Spherical varieties: Lecture 22

Local geometry of sph. varieties

X sph. G-var.

$\cup$ Y_0 open G-orbit

Y any G-orbit

$X_Y^\circ \subset X$ open B-chart

Note: X covered by $g \cdot X_Y^\circ$ over all $Y \subset X$, $g \in G$
(may choose finite subcover)

$P = P_Y = \{ g \in G \mid g \cdot X_Y^\circ = X_Y^\circ \} \supset B$ parabolic subgrp. of G

Connected

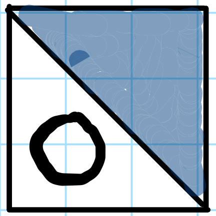
$= \{ g \in G \mid g \cdot D = D, \ \forall B\text{-stable } D \underset{\not\supset Y}{\subset} X \}$

Reference on parabolic subgrps.:

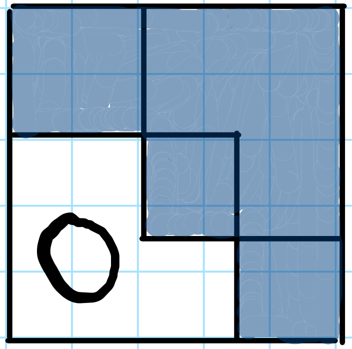
[Humphreys, LAG, § 30]

Levi decomposition: $$\underset{\substack{\text{unipot.}\\ \text{radical}}}{P = P_u} \rtimes \underset{\substack{\text{red.}\\ \text{Levi subgrp.}}}{L}, \quad T \subset L = \underbrace{[L,L]}_{\text{semisimple}} \cdot T$$

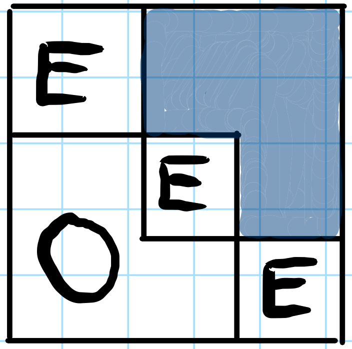
Example: $G = Gl_n$, $B =$



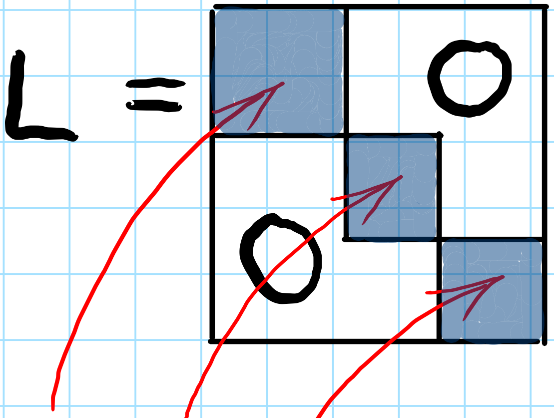
$P =$



, $P_u =$



, $L =$

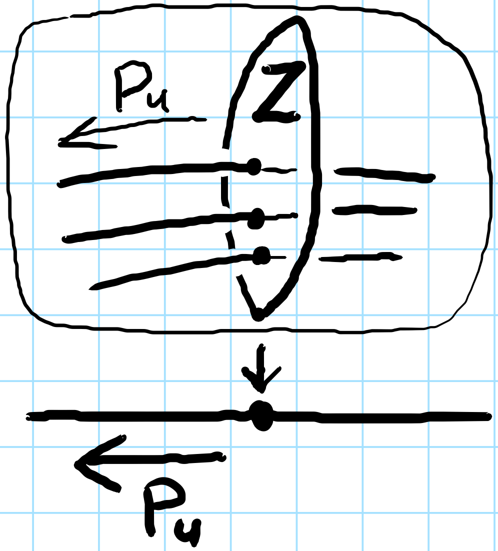


det = 1 : equations for [L,L]

Local Structure Thm. (Brion–Luna–Vust '1986)

∃ closed L-stable $Z \subset X^\circ_Y$ s.t. :

- $X^\circ_Y = P\cdot Z \simeq P \times^L Z \simeq P_u \times \underset{\substack{\uparrow \\ \text{(BLV-) slice}}}{Z}$



$$\begin{tikzcd}
X^\circ_Y \arrow[d, "\varphi"] \\
P/L \simeq P_u
\end{tikzcd}$$

- Z is aff. sph. L-variety w. closed L-orbit $Y \cap Z$
- $\Lambda_Z = \Lambda_X$

  $\mathcal{E}_{Y\cap Z} = \mathcal{E}_Y$

  $\mathcal{D}^{Y\cap Z} \simeq \{D \in \mathcal{D}^Y \mid P.D \neq D\}$

Proof: 1) May assume $X = X_Y$

Then $X \setminus X_Y^\circ = \bigcup_{D \in \mathcal{D} \setminus \mathcal{D}^Y} D$ ← contain no G-orbit

$$\Longrightarrow \delta = m \cdot \sum_{D \in \mathcal{D} \setminus \mathcal{D}^Y} D \quad (m \gg 0) \text{ is Cartier}$$

w. stabilizer P

$$= \operatorname{div}(s), \quad s \in H^0(X, \mathcal{O}(\delta))^{(P)}_\lambda$$

$$\varphi : X \dashrightarrow \mathfrak{g}^*, \quad \langle \varphi(x), \xi \rangle := \frac{\xi \cdot s(x)}{s(x)}, \quad \forall \xi \in \mathfrak{g}$$

- P- equivar. ← Exercise 1: Check this.
- regular on $X_Y^\circ = \{s \neq 0\}$

because G reductive

2) $\mathfrak{g}^* \simeq \mathfrak{g}$ via inv. non-deg. inner prod.:

$$(\xi | \eta) = \operatorname{tr}(\xi \cdot \eta) \quad \text{for } G \hookrightarrow GL_n$$

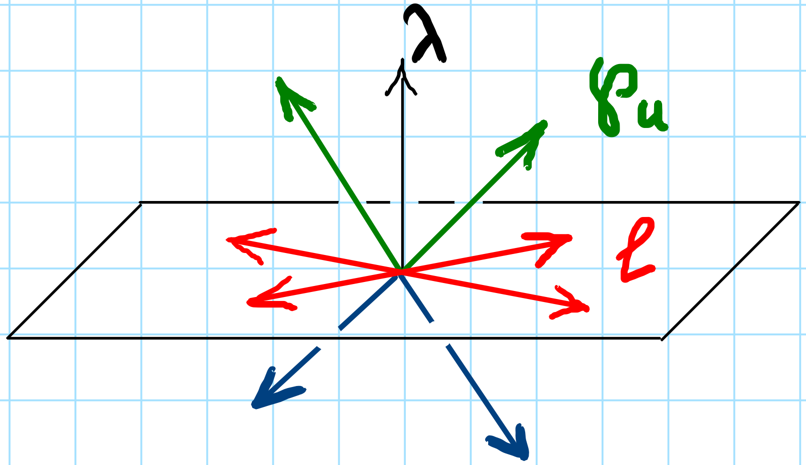
[Onishchik–Vinberg, LG & AG, Chap. 4, §1] $\quad \mathfrak{g} \hookrightarrow \mathfrak{gl}_n$

Properties:

- $(\cdot|\cdot)$ non-deg. on $\mathfrak{t}$
- $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta)=0$ unless $\beta=-\alpha$
- $(\mathfrak{t}|\mathfrak{g}_\alpha)=0$
- $(\cdot|\cdot)$ is $\mathbb{Q}$-valued and pos. definite on $\Lambda(T)\subset\mathfrak{t}^*\simeq\mathfrak{t}$

3) $\lambda\in\mathfrak{t}^*\simeq\mathfrak{t}$

$$\mathfrak{p}=\mathfrak{t}\oplus\bigoplus_{(\alpha|\lambda)\geq 0}\mathfrak{g}_\alpha$$



$$\mathfrak{p}_u=\bigoplus_{(\alpha|\lambda)>0}\mathfrak{g}_\alpha\ ,\quad \mathfrak{l}=\mathfrak{t}\oplus\bigoplus_{(\alpha|\lambda)=0}\mathfrak{g}_\alpha$$

4) $$\xi\cdot s=\begin{cases}\lambda(\xi)\cdot s\ , & \xi\in\mathfrak{t}\\ 0\ , & \xi\in\mathfrak{g}_\alpha\subset\mathfrak{p}\end{cases}\ ,\qquad (\varphi(x)|\xi)=\frac{\xi\cdot s(x)}{s(x)}$$

Hence: $\varphi(x)\in\lambda+\mathfrak{p}_u\ ,\quad\forall x\in X^{\circ}_{\underline{Y}}$

$\varphi : X_y^\circ \longrightarrow \lambda + \mathfrak{p}_u \subset \mathfrak{g} \simeq \mathfrak{g}^*$ P-equivar.

Claim $\lambda + \mathfrak{p}_u = \mathrm{Ad}(P)\cdot\lambda \simeq P/L$

Proof of Claim: $\mathrm{ad}(\lambda) = 0$ on $\mathfrak{l}$

$= (\alpha|\lambda)\cdot \mathrm{id}$ on $\mathfrak{g}_\alpha$

$\Longrightarrow [\mathfrak{l},\lambda] = 0, \quad [\mathfrak{p}_u,\lambda] = \mathfrak{p}_u$

$\Longrightarrow P \underset{\mathrm{Ad}}{\curvearrowright} \lambda + \mathfrak{p}_u, \quad P_\lambda = L$

$\Longrightarrow \mathrm{Ad}(P)\cdot\lambda = \mathrm{Ad}(P_u)\cdot\lambda \simeq P_u \xleftarrow[\sim]{\exp} \mathfrak{p}_u$

$\subseteq \lambda + \mathfrak{p}_u$

$\Longrightarrow \mathrm{Ad}(P)\cdot\lambda = \lambda + \mathfrak{p}_u$

Because $\mathbb{A}^n \cancel{\xhookrightarrow[\text{open}]{}} \mathbb{A}^n$

Thus $\varphi : X^\circ_Y \longrightarrow \lambda + \mathfrak{p}_u \simeq P/L$

$\Rightarrow X^\circ_Y \simeq P \times^L Z, \quad Z = \varphi^{-1}(\lambda)$

5) $X^\circ_Y \supset Y^\circ$ open B-orbit, $\quad B = P_u \rtimes (B \cap L)$

$\Rightarrow Z \supset Y^\circ \cap Z$ open $(B \cap L)$-orbit

$\Rightarrow Z$ spherical for L

6) $f \in k(X)^{(B)}_\lambda \Rightarrow f' = f|_Z \in k(Z)^{(B \cap L)}_\lambda$

Conversely: $f' \in k(Z)^{(B \cap L)} \rightsquigarrow f \in k(X)^{(B)}$

$f(u \cdot z) = f'(z)$

$\forall u \in P_u, \; z \in Z$

Hence: $\Lambda_Z = \Lambda_X$

7) $D \subset X$ B-stable, $\underbrace{D \cap X_Y^\circ}_{\substack{\parallel \\ P_u \times \underbrace{(D \cap Z)}_{\substack{\parallel \\ D' \subset Z \ (B \cap L)\text{-stable}}}}} \neq \emptyset \iff D \supset Y$

Conversely: $D' \subset Z$ $(B \cap L)$-stable

$$\Rightarrow D = \overline{P_u \cdot D'} \subset X \quad \text{B-stable}$$

$$\overline{v_D} = \overline{v_{D'}} \in \Lambda^*$$

$$\Rightarrow D \supset Y \implies D' \supset Y \cap Z$$

Hence: • $Y \cap Z \subset Z$ unique closed L-orbit

- $$\mathcal{E}_{Y\cap Z} = \mathrm{cone}\left(\underset{\|}{\overline{v_{D'}}} \;\middle|\; D' \subset Z \ (B\cap L)\text{-stable} \right)$$
  $$= \mathrm{cone}\left(\overline{v_D} \;\middle|\; D \underset{\supset Y}{\subset X} \ B\text{-stable} \right) = \mathcal{E}_Y$$

- $D' \in \mathcal{D}^{Y\cap Z} \iff L\cdot D' \neq D' \iff P\cdot D \neq D$

∎