Spherical varieties: Lecture 24

Tensor product decomposition

G connected red. grp.

$G \circlearrowright V(\lambda)\otimes V(\mu)$: decompose into irr. G-modules

Approach: $V(\lambda) = H^0(G/P, \mathcal{L})$

$V(\mu) = H^0(G/Q, \mathcal{M})$

$X = G/P \times G/Q$ double flag variety

$\uparrow$

$\mathcal{N} = \mathcal{L} \boxtimes \mathcal{M}$, $\mathcal{N}_{(x,y)} = \mathcal{L}_x \otimes \mathcal{M}_y$

$H^0(X, \mathcal{N}) = V(\lambda)\otimes V(\mu)$

Ψ

$s_0 = s_\lambda \otimes s_\mu$ ht. wt. vector of ht. wt. $\lambda+\mu$

If $G \circlearrowright X$ spherical, then

$$V(\lambda) \otimes V(\mu) \simeq \bigoplus_{\nu = \lambda + \mu + (\mathcal{P} \cap \Lambda)} V(\nu)$$

polyhedral domain determined by $\operatorname{div}(s_0)$

Example: $GL_n \circlearrowright S^m(k^n)^* \simeq V(-m \cdot \varepsilon_n)$

$$= H^0(\mathbb{P}^{n-1}, \mathcal{O}(m))$$

$S^m(k^n)^* \ni x_n^m$ ht. wt. vector

$$S^k(k^n)^* \otimes S^l(k^n)^* = H^0(\underbrace{\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}}_{X \ni (x,y)}, \underbrace{\mathcal{O}(k) \boxtimes \mathcal{O}(\ell)}_{\mathcal{N}})$$

$$\ni s_0 = x_n^k \cdot y_n^\ell$$

ht. wt. $= -(k+\ell) \cdot \varepsilon_n$

B-stable prime divisors:

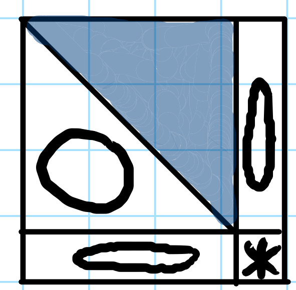
$$D_1 = \{x_n = 0\}, \quad D_2 = \{y_n = 0\}$$

$$D_3 = \left\{ \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} =: \Delta(x,y) = 0 \right\}$$

$$\mathrm{div}(s_0) = k \cdot D_1 + \ell \cdot D_2$$

$$X \setminus (D_1 \cup D_2 \cup D_3) = Y^0 \quad \text{single } B\text{-orbit}$$

$$x = (x_1 : \cdots : x_n) \underset{x_n \neq 0}{\overset{B}{\rightsquigarrow}} (0 : \cdots : 0 : 1) =: x_0$$

$B_{x_0} =$ 

$$y = (y_1 : \cdots : y_{n-1} : y_n) \underset{y_n \neq 0}{\overset{B_{x_0}}{\rightsquigarrow}} (0 : \ldots : 0 : 1 : 1) =: y_0$$

$$\Delta \neq 0 \Rightarrow y_{n-1} \neq 0$$

Hence X spherical

$$k(X)^{(B)} \ni f_\nu(x,y) = x_n^{k_1} \cdot y_n^{k_2} \cdot \Delta(x,y)^{k_3}$$

$$\bigcup \qquad \operatorname{div}(f_\nu) = k_1 \cdot D_1 + k_2 \cdot D_2 + k_3 \cdot D_3$$

$$k_1 + k_3 = k_2 + k_3 = 0$$

$$\frac{x_n \cdot y_n}{\Delta} = f_{\varepsilon_{n-1} - \varepsilon_n}$$

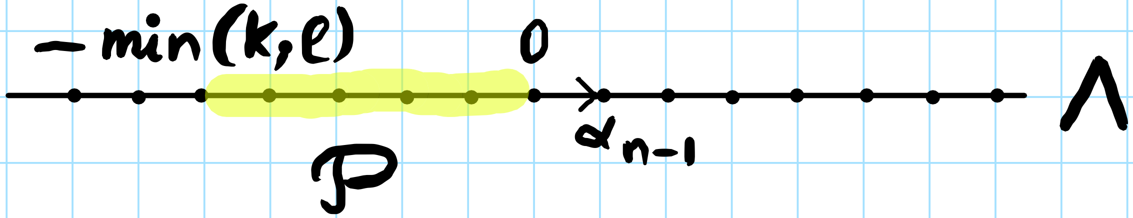
eigenvcts. of x_n, y_n, Δ are $-\varepsilon_n$, $-\varepsilon_n$, $-\varepsilon_{n-1} - \varepsilon_n$

$$\Longrightarrow \Lambda = \mathbb{Z} \cdot \underbrace{(\varepsilon_{n-1} - \varepsilon_n)}_{\alpha_{n-1}}$$

$$v_i \in \overline{v_{D_i}} \in \Lambda^* = \mathbb{Z} \cdot \alpha_{n-1}^*$$

$$v_1 = v_2 = \alpha_{n-1}^*, \quad v_3 = -\alpha_{n-1}^*$$

$$\mathcal{P} = \left\{ \omega = \underset{\mathbb{Q}}{\underset{\cap}{m}} \cdot \alpha_{n-1} \;\middle|\; \begin{array}{l} \langle v_1, \omega \rangle = m \geq -k \\ \langle v_2, \omega \rangle = m \geq -\ell \\ \langle v_3, \omega \rangle = -m \geq 0 \end{array} \right\}$$



Conclude:

$$S^k(\Bbbk^n)^* \otimes S^\ell(\Bbbk^n)^* \simeq \bigoplus V(\nu)$$

$$\begin{aligned} \nu &= -k\cdot\varepsilon_n - \ell\cdot\varepsilon_n - m\cdot\alpha_{n-1} \\ &= -m\cdot\varepsilon_{n-1} + (m-k-\ell)\cdot\varepsilon_n \end{aligned}$$

$$0 \leq m \leq \min(k,\ell)$$

Exercise 1: Decompose into irr. reps.:

(a) $Sp_{2n} \curvearrowright S^k(\Bbbk^{2n}) \otimes S^\ell(\Bbbk^{2n})$

(b) $GL_n \curvearrowright \Lambda^k(\Bbbk^n) \otimes \Lambda^\ell(\Bbbk^n)$

(Use sph. varieties)

Spherical double flag varieties :

Classification	Tensor product decompositions
P. Littelmann' 1994 ←	$P, Q \subset G$ maximal parabolics
J. Stembridge' 2003	E. Ponomareva' 2015-17

Exam on the course : June 18, 14:00 in VooV meeting

1) Homework (50% of final grade)
   Send solutions to e-mail by June 15
2) Theoretical part (oral, w. preparation) on June 18