Spherical varieties: Lecture 9

Thm 2. (D. Akhiezer '1985)

$Y = G/H$ spherical $\Leftrightarrow$ $\forall$ G-embedding $X \hookleftarrow Y$ has fin. many G-orbits

Proof: $\Rightarrow$ by Thm. 1 (Lecture 8)

$\Leftarrow$ 1) Suppose: Y not spherical

Then: $\exists\ \mathcal{L} \to Y$, $G \curvearrowright \mathcal{L}$

$$\begin{tikzcd} H^0(Y,\mathcal{L}) \supset \underset{G\text{-submodule}}{M} = & M' \oplus M'' & \\ & V(\lambda) \arrow[u, "\psi'", "\sim"'] \arrow[u, shift left=4, "\psi''"', "\sim"] & \end{tikzcd}$$

$\rightsquigarrow\ \varphi : Y \to \mathbb{P}(M^*)$, $M^* \simeq V(\lambda^*) \otimes k^2$

Chevalley's thm. $\Rightarrow$

$$\begin{tikzcd} Y \arrow[d, hook, "\text{open}"] \simeq G\cdot[v] \subset \mathbb{P}(V) \\ X = \overline{G\cdot[v]} \subset \mathbb{P}(V) \end{tikzcd}$$

proj.

$\varphi : X \dashrightarrow \mathbb{P}(M^*)$

$\Phi = \mathrm{id}_X \times \varphi : X \dashrightarrow X \times \mathbb{P}(M^*)$, $\Phi(x) = (x, \varphi(x))$

$$\begin{tikzcd} X \arrow[r, squiggly] & \overline{\Phi(X)} \subset X \times \mathbb{P}(M^*) \arrow[r, "\text{projection}"] & \mathbb{P}(M^*) \\ Y \arrow[u, phantom, "\cup"] \arrow[ur, hook, "\text{open}"] \arrow[urr, bend right, "\varphi"'] & & \end{tikzcd}$$

$\Rightarrow$ may assume $\varphi : X \longrightarrow \mathbb{P}(M^*)$
a morphism

2) $T \circlearrowright V(\lambda)$ diagonalizable

Choose $v_0, v_1, \dots, v_n$ basis of $V(\lambda)$ s.t.

v_i eigenvector for T w. eigenweight $\lambda_i \in \Lambda(T)$

$\lambda_0 = \lambda$ ht. wt., v_0 ht. wt. vector

Lemma. $\lambda_i = \lambda - \alpha_1 - \dots - \alpha_k$, $\alpha_j \in \Delta^+$, $k \in \mathbb{N}$

$\forall i = 1, \dots, n$

Proof of Lemma: $G \supset \underbrace{B^- \cdot B}_{\text{dense open}} = U^- \cdot B$

$$\Rightarrow V(\lambda) = \underbrace{\langle G \cdot v_0 \rangle_k}_{G\text{-stable subspace}} = \langle U^- \cdot v_0 \rangle_k$$

$$\| $$

$$\langle \underbrace{e_{-\alpha_1} \cdot \dots \cdot e_{-\alpha_k} \cdot v_0}_{T\text{-eigenvector w. eigenwt. } \lambda - \alpha_1 - \dots - \alpha_k} \mid k \geq 0,\ \alpha_1, \dots, \alpha_k \in \Delta^+ \rangle$$

$U^- = \exp \mathfrak{u}^-$

$\mathfrak{u}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha} \ni \xi = \sum_\alpha c_\alpha \cdot e_{-\alpha}$

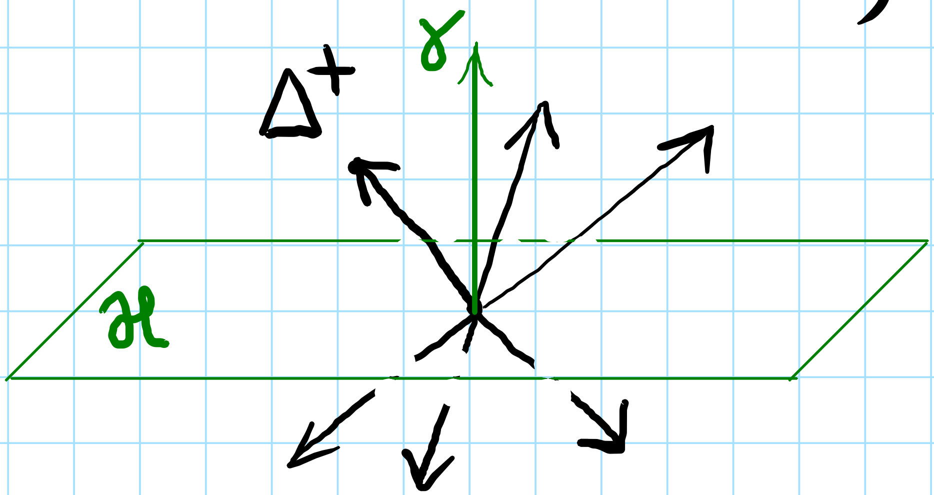
$\mathfrak{g}_{-\alpha} = k \cdot e_{-\alpha}$

$(\exp \xi) \cdot v_0 = \exp\left(\sum_\alpha c_\alpha \cdot e_{-\alpha}\right) \cdot v_0$

$\Longrightarrow$ all T-eigenweights are of the form $\lambda - \alpha_1 - \ldots - \alpha_k$

$\blacksquare$

Choose $\gamma \in \Lambda^*$ s.t. $\langle \gamma, \alpha \rangle > 0$, $\forall \alpha \in \Delta^+$



$\rightsquigarrow$ 1-parameter mult. subgrp.

$$k^\times \longrightarrow T$$
$$t \longmapsto t^\gamma \quad \text{s.t.}$$
$$\mu(t^\gamma) = t^{\langle \gamma, \mu \rangle}$$
$$\forall \mu \in \Lambda(T)$$

$$T = \underbrace{k^\times \times \ldots \times k^\times}_{m}$$

$$\Longrightarrow t^\gamma = (t^{k_1}, \ldots, t^{k_m})$$

$k_i = \langle \gamma, \varepsilon_i \rangle$, ε_i coords. on T
$\in \mathbb{Z}$

3) **Recall:** $\underset{\text{open}}{Y \subset} \underset{\text{proj.}}{X} \xrightarrow{\varphi} \mathbb{P}(M^*)$

$$H^0(X,\mathcal{L}) \supset M = M' \oplus M''$$

$$\begin{tikzcd}
M' & & M'' \\
& V(\lambda) \arrow[ul, "\psi' \wr"] \arrow[ur, "\wr \psi''"'] &
\end{tikzcd}$$

$$M^* \simeq V(\lambda^*) \otimes k^2$$

$s_i' := \psi'(v_i) \in M'$

$s_i'' := \psi''(v_i) \in M''$ $\Longrightarrow$ $s_0', s_0'', s_1', s_1'', \dots, s_n', s_n''$ basis of M

$t^\gamma \cdot s_i' = t^{l_i} \cdot s_i'$, $t^\gamma \cdot s_i'' = t^{l_i} \cdot s_i''$

$l_i \in \langle \gamma, \lambda_i \rangle \in \mathbb{Z}$, $l_0 > l_1, \dots, l_n$

$\forall x \in X :\ \varphi(t^\gamma \cdot x) = t^\gamma \cdot \varphi(x) =$

$= [t^{-l_0} \cdot s_0'(x) : t^{-l_0} \cdot s_0''(x) : \dots : t^{-l_n} \cdot s_n'(x) : t^{-l_n} \cdot s_n''(x)]$

$= [s_0'(x) : s_0''(x) : t^{l_0-l_1} s_1'(x) : t^{l_0-l_1} s_1''(x) : \dots : t^{l_0-l_n} s_n'(x) : t^{l_0-l_n} s_n''(x)]$

$\xrightarrow[t \to 0]{} [s_0'(x) : s_0''(x) : 0 : \dots : 0]$ unless $s_0'(x) = s_0''(x) = 0$

In coord. free terms: $\varphi(t^{\gamma}\cdot x) \xrightarrow[t\to 0]{} [v^* \otimes w]$

$$\begin{array}{c} [v^*\otimes w] \\ \Cap \\ \mathbb{P}(V(\lambda^*)) \times \mathbb{P}^1 \\ \int \text{Segre} \\ \mathbb{P}(M^*) \end{array}$$

$v^* \in V(\lambda^*)$, $\langle v^*, v_i \rangle = \begin{cases} 1, & i=0 \\ 0, & i\neq 0 \end{cases}$

$w \in \mathbb{R}^2$

Segre embedding:

$$\mathbb{P}(V) \times \mathbb{P}(W) \xhookrightarrow{\text{closed}} \mathbb{P}(V \otimes W)$$
$$([v],[w]) \longmapsto [v \otimes w]$$

X proj. $\Rightarrow$ $t^{\gamma}\cdot x \xrightarrow[t\to 0]{} x_0 \in X$, $\varphi(x_0) = [v^* \otimes w]$

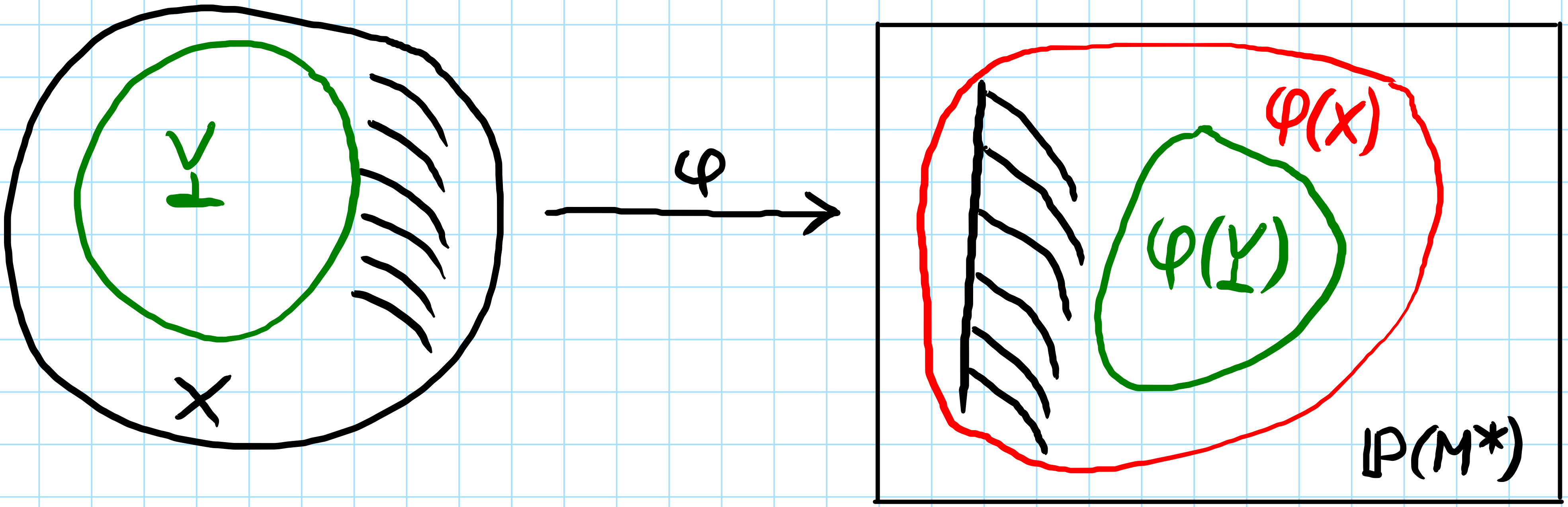
$\forall x \in X \setminus \{s_0' = s_0'' = 0\}$

dense open subset

$X \dashrightarrow \mathbb{P}^1$ non-const. $\Leftarrow$ $s_0' \not\sim s_0''$

$x \longmapsto [s_0'(x) : s_0''(x)]$

Hence: $\varphi(X) \supset [v^*] \times \mathbb{P}^1 \subset \mathbb{P}(V(\lambda^*)) \times \mathbb{P}^1 \subset \mathbb{P}(M^*)$



$$\Rightarrow \quad \varphi(X) \supset \underbrace{G\cdot[v^*] \times \mathbb{P}^1}_{\text{inf. family of } G\text{-orbits}}$$

$$\Rightarrow \quad X \supset \text{inf. many } G\text{-orbits}$$

Contradiction.

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