Spherical varieties: Lecture 4

Generalization: $G \supset P \supset B \rightsquigarrow Y = G/P$ (gen.) partial flag var.

parabolic subgrp.

Suppose λ extends to $P \to k^\times \rightsquigarrow \mathcal{L}_{P,-\lambda} = G \times^P k_{-\lambda}$

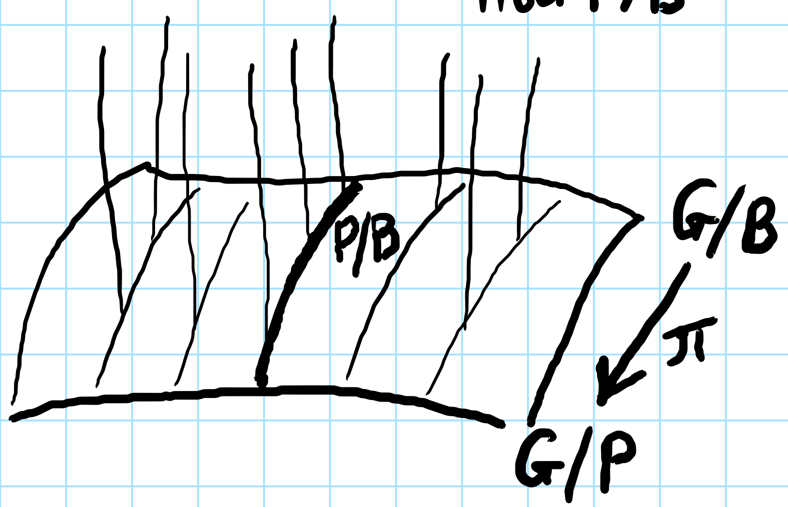
Cor. $H^0(G/P, \mathcal{L}_{P,-\lambda}) = \begin{cases} 0, & \lambda \notin C^+ \\ V(\lambda^*), & \lambda \in C^+ \end{cases}$

Proof: $G/B \xrightarrow[\text{fiber } P/B]{\pi} G/P$

Ex. 1 (Lect. 3)

$\mathcal{L}_{-\lambda}\big|_{P/B} = P \times^B k_{-\lambda} \simeq \underbrace{P/B}_{\text{proj. var.}} \times k_{-\lambda}$

$\Rightarrow H^0(P/B, \mathcal{L}_{-\lambda}) = k_{-\lambda}$



$\mathcal{L}_{-\lambda} = \pi^* \mathcal{L}_{P,-\lambda}$

$H^0(G/B, \mathcal{L}_{-\lambda}) = \pi^*(G/P, \mathcal{L}_{P,-\lambda})$

Now apply BW thm.

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Example. $Y = \mathbb{P}^n = GL_{n+1}/P \longleftarrow \mathcal{O}(m) = \mathcal{L}_{P,-m\varepsilon_1}$

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$H^0(\mathbb{P}^n, \mathcal{O}(m)) = k[x_1, \ldots, x_{n+1}]_m = V(m\varepsilon_1^*)$$

$$\varepsilon_1^* = -\varepsilon_{n+1}$$

$$\varepsilon_i \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{n+1} \end{pmatrix} = t_i$$

h.t. wt. vector is x_{n+1}^m

$k \cdot x_{n+1}^m$ preserved by B

$$(b \cdot f)(x) = f(b^{-1} \cdot x)$$

$$(b \cdot x_{n+1})(x) = (b^{-1} \cdot x)_{n+1}$$

In general:

$$G \curvearrowright H^0(G/H, \mathcal{L}_\chi) = \mathrm{Ind}_H^G(k_\chi) \quad \text{reducible}$$

Multiplicities: $[H^0(G/H, \mathcal{L}_\chi) : V(\lambda)] =$

$$= \dim \mathrm{Hom}_G(V(\lambda), \mathrm{Ind}_H^G(k_\chi)) =$$

$$= \dim \mathrm{Hom}_H(V(\lambda), k_\chi)$$

$$= \dim \mathrm{Hom}_H(k_{-\chi}, V(\lambda^*)) = \dim V(\lambda^*)^{(H)}_{-\chi} < \infty$$

Harmonic analysis: decompose $G \curvearrowright H^0(G/H, \mathcal{L})$ into irr. reps.

Good situation: all multiplicities ≤ 1 mult. free case

($\Rightarrow$ decomposition is unique)

Spherical homogen. spaces

G connected red. alg. grp.

$H \subset G$ closed subgrp., $B \subset G$ Borel subgrp.

$\mathfrak{g} = T_e G$ Lie algebra of G

$G \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}$ adjoint rep.

(Alg. groups = capital Latin letters
Lie algebras = lowercase German letters)

$G \subset GL_n \implies \mathfrak{g} \subset \mathrm{Mat}_n$ Lie subalg.

$[\xi, \eta] = \xi \cdot \eta - \eta \cdot \xi, \quad \forall \xi, \eta \in \mathfrak{g}$

$\mathrm{Ad}(g)\xi = g \cdot \xi \cdot g^{-1}, \quad \forall g \in G$

Def.–Thm. Hom. space $Y = G/H$ is **spherical** if any of equivalent properties holds:

(1) $B \curvearrowright Y$ w. open orbit

(2) $\exists g \in G : \quad \mathfrak{b} + \mathrm{Ad}(g)\mathfrak{h} = \mathfrak{g}$ — Lie alg. of gHg^{-1}

(3) $\exists$ Borel $B' \subset G : \quad \mathfrak{b}' + \mathfrak{h} = \mathfrak{g}$

(4) $k(Y)^B = k$

(5) $\forall$ homogen. line bundle $\mathcal{L} \to Y$:

$G \curvearrowright H^0(Y, \mathcal{L})$ mult. free

(6) $\forall \mu \in \Lambda(T) \cap C^+ \quad \forall \chi \in \Lambda(H) : \quad \dim V(\mu)_\chi^{(H)} \leq 1$

if Y quasiaff. var.:

(7) $G \curvearrowright k[Y]$ mult. free

(8) $\forall \mu \in \Lambda(T) \cap C^+ : \quad \dim V(\mu)^H \leq 1$

Proof: (1) $\Leftrightarrow$ (2)

$$B\cdot y \subset Y \text{ open}, \quad y = g\cdot x, \quad G_x = H$$
$$\Updownarrow$$
$$B\cdot g\cdot H \subset G \text{ open}$$
$$\Updownarrow$$
$$B\cdot (gHg^{-1}) \subset G \text{ open}$$
$$\Updownarrow$$
$$\mathfrak{b} + \mathrm{Ad}(g)\mathfrak{h} = \mathfrak{g}$$

Differentiate $B \times gHg^{-1} \xrightarrow[\text{mult.}]{} G$ at (e,e)

(2) $\Leftrightarrow$ (3) Take $B' = g^{-1}\cdot B\cdot g$.

(1) $\Rightarrow$ (4) $f \in k(Y)^B \Rightarrow f = \text{const on } B\cdot y$

$\Longrightarrow$ on $\overline{B\cdot y} = Y$

(4) $\Rightarrow$ (1) Rosenlicht thm: $\underset{\text{alg. grp.}}{B} \curvearrowright \underset{\text{irr. var.}}{Y} \Longrightarrow$

$\exists$ open $U \subset Y$ s.t. $\forall x, y \in U$, $B\cdot x \neq B\cdot y$ $\exists f(Y)^B : f(x) \neq f(y)$

In our case: $k(Y)^B = k \Rightarrow \forall y \in U : B\cdot y \supset U$ open in Y

(4) $\Rightarrow$ (5) By contradiction:

assume that $M_1, M_2 \subset H^0(Y, \mathcal{L})$

$G \curvearrowright M_i \simeq V(\lambda)$, $M_1 \neq M_2$, $M_1 \cap M_2 = 0$

Choose ht. wt. vectors $s_i \in (M_i)^{(B)}_\lambda$

$\Rightarrow f = \frac{s_1}{s_2} \in k(Y)^B$
$\neq$ const Contradiction

(5) $\Rightarrow$ (7) Take $\mathcal{L} = \mathcal{O}_Y$ trivial line bundle over Y

(7) $\Rightarrow$ (4) Y quasiaff. $\Rightarrow k(Y) = \mathrm{Frac}\, k[Y]$

$f \in k(Y)^B \Rightarrow f = \frac{p_1}{p_2}$, $p_i \in k[Y]$

$p_1 = f \cdot p_2$

$M_i := \langle B\cdot p_i \rangle_k \subset k[Y]$

B-stable subspace of $\dim < \infty$

Lie-Kolchin thm. $B \circlearrowright M$ lin. rep. of connected solvable alg. grp.

$$\Longrightarrow \quad \exists \text{ B-stable line } k\cdot m \subset M$$

Choose $q_2 \in (M_2)_\lambda^{(B)}$, $q_2 = \sum_i \alpha_i \cdot b_i \cdot p_2$

$\alpha_i \in k$, $b_i \in B$

$\Rightarrow q_1 := \sum_i \alpha_i \cdot b_i \cdot p_1 = f \cdot q_2$ (because $p_1 = f \cdot p_2$)

$\in (M_1)_\lambda^{(B)}$

q_i are ht. wt. vectors in $N_i = \langle G \cdot q_i \rangle_k \subset k[Y]$

$\simeq V(\lambda)$

$\Rightarrow N_1 = N_2$

$\Rightarrow f = \dfrac{q_1}{q_2} = \text{const}$

$(5) \Rightarrow (4)$ $\quad \exists\, G \curvearrowright V$ s.t. $Y = G\cdot[v] \subseteq \mathbb{P}(V)$

Chevalley's thm.

$f \in k(Y)^B \Rightarrow f = \dfrac{p_1}{p_2}$

$p_i \in k[V]_m$

As before, replace p_i w. $q_i \in k[V]_m^{(B)}$

$q_i|_Y = s_i \in H^0(Y, \mathcal{L})^{(B)}_\lambda$, $\mathcal{L} = \mathcal{O}(m)|_Y$

ht. wt. vectors in $N_i \subset H^0(Y,\mathcal{L})$

$\simeq V(\lambda)$

$\Rightarrow N_1 = N_2 \Rightarrow f = \dfrac{s_1}{s_2} = \text{const}$

$\left.\begin{array}{l} (5) \Leftrightarrow (6) \\ (7) \Leftrightarrow (8) \end{array}\right\}$ Computation of multiplicities:

$[H^0(G/H, \mathcal{L}_\chi) : V(\lambda)] = \dim V(\lambda^*)^{(H)}_{-\chi}$

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