Spherical varieties: Lecture 2

Prerequisites: Basic knowledge of:

- Alg. geometry
- Alg. groups and their representations

References:

- I. Shafarevich. Basic alg. geometry
- R. Hartshorne. Alg. geometry. Chap. 1.
- J. Humphreys. Linear alg. groups.
- T. Springer. — " — " — " — " —

Digest

Alg. geometry: ground field k alg. closed, char $k = 0$

May assume $k = \mathbb{C}$

$\mathbb{A}^n$ = affine space of dim $= n$

$\mathbb{P}^n$ = projective space of dim $= n$

Affine variety $X \subset \mathbb{A}^n$ given by equations

$$f_1 = \ldots = f_m = 0$$

where $f_i \in k[x_1, \ldots, x_n]$

Zariski topology: closed subsets = aff. subvarieties

$k = \mathbb{C}$: classical **Hausdorff topology** induced from $\mathbb{A}^n = \mathbb{C}^n$

Topological terms refer to Zariski topology

Regular functions on open $U \subset X$:

$f : U \to k$ s.t. locally $f = \frac{p}{q}$, $p, q \in k[x_1, \ldots, x_n]$

Structure sheaf: $\mathcal{O}_X(U) = \{\text{reg. } f : U \to k\}$

General alg. variety = top. space X w. sheaf of functions

$\mathcal{O} = \mathcal{O}_X$: open $U \subset X$ $\rightsquigarrow$ $\mathcal{O}(U)$

s.t. $X = U_1 \cup \ldots \cup U_s$

U_i open, $(U_i, \mathcal{O}|_{U_i}) \simeq$ aff. variety

Example: $\mathbb{P}^n = \mathbb{A}^n_0 \cup \mathbb{A}^n_1 \cup \ldots \cup \mathbb{A}^n_n$

projective varieties = closed subsets in $\mathbb{P}^n$

X **quasiaffine** if X open in some aff. variety

quasiproj. if X open in some proj. var.

Exercise 1: $X = \mathbb{A}^n \setminus pt$ quasiaff. but not affine

Notation: $\mathcal{O}(X) =: k[X]$ used for (quasi)affine X

X irreducible if $X \neq Z_1 \cup Z_2$, $\underset{\text{closed}}{Z_i} \subsetneqq X$

In this case: rational function field $\Bbbk(X) =$

$= \{ f \text{ reg. on some open } U \subset X \}$

In general: $\exists!$ irr. decomposition $X = X_1 \cup \ldots \cup X_k$

X_i closed, irr., $X_i \not\subseteq X_j$

Morphism of alg. varieties $\varphi : X \longrightarrow Y$ continuous map

s.t. $\mathcal{O}_X(\pi^{-1}(U)) \xleftarrow{\varphi^*} \mathcal{O}_Y(U)$

$Y = \mathbb{A}^n \Rightarrow \varphi = (f_1, \ldots, f_n)$, $f_i \in \mathcal{O}(X)$

Alg. groups

Alg. group = group + aff. alg. variety G

s.t. $G \times G \longrightarrow G, \quad (g,h) \mapsto g \cdot h$

$G \longrightarrow G, \quad g \mapsto g^{-1}$

are morphisms

~~Abelian variety~~

$k = \mathbb{C}$: G is a Lie group$/\mathbb{C}$

Basic example: $G = GL_n(k) \underset{\text{open}}{\subset} Mat_n(k) \simeq \mathbb{A}^{n^2}$

$\underset{\text{closed}}{\hookrightarrow} \mathbb{A}^{n^2+1}$

$g \longmapsto \left(g, \frac{1}{\det g}\right)$

Fact: $\forall$ alg. group $G \underset{\text{closed}}{\hookrightarrow} GL_n(k)$

Alg. groups = linear alg. groups

Homomorphism of alg. groups = group homomorphism + morphism of alg. vars.

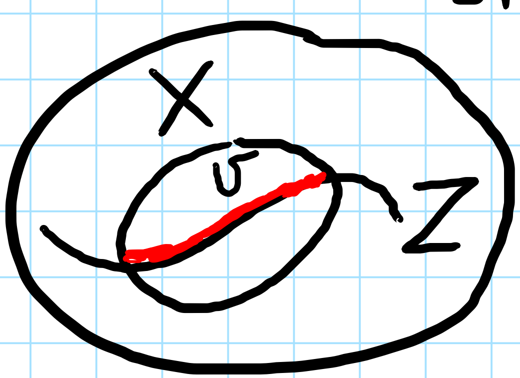
Representation (rational) = homomorphism $G \longrightarrow GL(V)$ of alg. varieties

Alg. group action $G \curvearrowright X$ group action s.t.

$$G \times X \longrightarrow X \quad \text{is morphism}$$
$$(g, x) \longmapsto g \cdot x$$

Thm. $\forall x \in X$: orbit $G \cdot x \subseteq X$ loc. closed subvar.

stabilizer $G_x \subseteq G$ closed subgroup

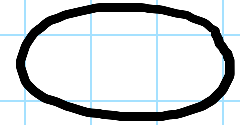


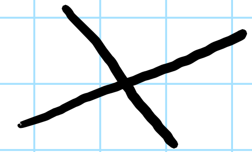
open in closed

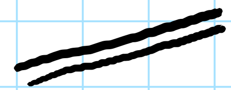
$$Z = \overline{G \cdot x} \subset X$$

$G \cdot x$ is open in Z

Example: $GL_3 \curvearrowright \mathbb{P}(\mathrm{Sym}_3(k)) = \mathbb{P}^5$

Orbits: $\mathrm{rk}\, q = 3$  $\mathcal{O}_3$

$\mathrm{rk}\, q = 2$  $\mathcal{O}_2$

$\mathrm{rk}\, q = 1$  $\mathcal{O}_1$

$\mathcal{O}_1$ closed, $\overline{\mathcal{O}_2} = \mathcal{O}_2 \cup \mathcal{O}_1$, $\mathcal{O}_3$ open in $\mathbb{P}^5$

$\overline{\mathcal{O}_3} = \mathbb{P}^5$

$G \curvearrowright X$

alg. variety — morphism

Geometric quotient $\quad Y = X/G \xleftarrow{\ \pi\ } X$

- $\forall y \in Y$: $\pi^{-1}(y)$ = single G-orbit
- U open in $Y \iff \pi^{-1}(U)$ open in X
- $f \in \mathcal{O}_Y(U) \iff \pi^* f \in \mathcal{O}_X(\pi^{-1}(U))^G$

does not always exist

Thm. $H \subset G$ closed subgrp. Then:

- $\exists$ geom. quot. $Y = G/H$ for $H \underset{\text{right}}{\curvearrowright} G$, $g \overset{h}{\longmapsto} g \cdot h^{-1}$
- $\exists$ rep. $G \curvearrowright V \ni v$ s.t. $H = G_{[v]}$, $\underset{\text{quasiproj.}}{Y \simeq G\cdot[v] \subseteq \mathbb{P}(V)}$
- $G \curvearrowright X$ transitive, $H = G_x \Rightarrow X \simeq Y$

Chevalley's thm.

$$H \curvearrowright Z \rightsquigarrow H \curvearrowright G\times Z, \quad (g,z) \xmapsto{h} (gh^{-1}, h\cdot z)$$

Thm. Z quasiproj. $\Rightarrow \exists\, (G\times Z)/H =: G\times^H Z$ ← homogen. fiber bundle

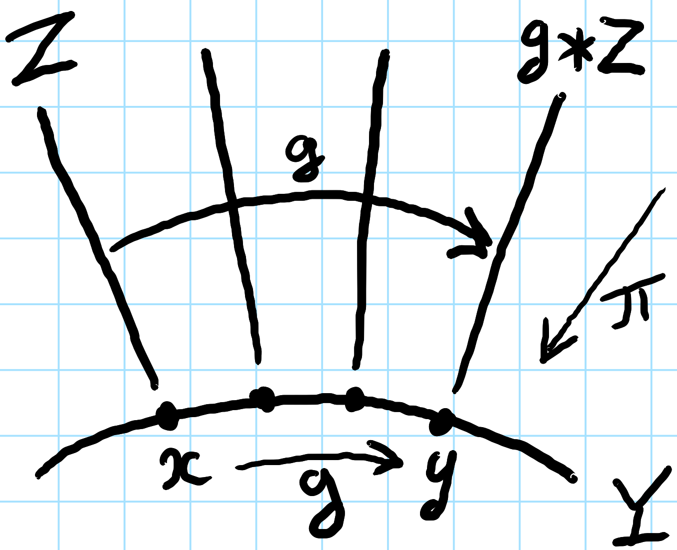
$$\rotatebox{90}{$\in$}$$

$$g*z \mapsfrom (g,z)$$

<u>Properties:</u>

- $\pi : G\times^H Z \longrightarrow Y = G/H$, $\quad g*z \longmapsto y = gH$ is morphism
- $G \curvearrowright G\times^H Z$, $\quad g\cdot(g'*z) = gg'*z$
- $x = eH \Rightarrow \pi^{-1}(x) = e*Z \simeq Z$

$$\pi^{-1}(y) = g*Z = g\cdot\pi^{-1}(x) \simeq Z$$

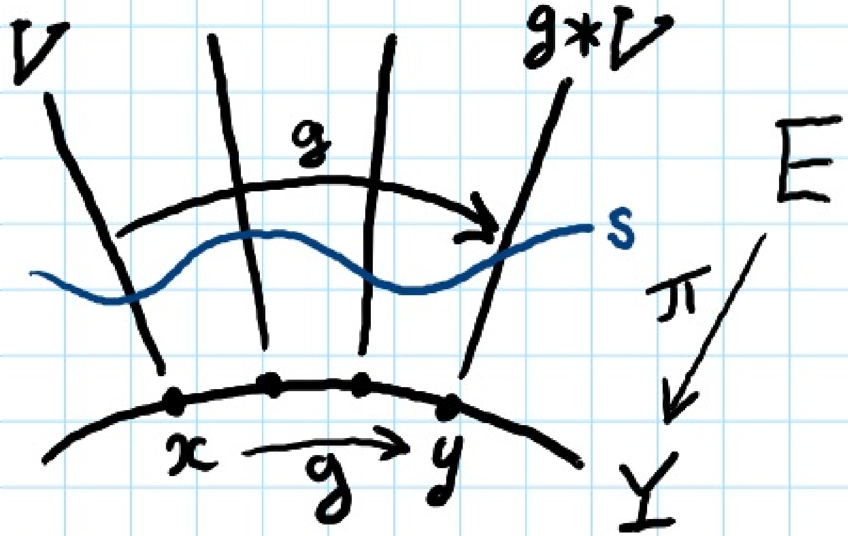


Conversely: $G \curvearrowright X \xrightarrow[\text{equivar.}]{\pi} Y$,

$$\pi^{-1}(x) \simeq Z \Longrightarrow X \simeq G\times^H Z$$

A. Białynicki-Birula. Ann. Inst. Fourier 43 (1993), no. 2, 365–368.

In particular: $H \curvearrowright V$ lin. rep. $\rightsquigarrow E = G \times^H V$



homogen. vector bundle

$$G \curvearrowright H^0(Y, E) =: \mathrm{Ind}_H^G(V)$$

induced representation

Lemma. $\mathrm{Ind}_H^G(V) \simeq \left(k[G] \otimes V\right)^H$, where $H \curvearrowright k[G]$, $(h \cdot f)(g) = f(g \cdot h)$

Proof:

$$\begin{tikzcd} E \arrow[d, shift right] & E \times_Y G \arrow[l] \arrow[d, shift right] & \simeq G \times V \\ Y \arrow[u, shift right, "s"'] & G \arrow[l, "\varphi"'] \arrow[u, shift right, "\varphi^* s"'] \arrow[ur, "f"'] & \end{tikzcd}$$

$Y \simeq G/H$

$x = eH \in Y$

$y = g \cdot x \longleftarrow\!\shortmid\; g$

$$\varphi^* s\,(g) = (\underbrace{s(g\cdot x)}_{\in\, g*V}, g)$$

$$f(g) = g^{-1} \cdot s(g \cdot x) \in V$$

$$f(g \cdot h^{-1}) = h \cdot f(g)$$

Conversely: $f \in \left(k[G] \otimes V\right)^H \rightsquigarrow s \in H^0(Y, E)$, $s(g \cdot x) = g * f(g)$

Homework	Exam	Final grade
50%	50 %	100 %