Spherical varieties: Lecture 8

$$\mathcal{L} = \mathcal{O}_X(\delta), \quad \underset{\dim<\infty}{M} \subset H^0(X,\mathcal{L})$$

$$\rightsquigarrow$$

$$\varphi : X \dashrightarrow \mathbb{P}(M^*)$$

$$x \longmapsto M(x) = \{ s \in M \mid s(x) = 0 \}$$

Claim 1: If $\mathcal{L}$ *globally generated*: $\forall x \in X \ \exists s \in H^0(X,\mathcal{L}) : s(x) \neq 0$

then $\exists \underset{\dim<\infty}{M} \subset H^0(X,\mathcal{L})$ s.t. $\varphi : X \longrightarrow \mathbb{P}(M^*)$ morphism

Claim 2: $\forall$ morphism $\varphi : X \longrightarrow \mathbb{P}^n$ comes from $\mathcal{L} = \varphi^* \mathcal{O}(1)$

$$M = \varphi^* H^0(\mathbb{P}^n, \mathcal{O}(1))$$

δ *very ample* if $\varphi : X \hookrightarrow \mathbb{P}^n$ for some $M \subset H^0(X, \mathcal{O}(\delta))$

ample if $\exists m \in \mathbb{N}$ s.t. $m \cdot \delta$ very ample

$$\mathcal{L} \rightsquigarrow \mathcal{L}^{\otimes m}$$

Back to proof of Sumihiro thm.:

$\underset{\text{conn.}}{G} \curvearrowright \underset{\text{norm.}}{X}$ Want to find G-stable quasiproj. open nbhd. $U \subset X$
$\ni x$

$X \supset \underset{\substack{\text{open}\\ \text{affine}}}{U'} \ni x$, $X \setminus U' = D = \underset{\uparrow}{D_1} \cup \ldots \cup \underset{\uparrow}{D_s}$
prime divisors

Lemma 2. D_i Cartier on $X \setminus \bigcap_{g \in G} g D_i$

Proof: May assume $\bigcap_g g D_i = \emptyset$

D_i Cartier on $X^{reg} \rightsquigarrow \mathcal{L}^{reg} = \mathcal{O}_{X^{reg}}(D_i \cap X^{reg})$

$s \in H^0(X^{reg}, \mathcal{L}^{reg})$
$\operatorname{div} s = D_i \cap X^{reg}$

$G \curvearrowright \mathcal{L}^{reg} \implies \operatorname{div}(g \cdot s) = g D_i \cap X^{reg}$

$\implies \mathcal{L}^{reg}$ trivial on $X^{reg} \setminus g D_i =: U_g^{reg}$
$\forall g$

$$\begin{tikzcd} \mathcal{L}^{reg}|_{U_g^{reg}} \simeq U_g^{reg} \times k^1 \supset (U_g^{reg} \cap U_h^{reg}) \times & k^1 \\ \mathcal{L}^{reg}|_{U_h^{reg}} \simeq U_h^{reg} \times k^1 \supset (U_g^{reg} \cap U_h^{reg}) \times & k^1 \arrow[u, "\wr"', "f_{gh} \in \mathcal{O}(U_g^{reg} \cap U_h^{reg})^{\times}"] \end{tikzcd}$$

transition functions

$$X = \bigcup_g U_g \quad , \quad U_g = X \setminus gD_i$$

Normality $\Longrightarrow$ f_{gh} extends over $U_g \cap U_h$

$\in \mathcal{O}(U_g \cap U_h)^{\times}$

$\Longrightarrow$ $\mathcal{L}^{reg}$ extends to $\mathcal{L} \longrightarrow X$

s extends over X

On X: $\operatorname{div}(s) = D_i$

■

$\delta = D_1 + \dots + D_s$ Cartier on $U = X \setminus \bigcap\limits_g g\cdot D = G\cdot U'$

$k[U'] = k[f_1,\dots,f_n]$, $\mathrm{div}_\infty(f_j) = \sum\limits_i k_{ij}\cdot D_i$

Put $m := \max\{k_{ij}\}$

$s_0 := s_{m\delta} \in H^0(U, \mathcal{O}(m\delta))$

$\Rightarrow f_j\cdot s_0 = s_j \in H^0(U, \mathcal{O}(m\delta))$

Choose G'-stable fin. dim. $M \subset H^0(U, \mathcal{O}(m\delta))$

[Knop-Kraft-Luna-Vust] $\qquad \ni s_0, s_1, \dots, s_n$

$$\varphi : U \hookrightarrow \mathbb{P}(M^*)$$

$$\boxed{f_j \longmapsfrom s_j/s_0} \qquad g\cdot U' \xhookrightarrow[\text{closed}]{} \mathbb{P}(M^*)_{g\cdot s_0 \neq 0}$$

Hence U quasi proj.

Proof complete

Rmk. X quasiproj. normal, δ ample $\Longrightarrow$ $\exists\, m \in \mathbb{N}$ s.t. $m\cdot\delta$ very ample

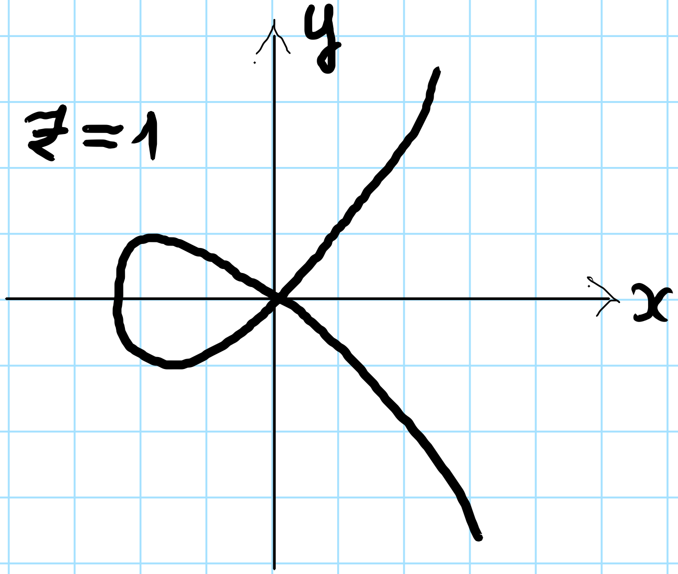
$$\mathcal{O}(m\delta) = \mathcal{O}(\delta)^{\otimes m} \quad G\text{-linearizable}$$

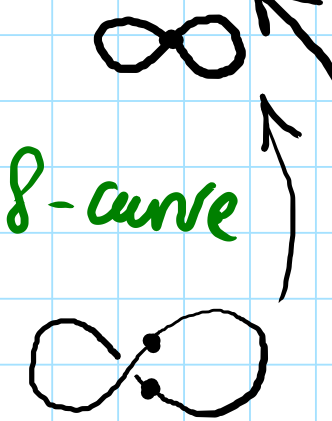
$\Longrightarrow$ $\exists$ G-stable fin. dim. $M \subset H^0(X, \mathcal{O}(m\delta))$ s.t.

$$X \hookrightarrow \mathbb{P}(M^*) \quad G\text{-equivariant embedding}$$

Example:

$$\begin{tikzcd} \{x^3 + x^2 z = y^2 z\} =: X \arrow[r, phantom, "\subset"] & \mathbb{P}^2 \\ & \tilde{X} = \mathbb{P}^1 \arrow[ul, "\nu"'] \end{tikzcd}$$





$G = k^\times \curvearrowright \mathbb{P}^1$, $[u_0 : u_1] \xrightarrow{t} [u_0 : t\cdot u_1]$

$0, \infty$ fixed points

$$\rightsquigarrow \quad G \curvearrowright X$$

Exercise 1:

$\nexists$ lin. rep. $G \curvearrowright V$

s.t. $X \overset{G}{\hookrightarrow} \mathbb{P}(V)$

Hint: $\overline{G\cdot[v]} \ni$ two fixed pts.

Finiteness thms.

G conn. red. grp., X spherical G-var.

Thm. 1. $G \curvearrowright X$ has fin. many orbits. Each orbit is a sph. hom. space

(Servedio' 1973, Luna-Vust' 1983)

Proof: 1) Sumihiro thm. $\Longrightarrow$ $X = \bigcup_{i=1}^{s} X_i$

X_i G-stable open quasiproj.

May assume $X \xhookrightarrow[G\text{-equiv.}]{} \mathbb{P}(V)$, $G \curvearrowright V$ lin. rep.

$X \rightsquigarrow \overline{X}$ $\Longrightarrow$ may assume $X \subset \mathbb{P}(V)$ G-stable proj. subvar.

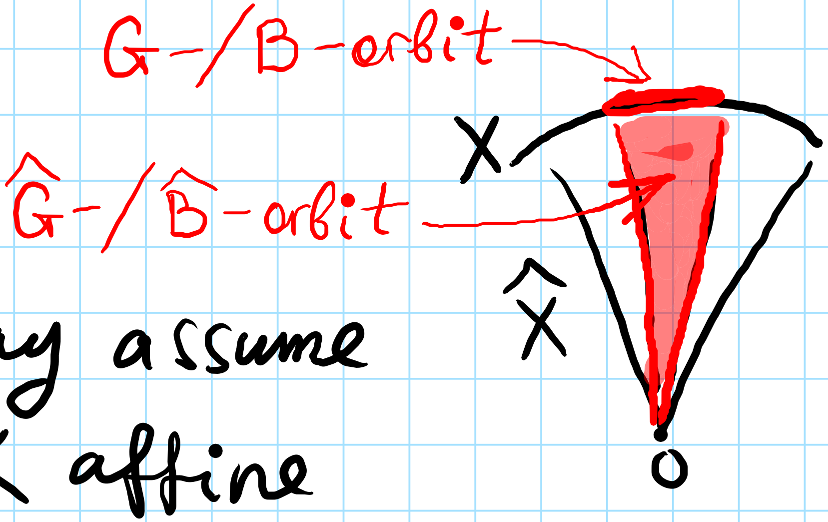
Affinization: $\hat{X} \subset V$ aff. cone over X

$\hat{G} = G \times k^\times \supset \hat{B} = B \times k^\times$ Borel subgrp.

↑ acts by scalar mult.

$$\{G\text{-}/B\text{-orbits in } X\} \overset{1:1}{\longleftrightarrow} \{\hat{G}\text{-}/\hat{B}\text{-orbits in } \hat{X}\setminus 0\}$$

$\Rightarrow$ $\hat{X}$ spherical for $\hat{G}$



$X \rightsquigarrow \hat{X}$, $G \rightsquigarrow \hat{G}$ $\Rightarrow$ may assume X affine

2) Let $Z \subset X$ closed irr. G-stable subvar.

Prove: $Z \supset$ open B-orbit (open in Z)

Otherwise: $G \circlearrowright k[Z]$ not mult. free

$$\Uparrow$$

$$\underset{G\text{-stable}}{\mathcal{I}(Z)} \lhd k[X] \underset{\overset{\Uparrow}{\text{complete reducibility}}}{\simeq} \mathcal{I}(Z) \oplus k[Z] \text{ as } G\text{-module}$$

$\Rightarrow$ $G \circlearrowright k[X]$ not mult. free. Contradiction.

Induction on dim X

: $X \supset Y_0$ open G-orbit

$$\underset{\text{closed}}{X \setminus Y_0} = Z_1 \cup \dots \cup Z_m ,$$

irr. cmpts.

Z_i closed irr. G-subvar.

$\dim Z_i < \dim X$

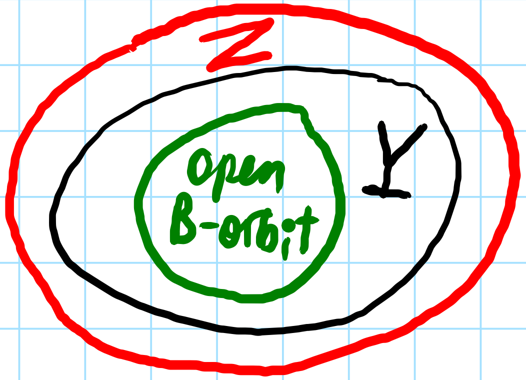
$\Rightarrow$ Z_i spherical for G

$\Rightarrow$ $G \curvearrowright Z_i$ has fin. many orbits

$\Rightarrow$ $G \curvearrowright X$ also.

3) $Y \subset X$ G-orbit $\Rightarrow$ $Z = \overline{Y}$ spherical

$\Rightarrow$ Y spherical



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