Spherical varieties: Lecture 3

Recall: G alg. grp., $H \subset G$ closed subgrp., $Y = G/H$

$H \curvearrowright V$ lin. rep. $\rightsquigarrow E = G \times^H V \xrightarrow{\pi} Y$

$$\mathrm{Ind}_H^G(V) := H^0(Y, E)$$
$$\simeq (k[G] \otimes V)^H$$

Exercise 1: $H \curvearrowright V$ comes from $G \curvearrowright V$

$$\Longrightarrow \quad E \simeq Y \times V$$
$$\mathrm{Ind}_H^G(V) = k[G/H] \otimes V$$

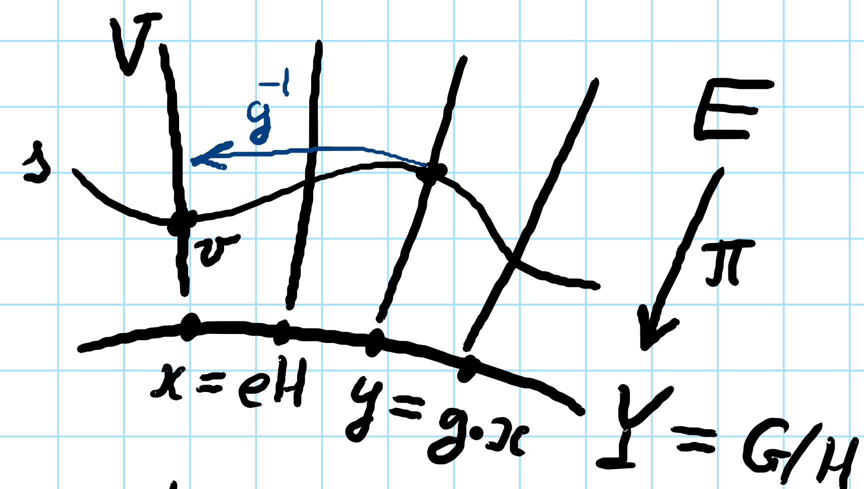
Exercise 2: $\left[\mathrm{Ind}_H^G(V)\right]^G \simeq V^H$

Frobenius reciprocity: $\mathrm{Hom}_G(W, \mathrm{Ind}_H^G(V)) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G W, V)$

Proof:

$$\begin{tikzcd} \varphi : W \arrow[r, "G"'] \arrow[d, rightsquigarrow] & \mathrm{Ind}_H^G(V) = H^0(Y, E) \\ \psi : W \arrow[r, "H"'] & V \end{tikzcd}$$

$$\varphi(w) = s$$



$$\psi(w) = s(x) = v$$

Recover s from v: $g^{-1}\cdot s(g\cdot x) = (g^{-1}\cdot s)(x)$

$$= \varphi(g^{-1} w)(x) = \psi(g^{-1} w)$$

$$s(g\cdot x) = g * \psi(g^{-1}\cdot w)$$

∎

Particular case:

1-dim rep. $H \circlearrowright k_\chi$ $\longleftrightarrow$ homogen. line bundle

$\chi : H \xrightarrow[\text{hom.}]{} k^\times$ character $\qquad \mathcal{L}_\chi = \mathcal{L} = G \times^H k_\chi$

$$h \cdot z = \chi(h) \cdot z$$

$$\mathrm{Ind}_H^G(k_\chi) = H^0(G/H, \mathcal{L}_\chi) \simeq (k[G] \otimes k_\chi)^H \simeq$$

$$\simeq k[G]^{(H)}_{-\chi} := \{ f \in k[G] \mid f(g \cdot h) = \chi(h)^{-1} \cdot f(g),\ \forall g \in G, h \in H \}$$

Notation: $H \circlearrowright V$ lin. rep.

eigenvector of weight χ, semiinvariant vector

$$V^{(H)}_\chi := \{ v \in V \mid h \cdot v = \chi(h) \cdot v,\ \forall h \in H \}$$

eigenspace of weight χ, weight subspace

Reductive groups

Def. Alg. group G is reductive if all its reps. are completely reducible (semisimple)

Examples: 1) ~~Finite groups~~

We shall consider connected red. grps.

2) $G = GL_n, SL_n, SO_n, Sp_n$ (n even)

3) Alg. tori $G = k^\times \times \dots \times k^\times$

2), 3) + $Spin_n, G_2, F_4, E_6, E_7, E_8$ $\xrightarrow[\text{quotients / finite central subgrps.}]{\text{products}}$ all connected red. groups

From now on: G a connected reductive alg. group

Important subgroups:

$G \supset B$ Borel subgroup = max. connected solvable closed subgrp.

$\quad\quad\; \cup$

$\quad\quad\; U$ max. unipotent subgrp.

i.e. $U \hookrightarrow GL_n$ consists of unipotent matrices (w. all eigenvalues $= 1$)

$\quad\quad\; T$ max. torus

$B = U \rtimes T$

B, U, T unique up to conj. in G

Basic example: $G = GL_n$, $B = \begin{pmatrix} * & & \\ & \ddots & \\ 0 & & * \end{pmatrix}$, $U = \begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$,

$T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

Representations of red. groups

Thm. Let $G \curvearrowright V$ irr. rep. Then:

- $\exists$ unique B-stable line $k\cdot v \subset V$

  $b \cdot v = \lambda(b)\cdot v$, $\quad \lambda : B \to k^\times$ highest weight

  $v = v_\lambda$ ht. wt. vector

- $G \curvearrowright V = V(\lambda)$ uniquely determined by λ

Weight lattice $\Lambda(B) = \{\lambda : B \xrightarrow[\text{hom.}]{} k^\times\}$ $\qquad \lambda|_U = 1$

$\simeq \Lambda(T)$

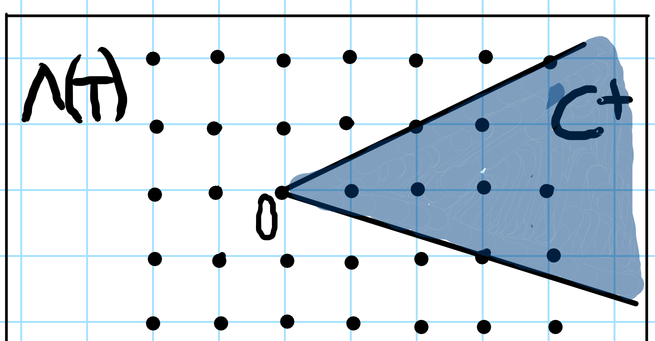
$T \simeq \underbrace{k^\times \times \dots \times k^\times}_{n} \implies \Lambda(T) \simeq \mathbb{Z}^n$

$\lambda(t) = t_1^{l_1} \cdots t_n^{l_n} =: t^\lambda$

$l_i \in \mathbb{Z}$

Write characters additively

- $\lambda \in \Lambda(T)$ ht. wt. of some irr. rep. $\iff \lambda \in \underbrace{\Lambda(T) \cap C^+}_{\text{dominant weights}}$, $C^+ \subset \Lambda(T)$, cosimplicial cone, called positive Weyl chamber



Example: $G = GL_n$, $V = \Lambda^k k^n \ni v = e_1 \wedge \ldots \wedge e_k$

ht. wt. vector

$$t = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \in T \implies t.v = t_1 \cdot \ldots \cdot t_k \cdot v$$

ht. wt. $\lambda = \varepsilon_1 + \ldots + \varepsilon_k$, $\varepsilon_i(t) = t_i$

$$C^+ = \{ \lambda = l_1 \varepsilon_1 + \ldots + l_n \varepsilon_n \mid l_1 \geq l_2 \geq \ldots \geq l_n \}$$

Geometric realization of irr. reps.

$X = G/B$ (gen.) flag variety

$G = GL(V) \Rightarrow X = \mathcal{F}\ell(V) = \{ V_\bullet = (0 = V_0 \subset V_1 \subset \ldots \subset V_n = V) \mid \dim V_k = k \}$

X is proj. var.

$\lambda \in \Lambda(T) = \Lambda(B) \rightsquigarrow \mathcal{L}_{-\lambda} = G \times^B k_{-\lambda}$

Borel–Weil thm. $H^0(G/B, \mathcal{L}_{-\lambda}) = \begin{cases} 0, & \lambda \notin C^+ \\ V(\lambda)^* = V(\lambda^*), & \lambda \in C^+ \end{cases}$

$\lambda \mapsto \lambda^*$ $\mathbb{Z}$-lin. involution on $\Lambda(T)$

Proof: $\mathrm{Hom}_G(V(\mu), \mathrm{Ind}_B^G(k_{-\lambda})) \simeq \mathrm{Hom}_B(V(\mu), k_{-\lambda})$

$\simeq \mathrm{Hom}_B(k_\lambda, V(\mu^*)) = \begin{cases} k, & \lambda = \mu^* \\ 0, & \text{otherwise} \end{cases}$

This means that $\exists$ unique irr. G-submodule in $\mathrm{Ind}_B^G(k_{-\lambda})$ namely $V(\lambda^*)$.

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