Thm. 1. $\left\{\begin{matrix}\text{sph. varieties} \\ \text{w. open orbit } Y_0\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{colored fans} \\ \text{in } \mathcal{E}\end{matrix}\right\}$

$$X \rightsquigarrow \mathcal{F}(X) = \{(\mathcal{C}_Y, \mathcal{D}^Y)\}_{Y \subset X}$$

Proof: Injectivity.

Suppose $\mathcal{F}(X) = \mathcal{F}(X') = \{(\mathcal{C}_i, \mathcal{B}_i)\}_{i=0,\dots,N}$

G-orbits

$$\Rightarrow X = \bigsqcup_{i=0}^{N} Y_i = \bigcup_i X_i, \qquad X_i = X_{Y_i} \supset Y_0$$

$$(\mathcal{C}_{Y_i}, \mathcal{D}^{Y_i}) = (\mathcal{C}_i, \mathcal{B}_i) = (\mathcal{C}_{Y_i'}, \mathcal{D}^{Y_i'}) \Longrightarrow \quad X_i \xrightarrow[\text{unique}]{\wr} X_i', \qquad Y_0 = Y_0$$

$$X' = \bigsqcup_{i=0}^{N} Y_i' = \bigcup_i X_i', \qquad X_i' = X_{Y_i'} \supset Y_0$$

open orbit

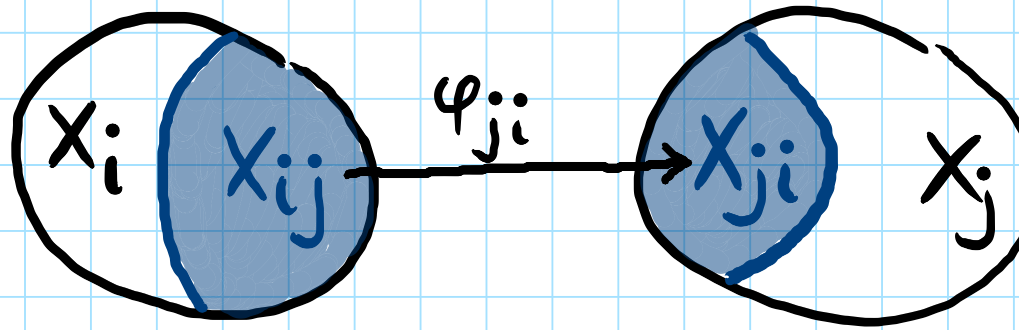
$$\Rightarrow X \xrightarrow{\sim} X'$$

Surjectivity. Let $\mathcal{F} = \{(\mathcal{C}_i, \mathcal{B}_i)\}_{i=0,\dots,N}$ colored fan

$$(\mathcal{C}_0, \mathcal{B}_0) = (0, \emptyset)$$

$$\begin{tikzcd}
(\mathcal{C}_i, \mathcal{B}_i) \arrow[r, rightsquigarrow] & X_i \arrow[d, phantom, "\cup" rotate=-90] & \text{simple sph. var.} \\
 & Y_0 & \text{open } G\text{-orbit}
\end{tikzcd}$$

$$\forall i,j: \quad \begin{tikzcd}
Y_0 \arrow[r, equal] \arrow[d, phantom, "\cap" rotate=-90] & Y_0 \arrow[d, phantom, "\cap" rotate=-90] \\
X_i \arrow[r, dashed, "\varphi_{ji}", "\sim"'] & X_j \\
X_{ij} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "\sim"'] & X_{ji} \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd} \quad \text{biregularity loci}$$



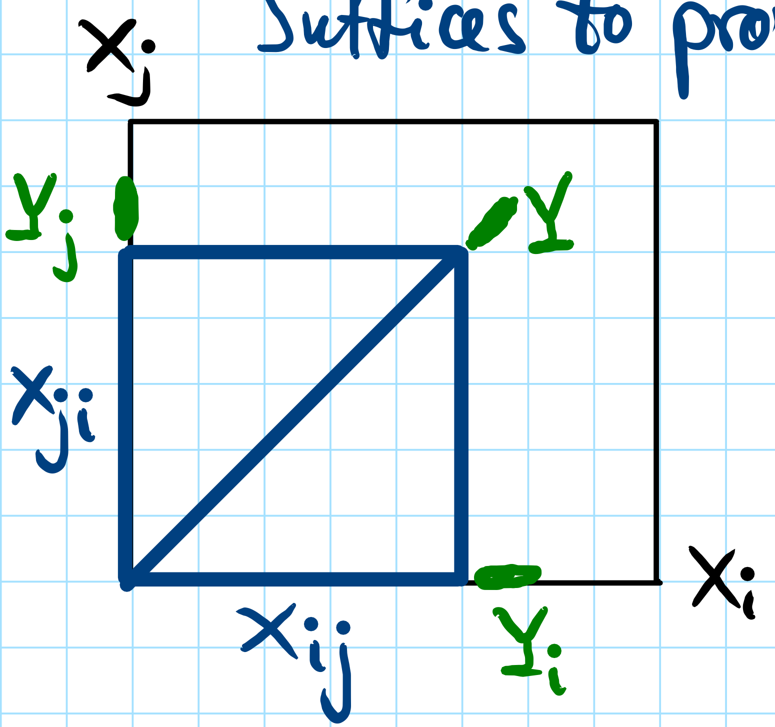
Exercise 1: Prove that X_{ij}, X_{ji} are simple

$$X = \bigsqcup_i X_i \Big/ \sim \,, \quad \text{where: } x_i \sim x_j \text{ if } x_i \in X_{ij},\ x_j \in X_{ji},\ \varphi_{ji}(x_i) = x_j$$

$$\mathcal{F}(X) = \mathcal{F}$$

Separation: $\operatorname{diag} X$ closed in $X \times X$

Suffices to prove: $\operatorname{diag} X_{ij}$ closed in $X_i \times X_j$, $\forall i, j$, $i \neq j$



Suppose $\overline{\operatorname{diag} X_{ij}} \setminus \operatorname{diag} X_{ij} \neq \emptyset$

$\supset Y$

G-orbit

$\exists v \in \mathcal{V}$ s.t.

$$\begin{tikzcd}
 & & X_i \arrow[d, phantom, "\cup"] \\
X_v \arrow[r] \arrow[d, phantom, "\cup"] & \overline{\operatorname{diag} X_{ij}} \arrow[ur] \arrow[dr] \arrow[d, phantom, "\cup"] & Y_i \\
D_v \arrow[r] & Y \arrow[ur] \arrow[dr] & X_j \arrow[d, phantom, "\cup"] \\
 & & Y_j
\end{tikzcd}$$

May assume Y_i closed in X_i

Y_j closed in X_j $\Rightarrow v \in \mathcal{E}_i^\circ \cap \mathcal{E}_j^\circ \cap \mathcal{V}$

Contradiction

$\blacksquare$

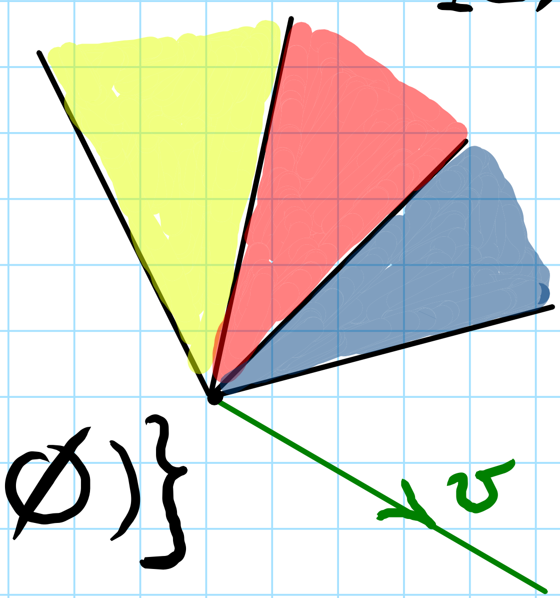
Closure ordering on G-orbits:

$$Y_i \subset \overline{Y_j} \iff (\mathcal{C}_j, \mathcal{B}_j) \text{ is a colored face of } (\mathcal{C}_i, \mathcal{B}_i)$$

Thm. 2. X complete $\iff \mathcal{F}(X)$ covers $\mathcal{V}$, i.e.

$$\bigcup_{Y \subset X} \mathcal{C}_Y \supseteq \mathcal{V}$$

Proof: $\Rightarrow$ Suppose $v \in \mathcal{V}$, $v \notin \bigcup_{Y \subset X} \mathcal{C}_Y$



$$\mathcal{F}' := \mathcal{F}(X) \cup \{(\mathbb{Q}_{\geq 0} v, \emptyset)\}$$
colored fan

$\Rightarrow \mathcal{F}' = \mathcal{F}(X')$, $X' = X \cup X_v$, $X' \setminus X = D_v$

open subvars.

$\Rightarrow X$ not complete

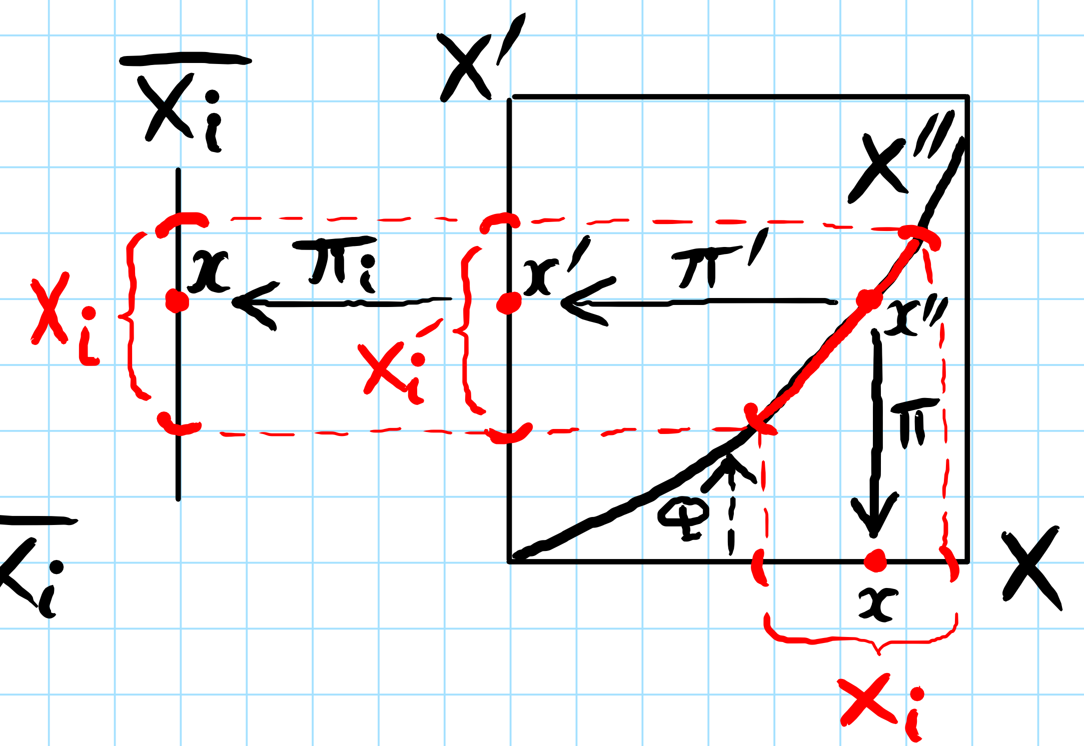
Contradiction

$$X = \bigcup_i X_i \quad \text{open}$$

simple $\Rightarrow$ quasiproj. $\Rightarrow$ $X_i \overset{\text{open}}{\hookrightarrow} \overline{X_i}$ proj. sph. vars.

$$\begin{tikzcd}
X \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\Phi := \mathrm{id}_X \times \varphi"'] & \prod_i \overline{X_i} & \supset X' = \overline{\varphi(X)} \\
X \times X' & \supset X'' = \overline{\Phi(X)} &
\end{tikzcd}$$

$$\begin{tikzcd}
& X'' \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & & \\
X & & X' \arrow[r, "\pi_i"] & \overline{X_i}
\end{tikzcd}$$



Claim: $\pi' : X'' \overset{\text{open}}{\hookrightarrow} X'$

Proof of Claim: $X_i' = \pi_i^{-1}(X_i) \subset X'$ open subsets

$x'' \in X'' \Rightarrow x = \pi(x'') \in X_i$ for some i

$\Rightarrow x'' = (x, x'),\quad \pi_i(x') = x$ identical on $\pi^{-1}(X_i)$

$\Rightarrow x' = \pi'(x'') \in X_i'$

Hence $\pi'(X'') \subset \bigcup_i X_i'$

$$X'' \supset \pi^{-1}(X_i) \xrightarrow[\sim]{\pi'} X_i'$$

$$\begin{array}{c} \Psi \\ x'' = (\underset{X}{\underset{\cap}{x}}, \overbrace{\underset{\overline{X_i}}{\underset{\cap}{x}}, \dots, \underset{\overline{X_j}}{\underset{\cap}{x_j}}, \dots}^{x'}) \longmapsto x' \end{array}$$

$$\Longrightarrow X'' \xrightarrow[\sim]{\pi'} \bigcup_i X_i' \subset X'$$

Claim proven.

$\mathcal{V}$ covered by $\mathcal{F}(X)$ also by $\mathcal{F}(\overline{X_i})$, $\forall i$ $\Longrightarrow$ $\forall v \in \mathcal{V}$ $\exists$ unique

$$\begin{tikzcd} Y_0 \arrow[r, equal] \arrow[d, phantom, "\cap"] & Y_0 \arrow[d, phantom, "\cap"] \\ X_v \arrow[r] \arrow[dr] \arrow[ddr] \arrow[dddr] & X \\ & \overline{X_i} \\ & X' \\ & X'' \end{tikzcd}$$

$\forall G$-orbit $Y \subset X'$ $\exists v \in \mathcal{V}$:

$$\begin{tikzcd} X_v \arrow[r] \arrow[d, phantom, "\cup"] & X' \arrow[d, phantom, "\cup"] \\ D_v \arrow[r] & Y \end{tikzcd}$$

$$\Longrightarrow Y \subset \pi'(X'')$$

Hence $X'' \simeq \pi'(X'') = X'$ proj.

$\Longrightarrow X = \pi(X'')$ complete

$\blacksquare$

Exercise 2: $\operatorname{cone}(v, \overline{v_D} \mid v \in \mathcal{V}, D \in \mathcal{D}) = \mathcal{E}$