Spherical varieties: Lecture 6

Symmetric $\Rightarrow$ spherical

Notation: $H \subset G$, $H^\circ = (G^\theta)^\circ$, $\theta \in \mathrm{Aut}(G)$, $\theta^2 = \mathrm{id}$
$\theta \neq \mathrm{id}$

Lemma 1. $\forall$ Borel subgroup $B \subset G$ $\exists$ θ-stable max. torus $T \subset B$

$$\mathfrak{t} = \mathfrak{t}^\theta \oplus \mathfrak{t}^{-\theta}$$

Lemma 2. $\exists$ θ-stable max. torus $T \subset G$ s.t. $\mathfrak{t}^{-\theta} \neq 0$

Proof: <u>Otherwise</u>: $\forall$ Borel $B \subset G$, $B \supset T$, $\theta|_T = \mathrm{id}$

$\Rightarrow \underset{\text{trivial}}{\theta \curvearrowright} \Lambda(T) \supset \Delta \Rightarrow \theta(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$, $\forall \alpha \in \Delta$

$\Rightarrow \theta(\mathfrak{b}) = \mathfrak{b} \Rightarrow \theta(B) = B$

<u>Hence</u>: $\forall$ max. torus $T' \subset G$, $T' = B \cap B^-$ θ-stable

$\Rightarrow \theta|_{T'} = \mathrm{id}$

$\bigcup_{T' \subset G} T'$ dense in G $\Rightarrow \theta = \mathrm{id}$. Contradiction. $\blacksquare$

Choose θ-stable max. torus $T \subset G$ s.t. $\dim \mathfrak{t}^{-\theta}$ is max. possible

Lemma 3. $\alpha \in \Delta$, $\theta(\alpha) = \alpha \Rightarrow \mathfrak{g}_\alpha \subset \mathfrak{g}^\theta$

Proof: $\Lambda(T) \hookrightarrow \mathfrak{t}^*$, $\lambda : T \to k^\times \rightsquigarrow d\lambda : \mathfrak{t} \to k$

$\theta(\alpha) = \alpha \iff \alpha|_{\mathfrak{t}^{-\theta}} = 0$

$[\xi, \eta] = \alpha(\xi) \cdot \eta$ $\forall \xi \in \mathfrak{t}$, $\eta \in \mathfrak{g}_\alpha$

centralizer

$$\mathfrak{l} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}^{-\theta}) = \mathfrak{t} \oplus \underbrace{\bigoplus \mathfrak{g}_\alpha}_{\theta(\alpha)=\alpha} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$$

reductive [Humphreys]

center

semisimple θ-stable

$\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{t}$, $\mathfrak{z}(\mathfrak{l}) \supset \mathfrak{t}^{-\theta}$

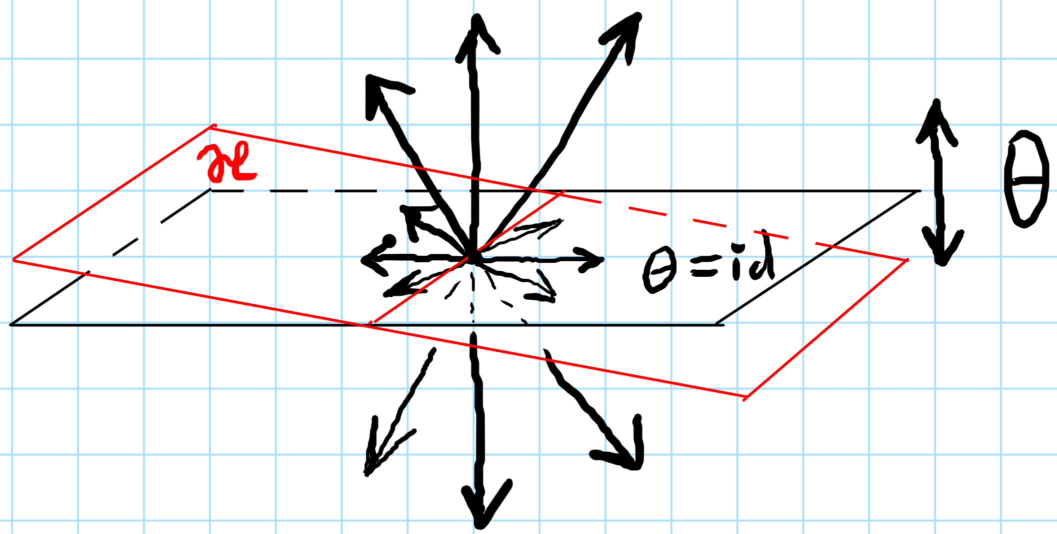
$\bigoplus_{\theta(\alpha)=\alpha} \mathfrak{g}_\alpha \subset [\mathfrak{l}, \mathfrak{l}]$

If $\theta|_{[\mathfrak{l},\mathfrak{l}]} \neq \mathrm{id}$, then $\exists$ θ-stable max. torus $S \subset [L, L]$, $\mathfrak{s}^{-\theta} \neq 0$

$T \rightsquigarrow T' = Z(L) \cdot S$, $(\mathfrak{t}')^{-\theta} = \mathfrak{t}^{-\theta} \oplus \mathfrak{s}^{-\theta}$, $\dim > \dim(\mathfrak{t}^{-\theta})$

Contradiction. ∎

End of proof of Thm.:



Choose $\mathcal{H} \subset \Lambda(T)_{\mathbb{Q}}$ s.t.

$\mathcal{H} \cap \Delta = \emptyset$ and:

$\beta \in \Delta^{\pm},\ \beta \neq \theta(\beta) \Rightarrow \theta(\beta) \in \Delta^{\mp}$

Then:
$$\mathfrak{h} = \mathfrak{g}^{\theta} = \mathfrak{t}^{\theta} \oplus \bigoplus_{\alpha = \theta(\alpha)} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\beta \neq \theta(\beta),\ \beta \in \Delta^{+}} k \cdot (e_{\beta} + \theta(e_{\beta}))$$
$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^{+}} \mathfrak{g}_{\alpha}$$
$$\Rightarrow \mathfrak{b} + \mathfrak{h} = \mathfrak{g}$$

Proof complete.

Exercise 1: $V = k^n$ vector space w. non-degenerate quad. form q

$$G = SO(V,q) \simeq SO_n(k)$$

$$\curvearrowright Y = \{ U \subset V \mid \dim U = m,\ q|_U \text{ non-deg.} \} \subset Gr_m(V)$$

Prove: Y symm. space for G

Exercise 2: $V = k^{2n}$ symplectic vector space

$$G = Sp(V) \simeq Sp_{2n}(k)$$

$$Y = \{ (U_1, U_2) \mid U_i \subset V \text{ Lagrangian},\ U_1 \oplus U_2 = V \}$$

Prove: Y symm. space for G

Spherical varieties

G connected red. alg. grp., $\quad Y \simeq G/H$ hom. space

Def. Equivariant open embedding: $G \curvearrowright X \supset Y$ open dense orbit
(G-embedding)

Spherical variety := open embedding of a sph. hom. space

Equivalently: Spherical variety = irr. alg. variety X w. action $G \curvearrowright X$ s.t. $B \curvearrowright X$ w. open orbit

Examples: 1) $S^{n-1} = SO_n / SO_{n-1} \subset X = \{x_0^2 + x_1^2 + \dots + x_n^2 = 0\} \subset \mathbb{P}^n$
smooth proj. quadric

2) $Q_n = PGL_{n+1}/PO_{n+1} \subset X = \mathbb{P}(\mathrm{Sym}_{n+1}) = \mathbb{P}^{\frac{n(n+3)}{2}}$
var. of smooth quadrics $\qquad$ var. of all proj. quadrics

3) $Q_2 \subset X$ variety of complete conics

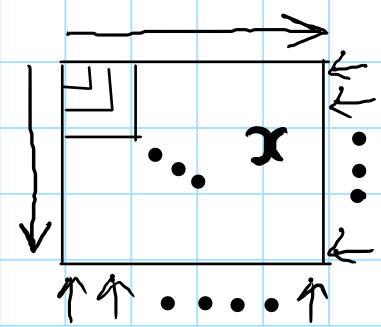
4) Determinantal varieties:

$$X_r = \{ x = (x_{ij}) \mid \mathrm{rk}\, x \leq r \} \subset \mathrm{Mat}_{m \times n}$$

$$G = GL_m \times GL_n \curvearrowright X_r$$

left / right mult.

$$B = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \times \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} :$$



$$\xrightarrow{\text{if}} \quad x_r = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & 0 \end{pmatrix}$$

$$\begin{cases} \det (x_{ij})_{i,j=1}^{k} \neq 0 \\ k = 1, \ldots, r \end{cases}$$

defines open orbit $B \cdot x_r$

Irreducibility: $X_r = G \cdot Z_r$, $Z_r = \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix} \simeq \mathbb{A}^r$

5) Toric varieties: $G = B = T$ alg. torus

$$X \supset Y \simeq \underset{\text{also torus}}{T/H}, \quad H \curvearrowright X \text{ trivial}$$

$\Rightarrow$ may assume $H = \{e\}$

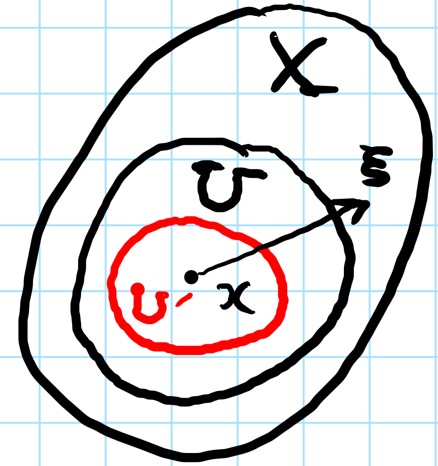
$$T \underset{\text{left mult.}}{\curvearrowright} Y \simeq T$$

Topics:

- Classification of sph. varieties → Classification of sph. hom. spaces; → Classification of open embeddings of given sph. hom. space
  ~~Combinatorial/geometric data~~
- Studying geometry of sph. varieties
- Applications of sph. varieties in geometry and rep. theory

Digression: singularities of alg. varieties

X alg. variety $/k$, $x\in X$



Local ring $\mathcal{O}_{X,x} = \mathcal{O}_x = \varinjlim_{U\ni x} \mathcal{O}(U)$

$$\mathcal{O}_x \supset \mathfrak{m}_x$$

$\mathfrak{m}_x$ vanishing ideal at x

Zariski tangent space: $T_xX := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$

$$\xi\in T_xX,\ f\in\mathcal{O}_x \rightsquigarrow \langle \underbrace{f-f(x)}_{\in\,\mathfrak{m}_x} \bmod \mathfrak{m}_x^2,\ \xi\rangle = \partial_\xi f$$

$$= d_xf(\xi)$$

derivative of f at x in direction ξ

$$\dim T_x X \geq \dim_x X := \max_{X_i \ni x} \dim X_i$$

$\uparrow$ irr. components of X

$$\dim X_i = \text{tr.deg. } k(X_i)/k$$

x smooth pt. (regular) if $\dim T_x X = \dim_x X$

singular pt. if $\dim T_x X > \dim_x X$

Smooth locus $X^{reg} \subset X$ open, dense

Singular locus $X^{sing} = X \setminus X^{reg}$ closed, smaller dim

$\supset X_i \cap X_j \quad (\forall i \neq j)$