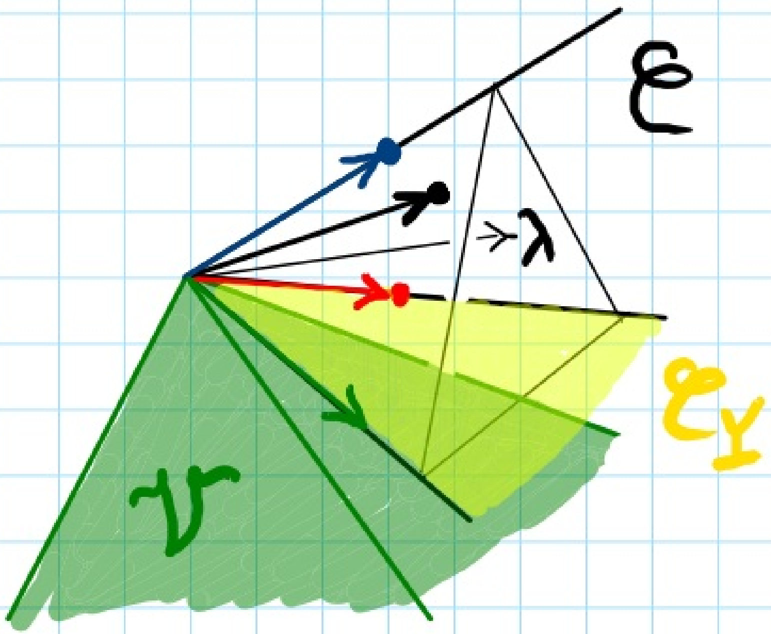
Spherical varieties: Lecture 21

Thm. 1. Let X be sph. G-var. Then:

X affine $\Longleftrightarrow$
- $X = X_Y$ simple
- $\mathcal{C} := \mathrm{cone}(\mathcal{C}_Y, \overline{v_D} \mid D \in \mathcal{D})$ pointed
- $\overline{v_D} \neq 0, \quad \forall D \in \mathcal{D}$
- $\mathcal{C}_Y$ = largest face of $\mathcal{C}$ w. rel. int. $\cap \mathcal{V} \neq \emptyset$
- $\mathcal{D}^Y = \mathcal{D} \cap \mathcal{C}_Y$



Proof: $\Leftarrow$ 1) $k[X] = \bigoplus_{\lambda \in \mathcal{C}^\vee \cap \Lambda} k[X]_{(\lambda)}$ f.g. algebra

gen. by $k[X]_{(\lambda_1)} \oplus \dots \oplus k[X]_{(\lambda_m)}$

$\lambda_1, \dots, \lambda_m$ generaters of $\mathcal{C}^\vee \cap \Lambda$

$\Rightarrow \exists$ affine G-var. X' w. $k[X'] = k[X]$

2) $\operatorname{Frac} k[X] = k(X)$

Suffices to prove: $\operatorname{Frac} k[X] \supset k[Y^\circ]$

$\mathcal{C}$ pointed $\Rightarrow \exists \lambda \in \Lambda : \langle \mathcal{C} \setminus 0, \lambda \rangle > 0$

$\overline{v_D} \neq 0 \Rightarrow \operatorname{ord}_D f_\lambda > 0, \; \forall B\text{-stable } D \subset X$

$f \in k[Y^\circ] \Rightarrow f \cdot f_\lambda^d \in k[X]$ for $d \gg 0$

$\Downarrow$

$f_\lambda \Rightarrow f \in \operatorname{Frac} k[X]$

3) Hence

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \cup & & \cup \\ Y_0 & = & Y_0 \end{array}$$

G-equivar. bir. isomorphism

By $\Rightarrow$ $X' = X'_{Y'}$, $\operatorname{cone}(\mathcal{C}_{Y'}, \overline{v_D} \mid D \in \mathcal{D}) = \mathcal{C}$

because dual cones coincide

$\Rightarrow (\mathcal{C}_{Y'}, \mathcal{D}^{Y'}) = (\mathcal{C}_Y, \mathcal{D}^Y) \Longrightarrow X \simeq X'$ ∎

Algebraic monoids

Alg. monoid = affine alg. variety + monoid M (= semigroup w. unity)

s.t. $M \times M \longrightarrow M$ is a morphism
$(x, y) \longmapsto x \cdot y$

Basic example: $M = \mathrm{Mat}_n(k)$

Fact. $\forall$ alg. monoid $M \xhookrightarrow{\text{closed}} \mathrm{Mat}_n(k)$ for some n

Cf. [Humphreys, LAG, 8.6]

Group of invertibles $G = G(M) := \{g \in M \mid \exists g^{-1} \in M\}$
$= M \cap GL_n$ open in M
alg. grp.

Exercise 1: Prove $\Rightarrow$ ($\subseteq$ obvious)

Assume: M irr. $\Longrightarrow$ G connected, dense in M

$$\underset{\text{left/right mult.}}{G\times G \curvearrowright} M \supset \underset{\text{open orbit}}{G} \simeq (G\times G)/\mathrm{diag}(G)$$

Thm. 2. $\forall$ aff. $(G\times G)$-equivar. $M \overset{\text{open}}{\hookleftarrow} G$ is an alg. monoid s.t. $G(M) \overset{\text{open}}{=} G$

Proof: $(g_1, g_2^{-1})\cdot x =: g_1\cdot x\cdot g_2$, $\forall g_1, g_2 \in G$, $x \in M$

extends mult. on G

In particular:

$$\begin{tikzcd}
G\times M \arrow[drr] & & \\
G\times G \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\cap"] \arrow[r, "\text{mult.}"] & G \subset & M \\
M\times G \arrow[urr] & &
\end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd}
k[G]\otimes k[M] & & \\
k[G]\otimes k[G] \arrow[u, phantom, "\cap"] \arrow[d, phantom, "\cup"] & k[G] \arrow[l] & \supset k[M] \arrow[ull] \arrow[dll] \\
k[M]\otimes k[G] & &
\end{tikzcd}$$

$$\rightsquigarrow \quad k[M] \longrightarrow (k[G] \otimes k[M]) \underset{\substack{\| \\ k[M] \otimes k[M]}}{\cap} (k[M] \otimes k[G])$$

$\rightsquigarrow$ mult. $M \times M \longrightarrow M$ extending $G \times G \longrightarrow G$

$$g \in G(M) \implies \begin{array}{ccc} M & \xrightarrow{\sim} & M \\ x & \longmapsto & g \cdot x \end{array}$$

$$\implies \underset{\uparrow}{g \cdot G} \cap \underset{\nearrow}{G} \neq \emptyset$$

open, $\neq \emptyset$

$$\implies g \cdot g_1 = g_2 \quad \text{for some } g_1, g_2 \in G$$

$$\implies g = g_1^{-1} \cdot g_2 \in G$$

$\square$

Def. Alg. monoid M is *reductive* if:

- M irr., normal
- $G = G(M)$ reductive

$$\boxed{\begin{array}{l}\text{Red. monoids} \overset{\text{Thm. 2}}{=} \text{aff. sph. } (G\times G)\text{-varieties} \\ \qquad\qquad\qquad\qquad \text{w. open orbit } G = G\times G/\operatorname{diag}(G)\end{array}}$$

Borel subgrp. $B^- \times B \subset G \times G$

Wt. lattice $\Lambda = \{(-\lambda, \lambda) \mid \lambda \in \Lambda(T)\} \simeq \Lambda(T)$

Colors: $\mathcal{D} = \{D_1, \dots, D_\ell\}$, $\overline{v_{D_i}} = \alpha_i^\vee$ simple coroots

Val. cone $\mathcal{V} = \{v \in \mathcal{E} \mid \langle v, \alpha_i \rangle \le 0,\ \forall i = 1, \dots, \ell\}$

Thms. 1, 2 $\Longrightarrow$

Thm. 3. (E. Vinberg '1995, A. Rittatore '1998)

$$\left\{\begin{array}{l}\text{Red. monoids} \\ \text{w. } G(M) = G\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{pointed cones} \\ \mathcal{E} = \operatorname{cone}(\alpha_1^\vee, \dots, \alpha_\ell^\vee, \\ \qquad v_1, \dots, v_k) \\ \text{s.t. } v_i \in \mathcal{V}\end{array}\right\}$$

Rmk. $k[M] = \bigoplus\limits_{\lambda \in \mathcal{C}^\vee \cap \Lambda} \underbrace{k[G]_{(\lambda)}}_{\substack{\| \\ \mathrm{Mat}\, V(\lambda)}}$

$$\lambda \in \mathcal{C}^\vee \cap \Lambda \iff \begin{array}{ccc} G & \longrightarrow & GL(V(\lambda)) \\ \cap & & \cap \\ M & \longrightarrow & \mathrm{End}\, V(\lambda) \end{array}$$

$\lambda_1, \dots, \lambda_m$ generators of $\mathcal{C}^\vee \cap \Lambda$

$\Downarrow \quad \cancel{\Uparrow}$

$k[M]$ generated by $\mathrm{Mat}\, V(\lambda_1) \oplus \dots \oplus \mathrm{Mat}\, V(\lambda_m)$

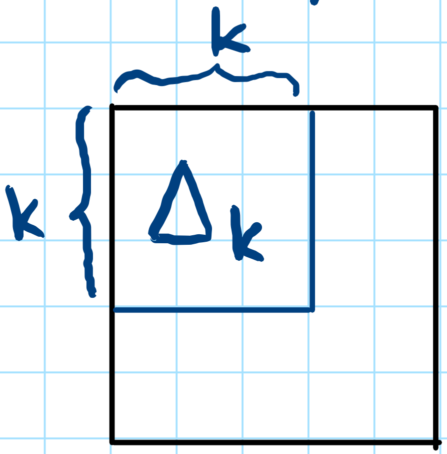
$\Updownarrow$

$\forall \lambda \in \mathcal{C}^\vee \cap \Lambda: \quad V(\lambda) \hookrightarrow V(\lambda_{i_1}) \otimes \dots \otimes V(\lambda_{i_N})$ for some $1 \le i_1, \dots, i_N \le m$

$\Updownarrow$

$M = \overline{G} \subset \mathrm{End}\left[V(\lambda_1) \oplus \dots \oplus V(\lambda_m) \right]$

Example. $M = \mathrm{Mat}_n(k)$, $G = GL_n(k)$



$$\Delta_k \in k[G]^{(B^- \times B)}$$

$$\text{eigenwt.} = (-\varepsilon_1 - \ldots - \varepsilon_k,\ \varepsilon_1 + \ldots + \varepsilon_k)$$

B-stable prime divisors : $D_1, \ldots, D_{n-1}, D_n$

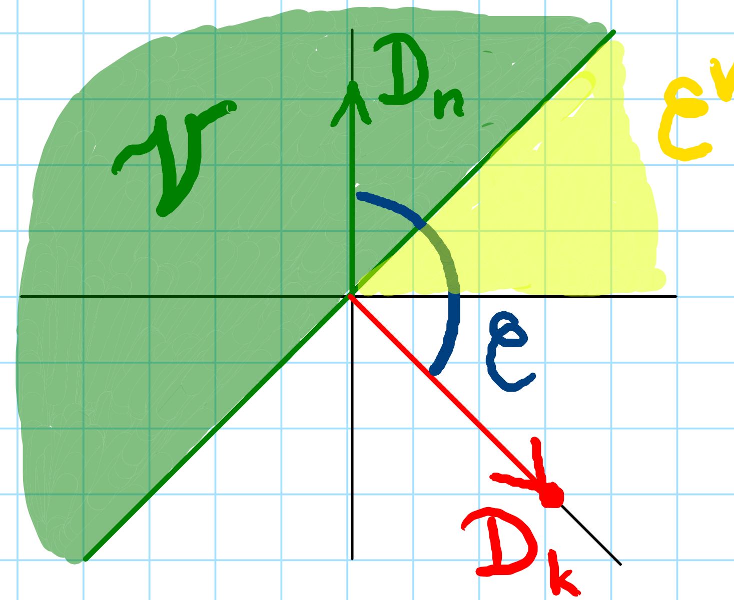
colors $\quad$ $(G\times G)$-stable

$$D_k = \{\Delta_k = 0\}$$

$$\langle \overline{v_{D_k}}, \varepsilon_1 + \ldots + \varepsilon_j \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

$$\Longrightarrow \overline{v_{D_k}} = \varepsilon_k^* - \varepsilon_{k+1}^*, \quad k < n$$

$$\overline{v_{D_n}} = \varepsilon_n^*$$



$$\mathcal{E} = \mathrm{cone}(\varepsilon_1^* - \varepsilon_2^*, \ldots, \varepsilon_{n-1}^* - \varepsilon_n^*, \varepsilon_n^*)$$

$$\mathcal{E}^\vee = \mathrm{cone}(\underbrace{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_1 + \ldots + \varepsilon_n})$$

generate $\mathcal{E}^\vee \cap \Lambda$

$k[M]$ gen. by $\mathrm{Mat}(V(\varepsilon_1)) \oplus \dots \oplus \mathrm{Mat}(V(\varepsilon_1 + \dots + \varepsilon_n))$

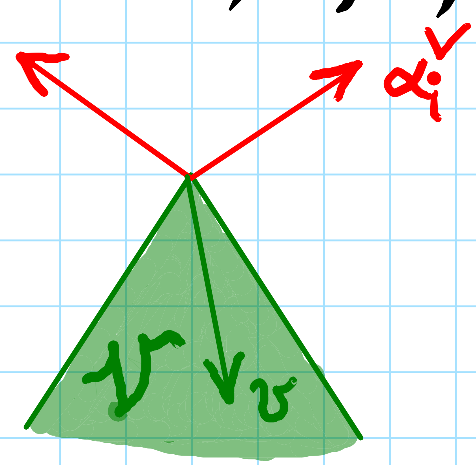
$V(\varepsilon_1) = k^n \ni e_1$ ht. wt. vector

$$V(\varepsilon_1)^{\otimes k} \supset \underset{\substack{\rotatebox{90}{$\in$} \\ e_1 \wedge \dots \wedge e_k}}{\Lambda^k} V(\varepsilon_1) = V(\varepsilon_1 + \dots + \varepsilon_k)$$

$\Longrightarrow$ $k[M]$ generated by $\mathrm{Mat}\, V(\varepsilon_1)$

Cor. The only alg. monoid w. $G(M) = G$ semisimple is $M = G$

Proof: For G s/s $\forall \underset{\neq 0}{v} \in \mathcal{V}$: $\mathrm{cone}(v, \alpha_1^\vee, \dots, \alpha_\ell^\vee) = \mathcal{E}$



∎

Exercise 2: $G = SL_2 \times (k^\times)^2 \curvearrowright V = k^2 \oplus k^2$

$\uparrow$ dilations

$M = \overline{G} \subset \mathrm{End}(V)$

(a) Check normality

Hint: Find ht. weights of the irr. summands in all $V^{\otimes k}$ and check that they form the semigroup of all lattice vectors in a polyhedral cone.

Then $k[M] = k[\text{some normal red. monoid}]$ by Thm. 3 and Rmk. after.

(b) Find $\mathcal{E}$

(c) Describe $(G \times G)$-orbits in M