Spherical varieties: Lecture 19

References on spherical varieties:

- M. Brion. Variétés sphériques. Expository notes (French), 1997, http:\\www-fourier.univ-grenoble-alpes.fr/~mbrion/spheriques.pdf

- N. Perrin. On the geometry of spherical varieties. Transform. Groups 19, 2014, no.1, 171-223.

- J. Gandini. Embeddings of spherical homogeneous spaces. Acta Math. Sinica (English series) 34 (2018), no.3, 299-340.

- D. Timashev. Homogeneous spaces and equivariant embeddings. Springer Encyclopadia of Math. Sciences 138 (2011).

Example. $Y_0 = \{\text{smooth conics in } \mathbb{P}^2\} \circlearrowleft G = GL_3$

$$G \supset B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

$$T = \left\{ t = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \right\}$$

$$\Lambda(T) = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle_{\mathbb{Z}} \simeq \mathbb{Z}^3$$
$$t^{\varepsilon_i} = t_i \quad (i = 1,2,3)$$

$$Y_0 \subset \mathbb{P}(\mathrm{Sym}_3) = \mathbb{P}^5$$
$$Y_0 = \{\det q \neq 0\}, \qquad [q] \in \mathbb{P}(\mathrm{Sym}_3), \quad q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = q^T$$

$$\Delta_k(q) = \det(q_{ij})_{i,j=1}^{k} \in H^0(\mathbb{P}^5, \mathcal{O}(k))^{(B)}$$
$k = 1,2,3$ Eigenwts.: $2(\varepsilon_1 + \ldots + \varepsilon_k)$

$$Y^\circ = \{\Delta_1, \Delta_2, \Delta_3 \neq 0\}$$
$$\mathbb{P}^5 \setminus Y^\circ = D_1 \cup D_2 \cup D_3, \qquad D_k = \{\Delta_k = 0\}$$

D_1, D_2: colors; D_3: G-stable

$$\Rightarrow \mathcal{D} = \{D_1, D_2\}$$

$$f \in k(Y_0)^{(B)} \Rightarrow \operatorname{div}_{\mathbb{P}^5}(f) = m_1 \cdot D_1 + m_2 \cdot D_2 + m_3 \cdot D_3$$

$$m_i \in \mathbb{Z}$$

$$\Rightarrow f \sim \Delta_1^{m_1} \cdot \Delta_2^{m_2} \cdot \Delta_3^{m_3}, \quad m_1 + 2m_2 + 3m_3 = 0$$

Generators of $k(Y_0)^{(B)}/k^\times$: $f_1 = \Delta_1^2/\Delta_2$, $f_2 = \Delta_2^2/\Delta_1\Delta_3$

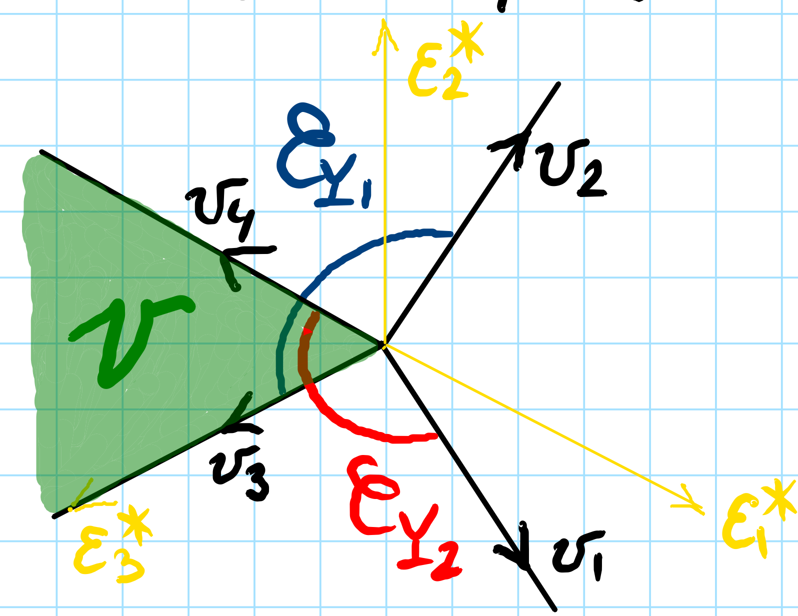
eigenwts. : $\underbrace{\lambda_1 = 2(\varepsilon_1 - \varepsilon_2),\ \lambda_2 = 2(\varepsilon_2 - \varepsilon_3)}_{\text{basis of } \Lambda \simeq \mathbb{Z}^2}$

$$v_k = \overline{v_{D_k}} \in \mathcal{E}$$

$$v_1 = (\varepsilon_1^* - \varepsilon_2^*)/2$$

$$v_2 = (\varepsilon_2^* - \varepsilon_3^*)/2$$

$$v_3 = (-\varepsilon_1^* - \varepsilon_2^* + 2\varepsilon_3^*)/6$$



Closed orbit $Y_1 = \{\operatorname{rk} q = 1\}$, $\mathcal{E}_{Y_1} = \operatorname{cone}(v_2, v_3)$, $\mathcal{D}^{Y_1} = \{D_2\}$

$$Y_0 \hookrightarrow \mathbb{P}(\mathrm{Sym}_3^*) = (\mathbb{P}^5)^* \simeq \mathbb{P}^5 \circlearrowleft G$$

$$[q] \longmapsto [q^{-1}] = [q^\vee]$$

twisted:

$$g \longmapsto (g^{-1})^T$$

$$q \longmapsto (g^{-1})^T . q . g^{-1}$$

$$q^{-1} \longmapsto g \cdot q^{-1} \cdot g^T$$

$$B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \longrightarrow \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} = B^-$$

$$\mathrm{Sym}_3^* \simeq \mathrm{Sym}_3 \ni q^* = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 \longmapsto -\varepsilon_1, -\varepsilon_2, -\varepsilon_3 \longmapsto -\varepsilon_3, -\varepsilon_2, -\varepsilon_1$$

$$v_1 \longleftrightarrow v_2$$

$$v_3 \longleftrightarrow v_4 := (-2\varepsilon_1^* + \varepsilon_2^* + \varepsilon_3^*)/6$$

$$(\mathbb{P}^5)^* \setminus Y^0 = D_1 \cup D_2 \cup D_4 \,, \quad \overline{v_{D_4}} = v_4$$

Closed orbit Y_2, $\mathcal{C}_{Y_2} = \mathrm{cone}(v_1, v_4)$

$$\mathcal{D}^{Y_2} = \{D_1\}$$

Completeness $\Longrightarrow \mathcal{C}_{Y_1}, \mathcal{C}_{Y_2} \supset \mathcal{V} \ni v_3, v_4$

$$\Longrightarrow \mathcal{V} = \mathcal{C}_{Y_1} \cap \mathcal{C}_{Y_2} = \mathrm{Cone}(v_3, v_4)$$

Exercise 1: Variety of <u>complete</u> conics

$$X = \overline{Y_0} \subset \mathbb{P}^5 \times (\mathbb{P}^5)^*$$

a) Prove that X is simple

b) Compute $(\mathcal{C}_Y, \mathcal{D}^Y)$ for the closed orbit $Y \subset X$

Exercise 2: $V = k^4$ symplectic vector space w. symple. form ω

$$\mathbb{P}(V) = \mathbb{P}^3, \quad G = Sp(V, \omega) = Sp_4$$

$$Y_0 = \{(p_1, p_2) \in \mathbb{P}^3 \times \mathbb{P}^3 \,/\, p_i = [v_i],\ U = \langle v_1, v_2 \rangle \subset V,\ \omega|_U \text{ non-degen.}\}$$

a) Prove that Y_0 is a G-spherical hom. space

b) Compute colored data

Exercise 3: $Y_0 = G/H$, $G = GL_2$, $H = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$

a) Prove that Y_0 is a sph. hom. space

b) Compute colored data

Toric varieties

Toric variety = sph. variety X for $G = T$ alg. torus

Open orbit $Y_0 = T \cdot y_0 \simeq T/T_{y_0}$

T abelian $\Longrightarrow$ $T_{y_0} \curvearrowright X$ trivial

$T \rightsquigarrow T/T_{y_0}$ $\Longrightarrow$ may assume $T_{y_0} = \{e\}$

$Y_0 \simeq T \curvearrowleft T$

left mult.

adopt this convention for the sequel

$$B = G = T \implies \mathcal{D} = \emptyset$$

B-semi-inv. functions:

$$f_\lambda(t) = t^{-\lambda} \cdot f(e) \sim t^{-\lambda} = t_1^{-l_1} \cdots t_n^{-l_n}$$

$t_1, \dots, t_n$ coordinates on T

$\lambda = l_1 \cdot \varepsilon_1 + \dots + l_n \cdot \varepsilon_n$, $l_i \in \mathbb{Z}$

$t^{\varepsilon_i} = t_i$

Weight lattice: $\Lambda = \Lambda(T) = \mathbb{Z} \cdot \varepsilon_1 \oplus \dots \oplus \mathbb{Z} \cdot \varepsilon_n$

$$Y^0 = Y_0 = T$$

$$k[T] = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \bigoplus_{\lambda \in \Lambda} \underbrace{k \cdot f_\lambda}_{\substack{\parallel \\ k[T]_{(\lambda)}}}$$

Inv. valuations:

$$k[T]_{(\lambda)} \cdot k[T]_{(\mu)} = k[T]_{(\lambda+\mu)}$$

$$\implies \text{no tails} \implies \mathcal{V} = \mathcal{E} = \Lambda^* \otimes_{\mathbb{Z}} \mathbb{Q}$$

$v \in \mathcal{E} \rightsquigarrow T\text{-inv. val.}: \quad f = \sum_i \underset{\neq 0}{c_i} \cdot f_{\lambda_i}$

$$\Rightarrow v(f) = \min_i \langle v, \lambda_i \rangle$$

$v \in \Lambda^* \rightsquigarrow$ 1-param. subgroup $\begin{array}{ccc} k^* & \longrightarrow & T \\ s & \longmapsto & s^v \end{array}$

$\|$

$k_1 \cdot \varepsilon_1^* + \ldots + k_n \cdot \varepsilon_n^* \Rightarrow s^v = (s^{k_1}, \ldots, s^{k_n})$ in coords.

$$f(s^v) = \sum_i c_i \cdot s^{-\langle v, \lambda_i \rangle}$$

$$\Rightarrow v(f) = \operatorname{ord}_{s=\infty} f(s^v)$$