Spherical varieties: Lecture 7

Def. Irr. alg. variety X is *normal* if:

- $\operatorname{codim} X^{sing} \geq 2$
- $\forall$ open $U \subset X$ $\forall f \in k(X)$: $f \in \mathcal{O}(U \setminus Z)$, $\operatorname{codim} Z \geq 2$ $\Rightarrow f \in \mathcal{O}(U)$

Smooth $\Rightarrow$ normal

Fact: Irr. affine var. X is normal $\iff$ $k[X]$ *integrally closed* in $k(X)$

$$f \in k(X),\ \overbrace{f^m + g_1 \cdot f^{m-1} + \ldots + g_m = 0,\ g_i \in k[X]}^{(*)} \Rightarrow f \in k[X]$$

Normalization: X irr. var. $\Rightarrow \exists$ unique normal $\tilde{X}$ w. $\nu : \tilde{X} \longrightarrow X$

ν finite (= proper w. finite fibers)

birational: $\exists$ open $U \subset X$ $\nu^{-1}(U) \xrightarrow[\sim]{\nu} U$

X affine $\Rightarrow \tilde{X}$ affine

$k[\tilde{X}] = \widetilde{k[X]} = \{ f \in k(X) \text{ satisfying } (*) \}$

Spherical variety := normal spherical variety

Separation property: $X \times X \supset \mathrm{diag}(X)$ closed

Algebraic variety := separated alg. variety

G-linearization of line bundles

G connected alg. group (not necessarily reductive)

X normal alg. variety, $G \circlearrowright X$

Thm. $\exists$ isogeny $G' \longrightarrow G$ s.t. $\forall$ line bundle $\mathcal{L} \to X$
connected

action $G \circlearrowright X$ lifts to fiberwise linear action $G' \circlearrowright \mathcal{L}$

$\|$
Surj. homomorphism w. finite fibers

G'-linearization of $\mathcal{L}$

Reference:

F. Knop, H. Kraft, D. Luna, Th. Vust. Local properties of alg. group actions. DMV Seminar, vol. 13, 1989.

Cor. $\exists\, m \in \mathbb{N}$ s.t. $G \curvearrowright X$ lifts to $G \curvearrowright \mathcal{L}^{\otimes m}$

Proof: $Z = \operatorname{Ker}(G' \to G)$ finite, $Z \curvearrowright X$ trivial

$\Rightarrow Z \curvearrowright \mathcal{L}$ preserves fibers: $\mathcal{L}_x \xrightarrow{Z} \mathcal{L}_x$

$\Rightarrow Z \curvearrowright \mathcal{L}_x$ via $\chi : Z \longrightarrow k^\times$

Put $m := |Z| \Rightarrow Z \curvearrowright \mathcal{L}_x^{\otimes m}$ via $\chi^m = 1$

$\Rightarrow G' \curvearrowright \mathcal{L}^{\otimes m}$ factors through $G \curvearrowright \mathcal{L}^{\otimes m}$

$\blacksquare$

Example: $G = PGL_{n+1} = PSL_{n+1} \curvearrowright X = \mathbb{P}^n$

$$\begin{array}{c} X = \mathbb{P}^n \\ \uparrow \\ \mathcal{L} = \mathcal{O}(-1) \end{array}$$

$p \in [v]$, $v \in k^{n+1}_{\neq 0} \Rightarrow \mathcal{O}(-1)_p = k \cdot v$ $\qquad \mathcal{L} = \mathcal{O}(-1)$

tautological line bundle

$PSL_{n+1} \cancel{\curvearrowright} \mathcal{L}$ but $PSL_{n+1} \longleftarrow SL_{n+1} \curvearrowright \mathcal{L}$

$\operatorname{Ker} = \mathbb{Z}_{n+1}$ cyclic

$PSL_{n+1} \curvearrowright \mathcal{L}^{\otimes(n+1)} = \mathcal{O}(-n-1)$

Similarly: $\forall \mathcal{L} \to \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(k) = \mathcal{O}(-1)^{\otimes(-k)}$ $\circlearrowleft SL_{n+1}$

$k \in \mathbb{Z}$

$$\mathcal{L}^{\otimes(n+1)} = \mathcal{O}(-n-1)^{\otimes(-k)} \circlearrowleft PSL_{n+1}$$

<u>Sumihiro thm.</u> $\forall x \in X$ $\exists$ G-stable quasiproj. open nbhd. $U \subset X$, $U \ni x$

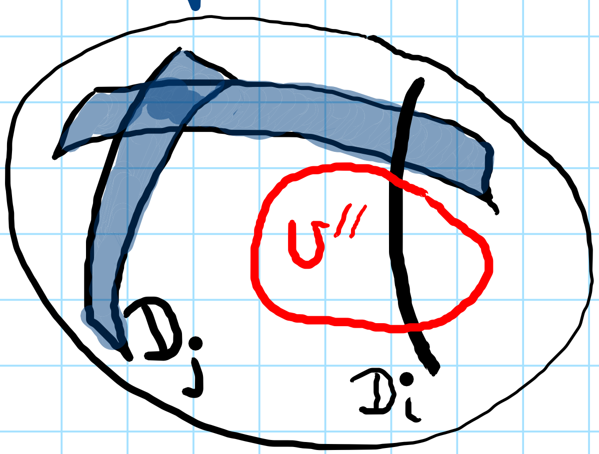
Proof: $\exists$ affine open nbhd. $U' \subset X$, $U' \ni x$

Lemma 1: $D := X \setminus U' = D_1 \cup \ldots \cup D_s$, D_i irr. compts. $\operatorname{codim}_X D_i = 1$

Proof of Lemma. Suppose $\exists D_i$, $\operatorname{codim} D_i \geq 2$

Choose aff. open $U'' \subset X$, $U'' \cap D_i \neq \emptyset$

$U'' \cap D_j = \emptyset$ $\forall j \neq i$



$$U''' = U'' \setminus D_i = \underbrace{U' \cap U''}_{\text{affine}} \simeq \underbrace{(U' \times U'')}_{\text{affine}} \cap \underbrace{\operatorname{diag} X}_{\text{closed}}$$

$$U''' \subset U'' \rightsquigarrow k[U'''] \supseteq k[U'']$$

$$\begin{array}{l} \cup\!\!\!\mid \\ f \text{ regular outside } D_i \cap U'' \\ \quad \operatorname{codim}(D_i \cap U'') \geq 2 \end{array}$$

normality $\Rightarrow$ f regular on U''

Hence: $k[U'''] = k[U''] \Longrightarrow U''' = U''$

Contradiction.

$\blacksquare$

Digression: divisors and line bundles

X normal variety

Prime divisor = irr. closed subvar. $D \subset X$, $\operatorname{codim} D = 1$

(Weil) divisor = formal $\mathbb{Z}$-linear combination

$$\delta = k_1 \cdot D_1 + \ldots + k_s \cdot D_s$$

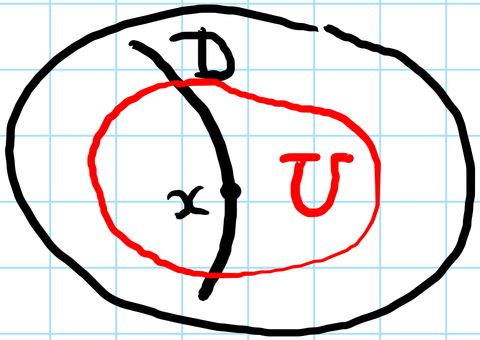
$k_i \in \mathbb{Z}$, D_i prime divisors

Effective divisor:

$\delta \geq 0$ if $k_1, \ldots, k_s \geq 0$

Claim: $D \cap X^{reg} \neq \emptyset$

$\forall x \in D \cap X^{reg}$ $\exists$ aff. open nbhd. $U \subset X$, $U \ni x$ s.t.



$$k[U] \rhd \underbrace{\mathcal{I}(D)}_{\text{vanishing ideal}} = (\underbrace{t_D}_{\text{local equation of } D})$$

$$\forall f \in k(X) \ \exists k \in \mathbb{Z} : f = t_D^k \cdot g, \quad g \in \mathcal{O}_{X,x}, \ g|_D \neq 0$$

$k = \mathrm{ord}_D(f)$ vanishing order of f along D

$k > 0$: f has zero along D

$k < 0$: f has pole along D

Prinipal divisor: $\mathrm{div}(f) = \sum_{D \subset X} \mathrm{ord}_D(f) \cdot D$

$$= \underbrace{\mathrm{div}_0(f)}_{\text{divisor of zeroes}} - \underbrace{\mathrm{div}_\infty(f)}_{\text{divisor of poles}}$$

both effective

$f \in \mathcal{O}(X) \iff \mathrm{div}_\infty(f) = 0$

Cartier divisor = divisor δ s.t. $\forall x \in X$
$\exists$ open nbhd. $\underset{\ni x}{U \subseteq X}$ s.t. $\delta \cap U$ principal

X smooth $\Longrightarrow$ $\forall$ divisor is Cartier

Suppose: $\mathcal{L} \longrightarrow X$ line bundle
$s \in H^0(U, \mathcal{L})$, $U \subset X$ open

$$X = \bigcup_i \underset{\text{open}}{U_i}, \quad \mathcal{L}|_{U_i} \simeq U_i \times k^1$$

$$s|_{U_i} \longleftrightarrow f_i \in k(U_i) = k(X)$$

$$f_i / f_j \in \mathcal{O}(U_i \cap U_j)^\times$$

$$\operatorname{ord}_D(s) := \operatorname{ord}_D(f_i) \quad \text{if } D \cap U_i \neq \emptyset$$

does not depend on i

$$\operatorname{div}(s) = \sum_D \operatorname{ord}_D(s) \cdot D \quad \text{Cartier}$$

$$s \in H^0(X, \mathcal{L}) \Longleftrightarrow \operatorname{div}(s) \geq 0$$

Conversely: δ Cartier $\Rightarrow \exists (\mathcal{L}, s)$ s.t. $\mathrm{div}(s) = \delta$

unique up to isomorphism

$$\mathcal{L} = \mathcal{O}_X(\delta) = \mathcal{O}(\delta), \quad s = s_\delta$$

$$H^0(X,\mathcal{L}) \supset \underset{\dim < \infty}{M} \rightsquigarrow \text{rat. map } \varphi : X \dashrightarrow \mathbb{P}(M^*)$$

$$x \longmapsto M(x)$$

$$\| $$

$$\{s \in M \mid s(x) = 0\}$$

hyperplane in M (or whole M)

In proj. coordinates:

$s_0, \dots, s_n$ basis of M

$$\varphi(x) = [\underbrace{s_0(x) : \dots : s_n(x)}_{\in \mathcal{L}_x \simeq k^1}]$$