Spherical varieties: Lecture 23

Local Structure Thm.

Let X a sph. G-var., $Y \subset X$ a G-orbit,

$$P = \{ g \in G \mid g \cdot X_Y^\circ = X_Y^\circ \} = P_u \rtimes L.$$

Then $\exists$ closed L-stable $Z \subset X_Y^\circ$ s.t.:

- $X_Y^\circ = P \cdot Z \simeq P \times^L Z \simeq P_u \times Z$
- Z is aff. sph. L-variety w. closed L-orbit $Y \cap Z$
- $\Lambda_Z = \Lambda_X$

  $\mathcal{C}_{Y \cap Z} = \mathcal{C}_Y$

  $\mathcal{D}^{Y \cap Z} \simeq \{ D \in \mathcal{D}^Y \mid P.D \neq D \}$

Locally: sph. var. $\simeq$ aff. space $\times$ aff. sph. var.

Def. A sph. var. X is *toroidal* (or *colorless*) if any color $D \not\supset$ a G-orbit, i.e.:

$$\mathcal{D}^Y = \emptyset, \ \forall G\text{-orbit } Y \subset X$$

In LST: X toroidal $\Longrightarrow Z$ toroidal, affine

$\Longrightarrow k[Z]^{(B\cap L)} = k[Z]^{(L)}$

$\quad \psi$

$\quad f \Longrightarrow \mathrm{div}(f)$ $B\cap L$-stable

$\qquad \Longrightarrow L$-stable

$\Longrightarrow [L,L] \curvearrowright k[Z]$

$\Longrightarrow L \curvearrowright Z \rightsquigarrow \underbrace{L/_{[L,L]}}_{\text{torus}} \curvearrowright Z$

$\Longrightarrow Z$ toric

Locally: toroidal var. $\simeq$ aff. space $\times$ aff. toric var.

$\simeq$ aff. toric var.

Thm. 1. Toroidal var. X smooth $\iff$

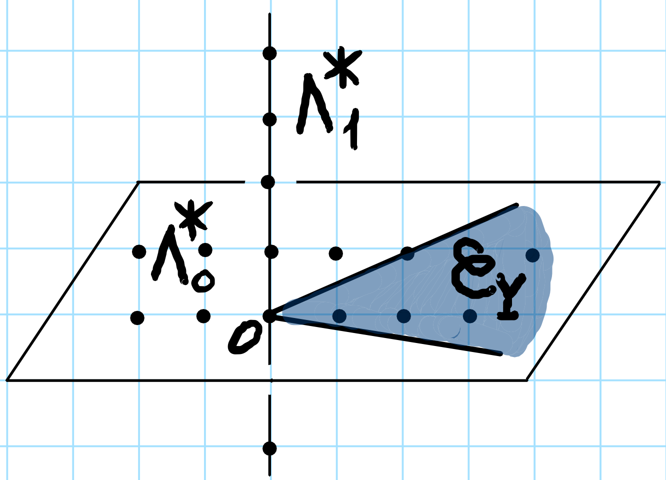
$\forall Y \subset X$: $\mathcal{C}_Y$ strictly simplicial, i.e.

$$\mathcal{C}_Y = \operatorname{cone}(\underbrace{v_1, \dots, v_m}_{\text{part of basis of } \Lambda^*})$$

Proof: By LST may assume: $X = X_Y = Z$

aff. toric var.

$$\langle \mathcal{C}_Y \rangle_{\mathbb{Q}} \cap \Lambda^* =: \Lambda_0^* \implies \Lambda^* = \Lambda_0^* \oplus \Lambda_1^* \quad \text{can choose}$$



$$\Lambda = \Lambda_0 \oplus \Lambda_1$$

$$T = T_0 \times T_1$$

$$X = X_0 \times T_1$$

$$k[X] = \bigoplus_{\lambda \in \mathcal{C}^\vee \cap \Lambda} k \cdot f_\lambda = \underbrace{\bigoplus_{\lambda \in \mathcal{C}^\vee \cap \Lambda_0} k \cdot f_\lambda}_{k[X_0]} \otimes \underbrace{\bigoplus_{\lambda \in \Lambda_1} k \cdot f_\lambda}_{k[T_1]}$$

$\implies$ may assume $\mathcal{C} = \mathcal{C}_Y$ solid

$\implies$ $\mathcal{C}^\vee$ pointed

$$k[X] = \bigoplus_{\lambda \in \mathcal{C}^\vee \cap \Lambda} k\cdot f_\lambda = k[f_{\lambda_1},\dots,f_{\lambda_m}]$$

$\lambda_1,\dots,\lambda_m \in \mathcal{C}^\vee \cap \Lambda$

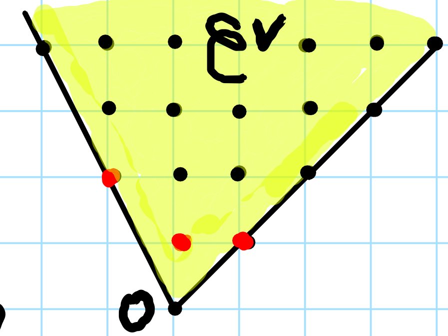
<u>all</u> indecomp. vectors

$$\bigtriangledown$$

$$\mathcal{I}(Y) = \bigoplus_{\lambda \neq 0} k\cdot f_\lambda$$

$$= (f_{\lambda_1},\dots,f_{\lambda_m})$$

T-stable maximal ideal



$$Y = \{y\}$$

$$\Rightarrow \quad T_y^* X = \mathcal{I}(Y)/\mathcal{I}(Y)^2 = \langle d_y f_{\lambda_1},\dots,d_y f_{\lambda_m}\rangle$$

basis

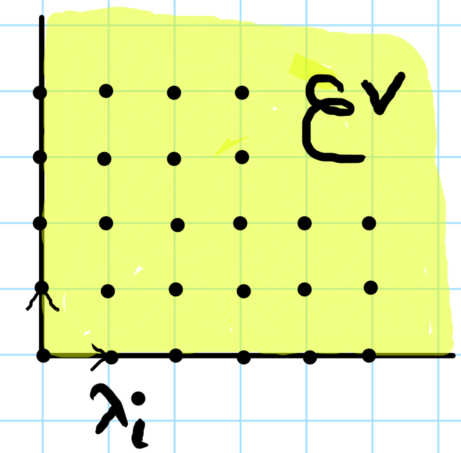
$$X \text{ smooth} \Rightarrow \underset{\displaystyle m}{\underset{\shortparallel}{\dim T_y X}} = \underset{\displaystyle \operatorname{rk}\Lambda}{\underset{\shortparallel}{\dim X}}$$

$\Rightarrow \{\lambda_1,\dots,\lambda_m\}$ basis of Λ

$\Rightarrow \mathcal{C}^\vee$ strictly simplicial $\Rightarrow \mathcal{C}$, too

$\mathcal{C}$ strictly simplicial $\Longrightarrow$ $\mathcal{C}^\vee$, too

Choose coordinates $(t_1,\dots,t_m)$ on T : $t_i = t^{\lambda_i}$



$$\Longrightarrow k[X] = k[t_1,\dots,t_m]$$

$$\Longrightarrow X \simeq \mathbb{A}^m$$

$\square$

Thm. 2. (M. Brion – F. Pauer '1987)

X sph. G-var. $\Longrightarrow$ $\exists$ G-equivar. proper bir. map

$$\begin{array}{ccc} X & \longleftarrow & X' \text{ toroidal} \\ \cup & & \cup \\ Y_0 & = & Y_0 \end{array}$$

Proof: $\delta = m\cdot\sum_{D\in\mathcal{D}} D$ Cartier on Y_0

$$\rightsquigarrow s_\delta \in H^0(Y_0, \mathcal{O}(\delta))^{(B)}$$

$$\rightsquigarrow M = \langle G\cdot s_\delta\rangle_k \subset H^0(Y_0, \mathcal{O}(\delta))$$

simple G-submodule

$$\rightsquigarrow \varphi:\ \begin{tikzcd}
Y_0 \arrow[rr] \arrow[d, phantom, sloped, "\subset"] \arrow[drr] & & \mathbb{P}(M^*) \\
X \arrow[urr, dashed] \arrow[rr, dashed, "\Phi = \mathrm{id}_X\times\varphi"'] & & X\times\mathbb{P}(M^*) \arrow[u, "\text{proj.}"'] \\
& & \overline{\mathrm{Im}\,\Phi} \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

$$\begin{tikzcd}
& X' \arrow[dl, "\text{proper bir.}"'] \arrow[dr, "\varphi'"] \arrow[r, "\text{normaliz.}"] & \overline{\mathrm{Im}\,\Phi} \\
X & & \mathbb{P}(M^*)
\end{tikzcd}$$

φ' extension of φ

$\Longrightarrow \mathcal{O}(\delta)$ extends to $\mathcal{L}\to X'$, $M\subset H^0(X',\mathcal{L})$, φ' reg.

$\Longrightarrow \forall x \in X' \ \exists g \in G : \ g \cdot s_\delta(x) \neq 0$

$\Longrightarrow \ s_\delta(g^{-1} x) \neq 0$

Hence $\{s_\delta = 0\} \not\supset G \cdot x \quad \Longrightarrow X'$ toroidal

$$= \bigcup_{D \in \mathcal{D}} \overline{D}$$

■

Cor. $\mathcal{V}$ is a polyhedral cone

Proof: Choose complete $X \supset Y_0$

By Thm. 2 may assume X toroidal

By completeness criterion: $\mathcal{V} = \bigcup_{Y \subset X} \mathcal{E}_Y, \ \mathcal{E}_Y \subset \mathcal{V}$

polyhedral cones

$\Longrightarrow \mathcal{V}$ polyhedral

■

Rmk. In fact, $\mathcal{V}$ is cosimplicial

$=$ fund. domain for finite refl. grp.

$W_X \subset GL(\Lambda_X^*)$ (M. Brion, F. Knop)

little Weyl grp.

Divisors and line bundles on sph. varieties

Let: X sph. G-var., $D_1, \dots, D_s \subset X$ the B-stable prime divisors

$$v_i = \overline{v_{D_i}} \in \Lambda^*$$

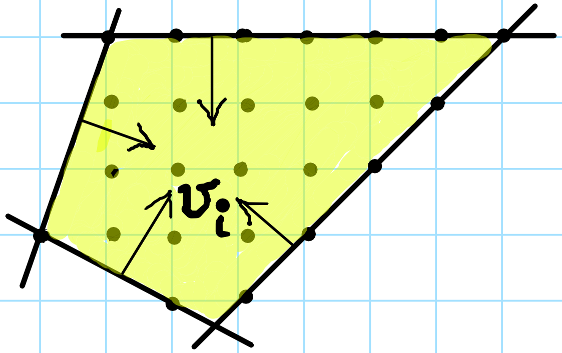
$\mathcal{L} \longrightarrow X$ G-line bundle, $s_0 \in H^0(X, \mathcal{L})^{(B)}_{\lambda_0}$

$$\delta = \mathrm{div}(s_0) = \sum_i m_i \cdot D_i$$

$$\mathcal{P} = \mathcal{P}(\delta) := \{\lambda \in \mathcal{E}^* = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle v_i, \lambda \rangle \geq -m_i, \ \forall i = 1, \dots, s\}$$

polyhedral domain

$\mathcal{P}$:



Prop. (Brion' 1989) $H^0(X,\mathcal{L}) \overset{\text{as G-module}}{\simeq} \bigoplus\limits_{\lambda\in\mathcal{P}\cap\Lambda} V(\lambda+\lambda_0)$

Proof: $G \circlearrowright H^0(X,\mathcal{L})$ mult. free

$\Longrightarrow$ suffices to find ht. wts.

$s\in H^0(X,\mathcal{L})^{(B)}_{\mu}$ ht. wt. vector $\Longrightarrow$ $s = f_\lambda \cdot s_0$

$\mu = \lambda+\lambda_0$

$$\operatorname{div}(s) = \operatorname{div}(f_\lambda)+\operatorname{div}(s_0) = \sum_i \langle v_i,\lambda\rangle\cdot D_i + \delta \geq 0$$

$$\Longleftrightarrow \langle v_i,\lambda\rangle + m_i \geq 0,\ \forall i \Longleftrightarrow \lambda\in\mathcal{P}$$

Conversely, any $\lambda\in\mathcal{P}\cap\Lambda$ corr. to $s\in H^0(X,\mathcal{L})^{(B)}_{\lambda+\lambda_0}$

$\square$

Exercise 1: $G = SO_n \circlearrowright S^k(\mathbb{R}^n)$: decompose into irr. reps.

Hint: $SO_n \circlearrowright \mathbb{P}^{n-1}$ spherical