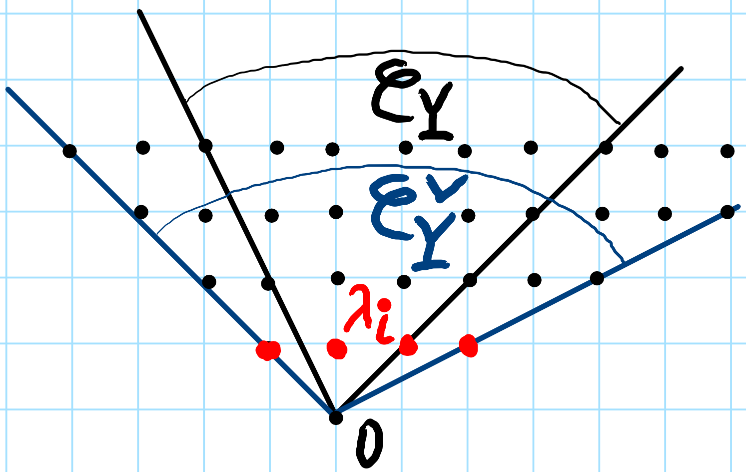
Spherical varieties: Lecture 20

Toric varieties (continued)

$X \supset T$ toric variety

$\forall$ orbit $Y \subset X$: $X_Y = X_Y^\circ$ affine

$$k[X_Y] = \bigoplus_{\lambda \in \mathcal{C}_Y^\vee \cap \Lambda} k \cdot f_\lambda = k[f_{\lambda_1}, \dots, f_{\lambda_m}]$$



$\lambda_1, \dots, \lambda_m$ = generators of $\mathcal{C}_Y^\vee \cap \Lambda$

Conversely: affine $\Rightarrow$ simple

Simple (= affine) toric vars. $\xleftrightarrow{1:1}$ pointed polyhedral cones in $\mathcal{E}$

Arbitrary toric vars. $\xleftrightarrow{1:1}$ ~~colored~~ fans in $\mathcal{E}$ = finite collections of ptd. phdl. cones intersecting along their faces, closed under taking faces

Toric var. X complete $\iff$ $\mathcal{F}(X)$ covers $\mathcal{E}$

Example: $X = \mathbb{P}^n \circlearrowleft T$

$$(x_0 : x_1 : \dots : x_n) \xmapsto{t} (x_0 : t_1 x_1 : \dots : t_n x_n)$$

$$X = \bigcup_{i=0}^{n} X_i, \quad X_i = \{x_i \neq 0\} \simeq \mathbb{A}^n \rightsquigarrow \mathcal{E}_i \subset \mathcal{E}$$

$$k[X_i] = k\left[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

eigenwts. : $\varepsilon_i, \varepsilon_i - \varepsilon_1, \dots, \varepsilon_i - \varepsilon_n$ for $i \neq 0$

$-\varepsilon_1, \dots, -\varepsilon_n$ for $i = 0$

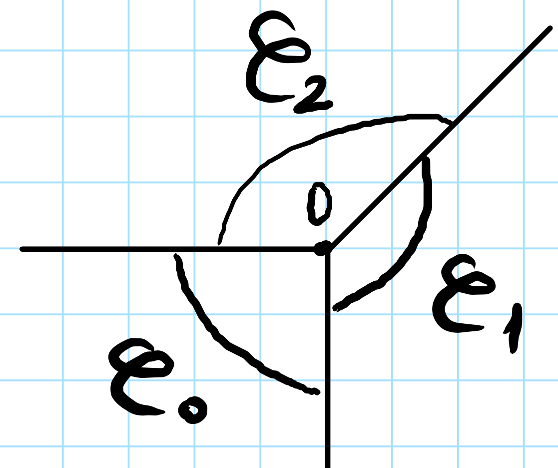
span $\mathcal{E}_i^\vee$

$$\Rightarrow \mathcal{E}_i = \mathrm{cone}(-\varepsilon_1^*, \dots, \underbrace{\varepsilon_1^* + \dots + \varepsilon_n^*}_{i}, \dots, -\varepsilon_n^*) \quad \text{for } i \neq 0$$

$$\mathcal{E}_0 = \mathrm{cone}(-\varepsilon_1^*, \dots, -\varepsilon_n^*)$$

$\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_n$ cover $\mathcal{E}$

n=2:



References on toric varieties:

- W. Fulton. Introduction to toric varieties.
- D. Cox, J. Little, H. Schenck. Toric varieties.

Exercise 1: Describe $\mathcal{F}(X)$ for:

a) $X = \mathbb{P}^1 \times \mathbb{P}^1 \circlearrowleft T = \Bbbk^\times \times \Bbbk^\times$, where $\Bbbk^\times \curvearrowright \mathbb{P}^1$:

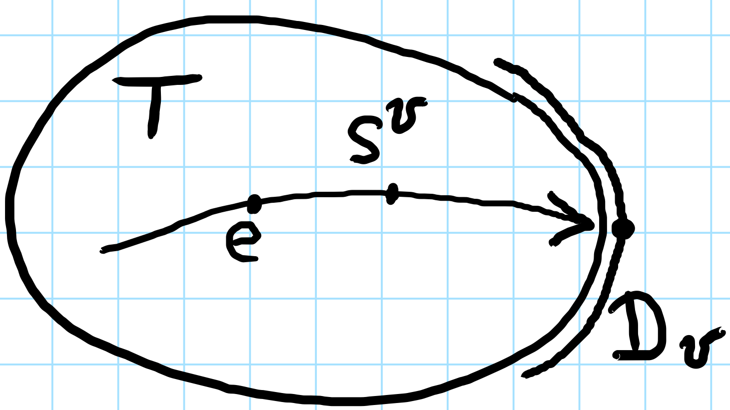
$$(x_0 : x_1) \overset{t}{\longmapsto} (x_0 : t x_1)$$

b) $X =$ blow-up of $(1:0:0)$ in $\mathbb{P}^2$

Exercise 2: For $\forall v \in \Lambda^*$ prove:

a) curve $\{ s \mapsto s^v \}$ intersects $D_v \subset X_v$ transversally at $s = \infty$

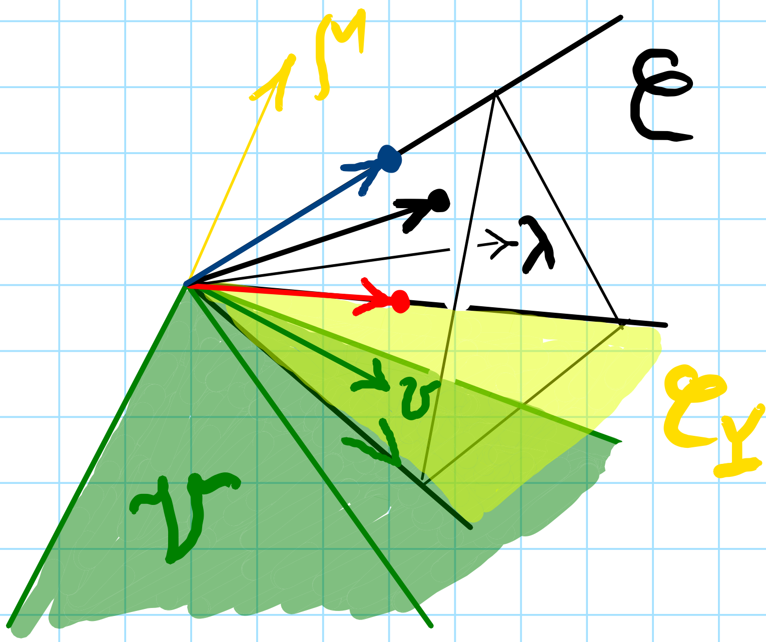
b) $v \in \bigcup_{\mathcal{C} \in \mathcal{F}(X)} \mathcal{C} \iff \exists \lim_{s \to \infty} s^v \in X$



Thm. Let X be sph. G-var. Then:

X affine $\Longleftrightarrow$

- $X = X_Y$ simple
- $\mathcal{C} := \mathrm{cone}(\mathcal{C}_Y, \overline{v_D} \mid D \in \mathcal{D})$ pointed
- $\overline{v_D} \neq 0, \quad \forall D \in \mathcal{D}$
- $\mathcal{C}_Y =$ largest face of $\mathcal{C}$ w. rel. int. $\cap \mathcal{V} \neq \emptyset$
- $\mathcal{D}^Y = \mathcal{D} \cap \mathcal{C}_Y$



Proof: $\Rightarrow$

$$k[X] = \bigoplus_{\lambda} k[X]_{(\lambda)} \ni f_\lambda \text{ ht. wt. vector}$$

$$v_D(f) \geq 0, \quad \forall B\text{-stable } D \subset X$$

$$v_D(f) = \langle \overline{v_D}, \lambda \rangle \Longleftrightarrow \lambda \in \mathcal{C}^\vee \cap \Lambda$$

$$\boxed{\mathcal{C} := \mathrm{cone}(\overline{v_D}, D \subset X \ B\text{-stable})}$$

1) $\exists f \in k[X]$, $f \neq 0$ s.t. $f = 0|_D$, $\forall$ B-stable $D \subset X$,

i.e. $v_D(f) > 0$

Lie-Kolchin thm. $\Longrightarrow$ may assume $f = f_\lambda$

$\Longrightarrow \langle \overline{v_D}, \lambda \rangle > 0$, $\forall D$,

$\Longrightarrow \mathcal{C}$ pointed

2) $v \in \mathcal{C} \cap \mathcal{V}$, $v \neq 0$ $\Longrightarrow v \geq 0$ on $k[X]$

$$k[X] \trianglerighteq \mathcal{I} = \{ f \in k[X] \mid v(f) > 0 \}$$

G-stable prime ideal

$\Longrightarrow \mathcal{I} = \mathcal{I}(Y)$, $Y \subset X$ G-orbit

$$\Longrightarrow \begin{array}{ccc} X_v & \longrightarrow & X \\ \cup & & \cup \\ D_v & \longrightarrow & Y \end{array} \Longrightarrow v \in \mathcal{C}_Y^\circ$$

3) $\exists f \in k[X]$ s.t. $f = 0|_D$, $\forall$ B-stable $D \subset X$
$\quad \notin \mathcal{I}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \not\supset Y$

2nd approx. lemma $\Rightarrow$ may assume $f = f_\mu$

$\Rightarrow \mu \in \mathcal{C}^\vee$

$\langle v, \mu \rangle = 0 \Rightarrow \langle \mathcal{C}_Y, \mu \rangle = 0$

$\langle \overline{v_D}, \mu \rangle > 0$, $\forall D \not\supset Y$

Hence $\mathcal{C}_Y \subset \mathcal{C}$ the face w. rel. int. $\ni v$

$$\mathcal{D}^Y = \mathcal{D} \cap \mathcal{C}_Y$$

4) $\exists$ largest face $\mathcal{C}_{max} \subset \mathcal{C}$ s.t. $\mathcal{C}^\circ_{max} \cap \mathcal{V} \neq \emptyset$

Indeed: $\mathcal{C}', \mathcal{C}'' \subset \mathcal{C}$ faces w. rel. int. $\cap \mathcal{V} \neq \emptyset$

$v' \in (\mathcal{C}')^\circ \cap \mathcal{V}$
$v'' \in (\mathcal{C}'')^\circ \cap \mathcal{V}$ $\Rightarrow v = v' + v'' \in (\mathcal{C}''')^\circ$

for some face $\mathcal{C}''' \subset \mathcal{C}$
$\supset \mathcal{C}', \mathcal{C}''$

Then $\mathcal{C}_{max} = \mathcal{C}_Y$, $Y \subset X$ unique closed orbit

5) $\forall$ G-stable $D \subset X$: $D \supset Y \Rightarrow \overline{v_D} \in \mathcal{C}_Y$

$$\Rightarrow \mathcal{C} = \text{cone}\left(\mathcal{C}_Y, \overline{v_D} \mid D \in \mathcal{D}\right)$$