Spherical varieties: Lecture 5

Recall: Hom. space $Y = G/H$ is spherical if any of equivalent properties holds:

(1) $B \curvearrowright Y$ w. open orbit

(2) $\exists g \in G$: $\mathfrak{b} + \mathrm{Ad}(g)\mathfrak{h} = \mathfrak{g}$ (Lie alg. of gHg^{-1})

(3) $\exists$ Borel $B' \subset G$: $\mathfrak{b}' + \mathfrak{h} = \mathfrak{g}$

(4) $k(Y)^B = k$

(5) $\forall$ homogen. line bundle $\mathcal{L} \to Y$: $G \curvearrowright H^0(Y, \mathcal{L})$ mult. free

(6) $\forall \mu \in \Lambda(T) \cap C^+$ $\forall \chi \in \Lambda(H)$: $\dim V(\mu)_\chi^{(H)} \leq 1$

if Y quasiaff. var.:

(7) $G \curvearrowright k[Y]$ mult. free

(8) $\forall \mu \in \Lambda(T) \cap C^+$: $\dim V(\mu)^H \leq 1$

Examples:

1) Sphere $S^{n-1} = \{x \in k^n \mid x_1^2 + \ldots + x_n^2 = 1\} = SO_n / SO_{n-1}$

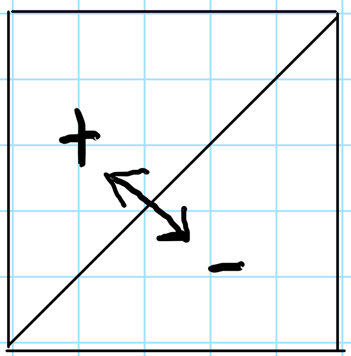
Change coordinates in k^n s.t. SO_n preserves quad. form w. matrix $q = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$

$G = SO_n$: $g \cdot q \cdot g^T = q$, $\det g = 1$

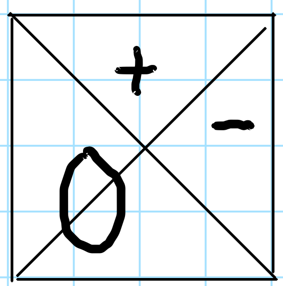
$\mathfrak{g} = \mathfrak{so}_n$: $\xi \cdot q + q \cdot \xi^T = 0$, $\operatorname{tr} \xi = 0$

$\Updownarrow$

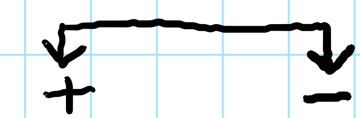
$\xi = - q \cdot \xi^T \cdot q$ $\Rightarrow$ $\operatorname{tr} \xi = 0$



skew-symm. matrices w.r.t. secondary diagonal

$\mathfrak{b} =$ 

$H = G_{e_1+e_n} \simeq SO_{n-1} \Rightarrow \mathfrak{h} = \{\xi \in \mathfrak{g} \mid \xi(e_1+e_n) = 0\} = \begin{pmatrix} 0 & v^T & 0 \\ v & \begin{smallmatrix} + & \\ & - \end{smallmatrix} & -v \\ 0 & -v^T & 0 \end{pmatrix}$



$\Rightarrow \mathfrak{b} + \mathfrak{h} = \mathfrak{g}$ (2)

Names: Spherical hom. space

M. Brion, D. Luna, Th. Vust '1986

Spherical subgroup (for H)

M. Krämer '1979

mult. free condition

E.B. Vinberg, B.N. Kimelfeld '1978

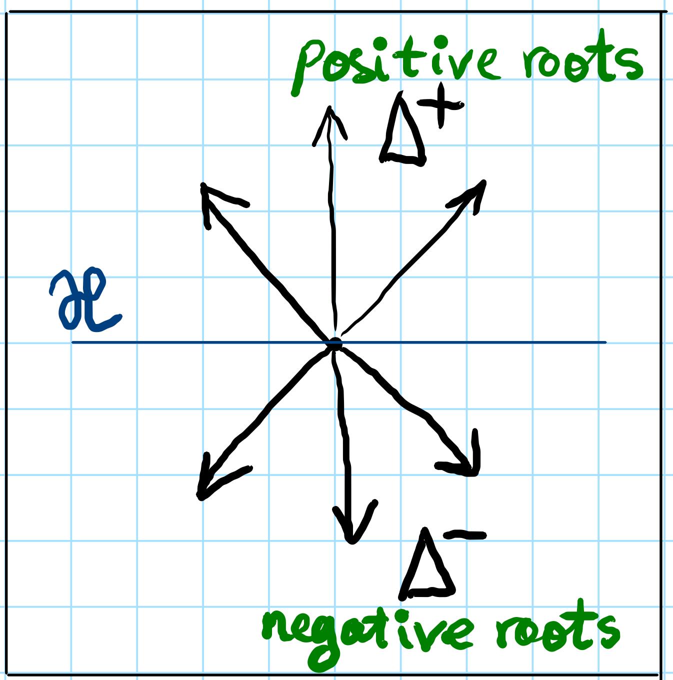
2) Flag var. $Y = G/B$

Mult. free (5) $\Leftarrow$ Borel-Weil thm.

Digression: root decomposition

$T \circlearrowright_{Ad} \mathfrak{g}$ diagonalizable $\Rightarrow$ $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ root subspaces

$\Delta \subset \Lambda(T) \setminus 0$ root system



$\mathfrak{g}_\alpha = k \cdot e_\alpha$, $Ad(t) e_\alpha = t^\alpha \cdot e_\alpha$, $\forall t \in T$

Choose hyperplane $\mathcal{H} \subset \Lambda(T)_{\mathbb{Q}}$, $\mathcal{H} \cap \Delta = \emptyset$

$\rightsquigarrow \Delta = \Delta^+ \sqcup \Delta^-$, $\Delta^- = -\Delta^+$

Then: $\mathfrak{b} = \mathfrak{t} \oplus \underbrace{\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha}_{\mathfrak{u}}$ for some $\mathcal{H}$

Opposite Borel subgrp. / subalg.: $\mathfrak{b}^- = \mathfrak{t} \oplus \underbrace{\bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha}_{\mathfrak{u}^-}$

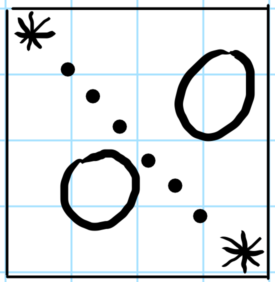
$B^- \cap B = T$

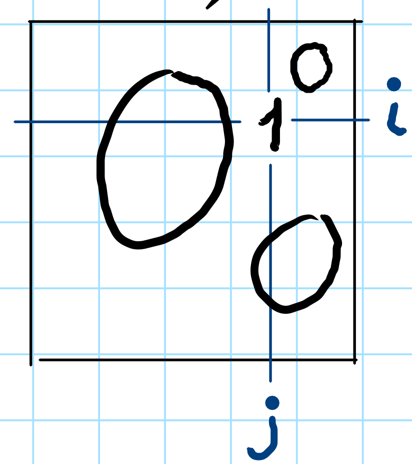
Then: $\mathfrak{b}^- + \mathfrak{b} = \mathfrak{g}$, $\quad B^- \cdot B$ open in G

big cell

$\Rightarrow \quad B^- \cdot x$ open in $Y \qquad$ (1), (3)

$x = eB \in G/B = Y$

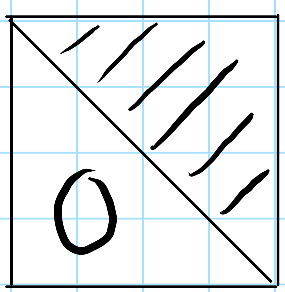
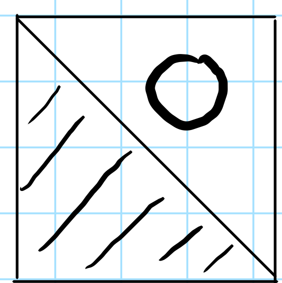
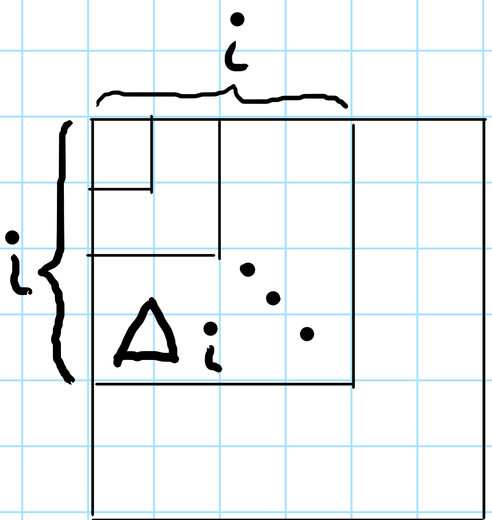
Basic example: $\quad G = GL_n$, $\quad T =$ 

Here: $\quad e_\alpha = e_{ij} =$ 

$\alpha = \varepsilon_i - \varepsilon_j$

$(i \neq j)$

$\alpha \in \Delta^+ \iff i < j$

$B =$  , $\quad B^- =$  , big cell: 

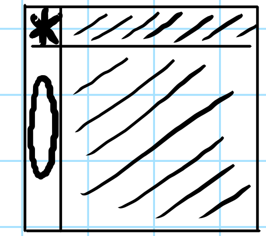
Exercise 1: Prove that big cell defined by $\Delta_i \neq 0$, $\forall i = 1, \dots, n$

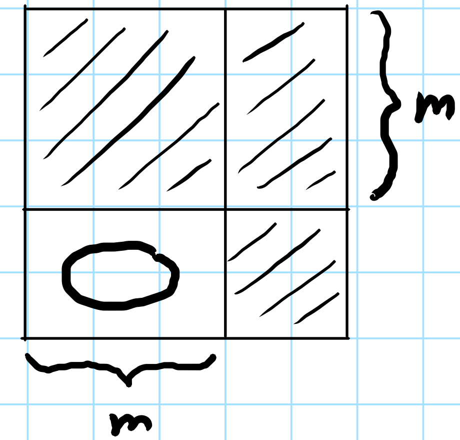
3) Partial flag vars. $Y = G/P \twoheadleftarrow G/B$

$\qquad\quad\cup \qquad\qquad\qquad\qquad \cup$

(1) open B-orbit $\longleftarrow$ open B-orbit

Particular cases:

- $\mathbb{P}^n$, $G = GL_{n+1}$, $P =$ 

- $Gr_m(k^n)$, $G = GL_n$, $P =$ 

4) $Q_n = \{\text{smooth quadric hypersurf.} \subset \mathbb{P}^n\}$

$= PGL_{n+1} / PO_{n+1}$

Base pt: $q_0 = \{x_0^2 + \dots + x_n^2 = 0\}$

Gram-Schmidt orthogonalization: $q \in Q_n$

$q = \{\sum_{i,j} q_{ij} x_i x_j = 0\}$, $\det(q_{ij})_{i,j=0,\dots,k} \neq 0$, $\forall k = 0,\dots,n$

$\Longrightarrow \exists b \in B : b \cdot q = q_0$. Hence $B \cdot q_0$ open in Q_n (1)

5) $G \times G \circlearrowright Y = G \simeq (G \times G)/\mathrm{diag}(G)$

left/right mult.

Borel subgrp.: $B^- \times B \subset G \times G$

$(B^- \times B)e = B^- \cdot B$ open in G (1)

Exercise 2: Are spherical? $Y = G/H$

(a) $GL_2 \times GL_2 \times GL_2 / \mathrm{diag}(GL_2)$

(b) $GL_3 \times GL_3 \times GL_3 / \mathrm{diag}(GL_3)$

(c) $GL_2 \times GL_2 \times GL_2 \times GL_2 / \mathrm{diag}(GL_2)$

Exercise 3: $H \subset G$ connected red. subgrp.

Prove: $G \times H / \mathrm{diag}(H)$ spherical for $G \times H$ $\Longleftrightarrow$

(MFR) $\forall$ irr. rep. $G \circlearrowright V$: $\mathrm{Res}^G_H(V)$ mult. free

Exercise 4: Does (MFR) hold for:

(a) $GL_n \supset GL_{n-1}$; (b) $SO_n \supset SO_{n-1}$; (c) $Sp_{2n} \supset Sp_{2n-2}$?

Def. $H \subset G$ symmetric subgrp. if

$\exists\, \theta \in \mathrm{Aut}(G)$, $\theta^2 = \mathrm{id}$, $\theta \neq \mathrm{id}$ s.t.

$$H^\circ = (G^\theta)^\circ$$

θ alg. group automorphism

H° ↑ identity component := connected component of H $\ni e$

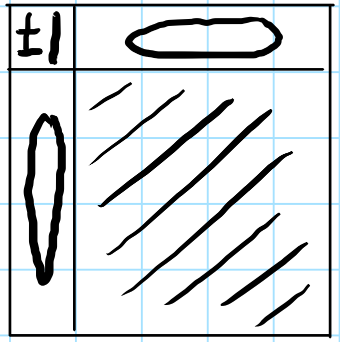
Equivalently: $\mathfrak{h} = \mathfrak{g}^\theta$

$Y = G/H$ symmetric space

Examples:

1) $S^{n-1} = SO_n / SO_{n-1}$

θ = conjugation w. $\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$, $G^\theta =$



2) $Q_n = PGL_{n+1} / PO_{n+1}$, $\theta(g) = (g^T)^{-1}$

3) $G = (G \times G)/\mathrm{diag}(G)$, $\theta(g_1, g_2) = (g_2, g_1)$

Thm. Symmetric spaces are spherical

Proof of thm: Suppose $H \subset G$ symm. subgrp.

$$\Rightarrow \mathfrak{g} = \underset{\substack{\| \\ \mathfrak{h}}}{\mathfrak{g}^{\theta}} \oplus \mathfrak{g}^{-\theta}, \quad \mathfrak{g}^{\pm\theta} = \theta\text{-eigenspaces of eigenvalue } \pm 1$$

Lemma 1. $\forall$ Borel subgrp. $B \subset G$ $\exists$ θ-stable max. torus $T \subset B$

Proof: $\theta(B) \subset G$ another Borel subgrp.

$R = B \cap \theta(B)$ θ-stable, solvable, connected, $\supset$ max. torus of G

[Humphreys, L.A.G., 28.3, Cor.] Bruhat decomposition

$$R = \underset{\text{torus}}{S} \ltimes \underset{\text{unipotent}}{N} \qquad \theta(N) = N$$

unipotent = set of all unipotent elts.

$$\theta(S) = n \cdot S \cdot n^{-1}, \quad n \in N$$

$$S = \theta(n) \cdot \theta(S) \cdot \theta(n)^{-1} = \underbrace{\theta(n) \cdot n}_{= e} \cdot S \cdot n^{-1} \cdot \theta(n)^{-1}$$

$$\Rightarrow \theta(n) = n^{-1}$$

$$\exp : \mathfrak{n} \xrightarrow{\ \sim\ } N, \quad \exp(\xi) = e + \xi + \frac{\xi^2}{2} + \ldots + \frac{\xi^k}{k!} + \ldots$$

$$\mathfrak{n} \subset \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, \qquad N \subset \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$n = \exp(\xi), \qquad \theta(n) = n^{-1} \Rightarrow \theta(\xi) = -\xi$$

Put $n^{1/2} := \exp(\xi/2)$, $T = n^{1/2} \cdot S \cdot n^{-1/2}$

$$\Rightarrow \theta(T) = \theta(n^{1/2}) \cdot \theta(S) \cdot \theta(n^{-1/2})$$

$$= n^{-1/2} \cdot n \cdot S \cdot n^{-1} \cdot n^{1/2} = T$$

$\square$