Spherical varieties: Lecture 1

Motivating problems

1) Representation theory: tensor product decomposition

Representations of $SL_2(\mathbb{C})$:

$\forall n \in \mathbb{Z}_{\geq 0}\ \exists$ unique irr. rep. of $\dim = n+1$

$SL_2 \curvearrowright \mathbb{C}[x,y]_n =: V(n)$

Clebsch-Gordan formula:

$$V(n) \otimes V(m) \simeq V(n+m) \oplus V(n+m-2) \oplus V(n+m-4) \oplus \dots \oplus V(n-m)$$

$(n \geq m)$

Geometric proof

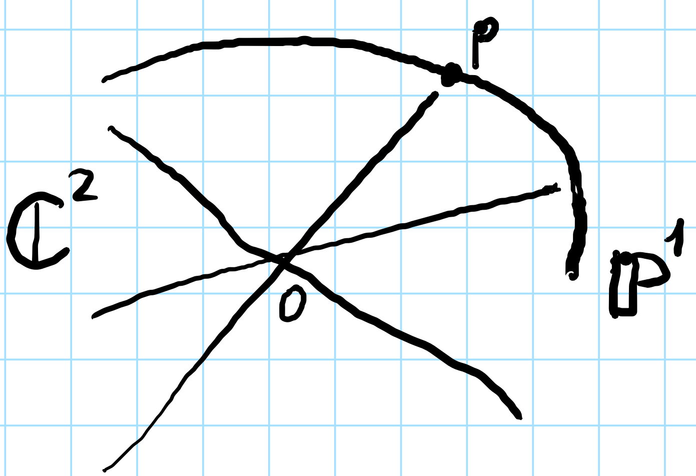
$$V(n) = H^0(\mathbb{P}^1, \mathcal{O}(n))$$

space of hol. sections

line bundle

$$p \in \mathbb{P}^1 \rightsquigarrow \mathcal{O}(n)_p = \{ f : p \to \mathbb{C},\ \deg f = n \}$$

$$= V(n) / \mathrm{Ann}\, p$$



Digression: line/vector bundles

$$X \xleftarrow{\pi} E$$

$\mathbb{C}$-mfld

- $\forall p \in X: \quad E_p := \pi^{-1}(p)$ vector space$/\mathbb{C}$
- Open cover $X = \bigcup_i U_i$ s.t.

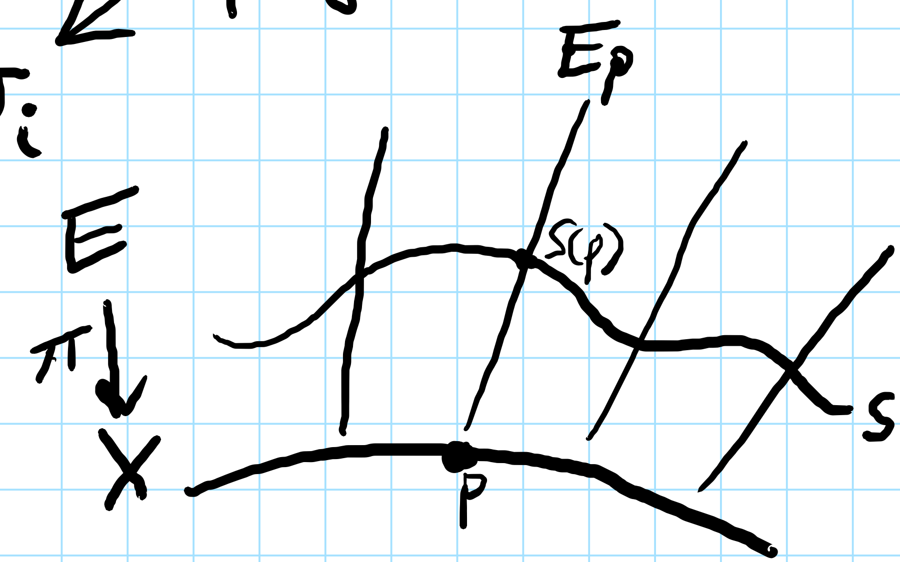
$$\begin{tikzcd}
E|_{U_i} := \pi^{-1}(U_i) \arrow[rr, "\text{fiberwise linear}", "\sim"'] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C}^r \arrow[dl, "\text{proj.}"] \\
& U_i &
\end{tikzcd}$$

vector bundle of rank r

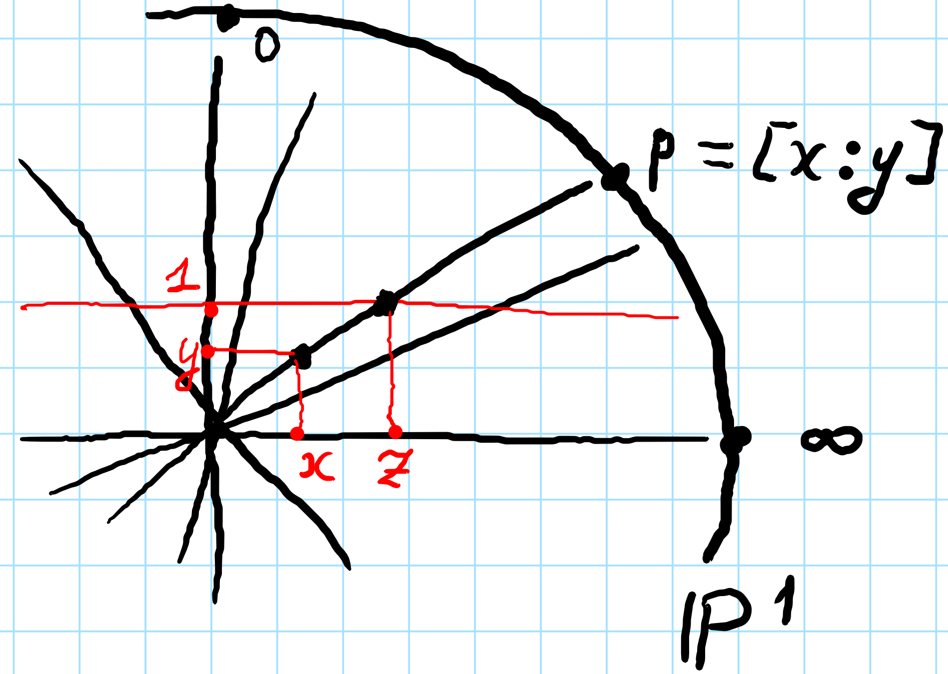
Line bundle: $r = 1$

Section $s : X \to E$

$s(p) \in E_p, \ \forall p \in X$



In our example:



$$\mathbb{P}^1 = \underset{x \neq 0}{\mathbb{A}^1_0} \cup \underset{y \neq 0}{\mathbb{A}^1_\infty}$$

$$p \in \mathbb{A}^1_\infty, \quad p = [z : 1], \quad z = x/y$$

$$\mathcal{O}(n)\big|_{\mathbb{A}^1_\infty} \simeq \mathbb{A}^1_\infty \times \mathbb{C}$$

$$f \in \mathcal{O}(n)_p \longmapsto f_\infty = f(z,1) \in \mathbb{C}$$

$$\mathcal{O}(n)\big|_{\mathbb{A}^1_0} \simeq \mathbb{A}^1_0 \times \mathbb{C}$$

$$f \longmapsto f_0(1 : 1/z)$$

$$p \in \mathbb{A}^1_0 \cap \mathbb{A}^1_\infty$$

$$\Longrightarrow f_\infty = z^n \cdot f_0$$

$$s \in H^0(\mathbb{P}^1, \mathcal{O}(n)) \Longrightarrow s_\infty = s_\infty(z) = z^n \cdot s_0(1/z)$$

$$\Longrightarrow s_\infty(z) \in \mathbb{C}[z]_{\leq n} \Longrightarrow s(x,y) = y^n \cdot s_\infty(z) \in \mathbb{C}[x,y]_n$$

$$SL_2 \curvearrowright \mathbb{P}^1 \times \mathbb{P}^1 \longleftarrow \mathcal{O}(n,m) = \mathcal{O}(n) \boxtimes \mathcal{O}(m)$$

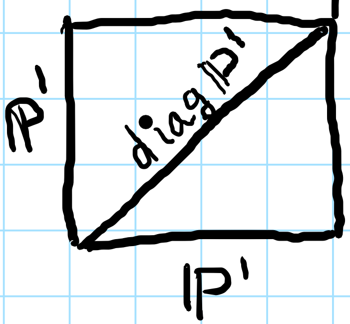
outer tensor product

$$\mathbb{P}^1 \times \mathbb{P}^1 \quad g(p,p') = (g(p), g(p'))$$

$$\mathcal{O}(n,m)_{(p,p')} = \mathcal{O}(n)_p \otimes \mathcal{O}(m)_{p'}$$

Two orbits:

$$\underbrace{\{(p,p') \mid p \neq p'\}}_{\text{open}} \sqcup \underbrace{\operatorname{diag} \mathbb{P}^1}_{\text{closed}}$$



$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n,m)) = H^0(\mathbb{P}^1, \mathcal{O}(n)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(m)) = V(n) \otimes V(m)$$

$$\Big\downarrow \operatorname{res}|_{\operatorname{diag}\mathbb{P}^1}$$

$$\operatorname{Ker} = \det\begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \times H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n-1, m-1))$$

$$H^0(\mathbb{P}^1, \mathcal{O}(n+m)) = V(n+m)$$

$$\mathcal{O}(n,m)\Big|_{\operatorname{diag}\mathbb{P}^1} = \mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m)$$

$$f(x,y,x',y') = 0\Big|_{\substack{x = x' \\ y = y'}} \iff f \vdots \det\begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \leftarrow \textbf{Exercise 1: } \text{Give elementary proof}$$

divisible

We get: $V(n) \otimes V(n) \simeq V(n+m) \oplus V(n-1) \otimes V(m-1)$

Conclude by induction.

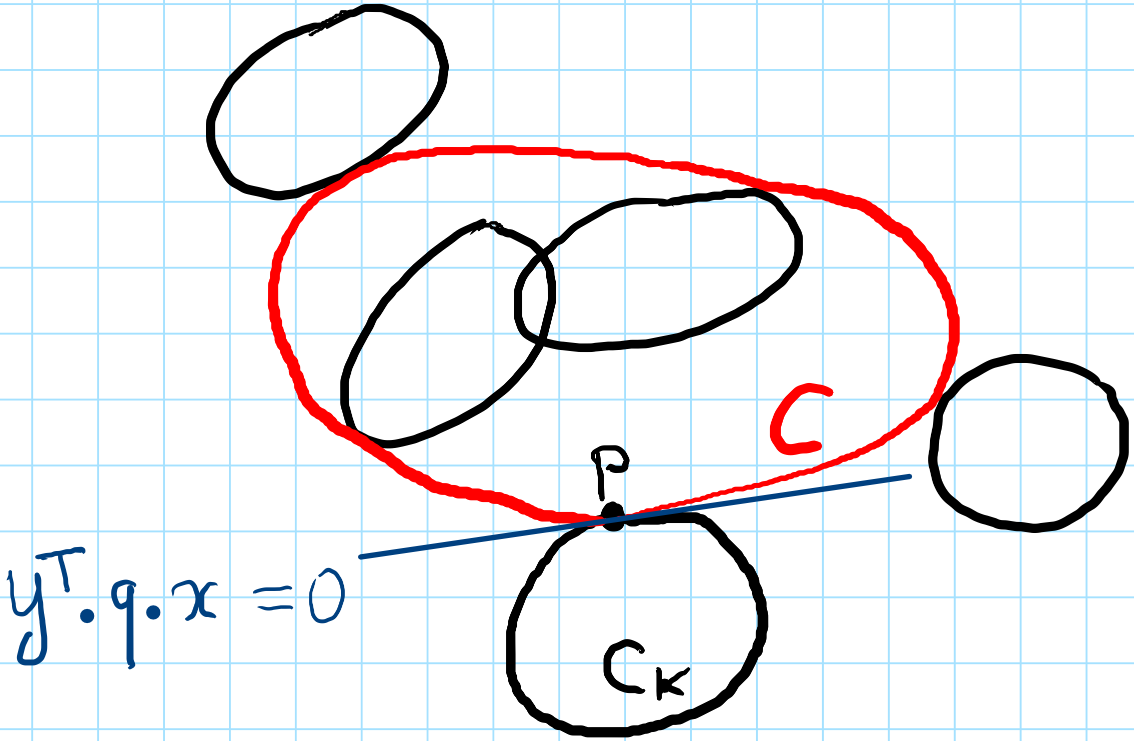
2) Enumerative geometry of conics

Conic $C = \{ p = [x_1 : x_2 : x_3] \in \mathbb{P}^2 \mid q(x_1, x_2, x_3) = \sum_{i,j=1}^{3} q_{ij} x_i x_j = 0 \}$

smoothness: $\det(q_{ij}) \neq 0$

Question (J. Steiner, 1848)

How many conics are tangent to 5 given conics $C_1, \ldots, C_5$ in general position?



$$q = (q_{ij})_{i,j=1,2,3} \in \mathrm{Sym}_3(\mathbb{C})$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{C}^3 \qquad q(x_1, x_2, x_3) = x^T \cdot q \cdot x$$

C tangent to C_k at p $\iff$ $\begin{cases} x^T \cdot q \cdot x = x^T \cdot q_k \cdot x = 0 \\ q \cdot x = \lambda \cdot q_k \cdot x \quad (\lambda \in \mathbb{C}) \end{cases}$

$\iff$ x is eigenvector of $A = q_k^{-1} \cdot q$ isotropic w.r.t. q_k

$\exists$ such x $\Longleftrightarrow$ A has multiple eigenvalue

Exercise 2: Prove it

$\Longleftrightarrow$ $\det(A-\lambda E)$ has multiple root

$$C \text{ tangent to } C_k \Longleftrightarrow \underbrace{\mathrm{Dis}\left[\det(q-\lambda q_k)\right]}_{\text{homogeneous of } \deg = 6 \text{ in } q_{ij}} = 0$$

Exercise 3: Prove it $\longrightarrow$

Space of conics $\mathcal{O} \subset \mathbb{P}(\mathrm{Sym}_3) = \mathbb{P}^5 \curvearrowleft GL_3$

open orbit

$$\overline{\{C \in \mathcal{O} \mid C \text{ tangent to } C_k\}} =: D_k \subset \mathbb{P}^5$$

hypersurface of $\deg = 6$ in $\mathbb{P}^5$

Bézout thm $D_1, \dots, D_n \subset \mathbb{P}^n$
hypersurfaces in gen. pos.
of degrees $d_1, \dots, d_n$

$$\Rightarrow |D_1 \cap \dots \cap D_n| = d_1 \cdot \ldots \cdot d_n$$

Expected answer: $6^5 = 7776$ (Steiner)

Wrong!

Reason: $D_1, \dots, D_5$ are not in gen. pos.

$\Uparrow$

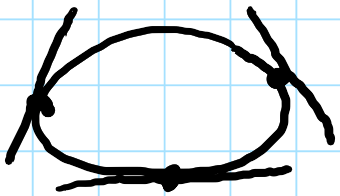
contain $\left\{ \begin{matrix} \text{double} \\ \text{lines} \end{matrix} \right\} \simeq (\mathbb{P}^2)^*$

$\operatorname{rk} q = 1$

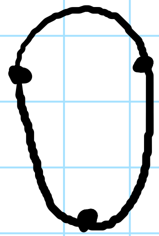
$C \rightsquigarrow$ dual conic $C^\vee = \{\text{lines tangent to } C\} \subset (\mathbb{P}^2)^*$

$q \rightsquigarrow q^\vee$ adjoint matrix

$q \cdot q^\vee = \det q \cdot E$



$\mathbb{P}^2$



$(\mathbb{P}^2)^*$

Exercise 4: Prove that $C^\vee$ is a conic given by $q^\vee$

$$\Theta \hookrightarrow \mathbb{P}^5 \times (\mathbb{P}^5)^* \supset X = \overline{\Theta}$$

$$C \longmapsto (C, C^\vee)$$

$$= \{([q],[q^*]) \mid q \cdot q^* = \lambda E\}$$

Space of Complete conics

Exercise 5: Prove it $\to$ Compact, Smooth

Redefine $D_k = \overline{\{C \in \Theta \mid C \text{ tangent to } C_k\}} \subset X$

Fact: $D_1, \dots, D_k$ are in gen. pos. in X

Answer: $|D_1 \cap \ldots \cap D_5| = \deg [D_1]\cdot \ldots \cdot [D_5] = 3264$ in $H^*(X)$

(M. Chasles, 1864)

Conclusion: It is important to study equivariant compactifications (more generally, open embeddings) of good homogeneous spaces.

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