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Elementary equivalence of Chevalley groups over local rings

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Abstract. It is proved that (elementary) Chevalley groups over local rings with invertible 2 are elementarily equivalent if and only if their types and weight lattices coincide and the initial rings are elementarily equivalent.

Bibliography: 25 titles.

Keywords: Chevalley groups, elementary equivalence, local rings.

Introduction

Two models $\mathscr{U}$ and $\mathscr{U}'$ of the same first order language $\mathscr{L}$ (for example, two groups or two rings) are called *elementarily equivalent* if every proposition φ of the language $\mathscr{L}$ is true in $\mathscr{U}$ if and only if it is true in $\mathscr{U}'$. Any two finite models of the same language are elementarily equivalent if and only if they are isomorphic. Any two isomorphic models are elementarily equivalent, but for infinite models the converse fact is not true. For example, the field $\mathbb{C}$ of complex numbers and the field $\overline{\mathbb{Q}}$ of algebraic numbers are elementarily equivalent, but not isomorphic since they have different cardinalities (for more detailed examples see [1]).

First results about the connection between elementary properties of some models and elementary properties of derivative models were obtained by Maltsev in 1961 [2]. He proved that the groups $G_n(K)$ and $G_m(L)$ (where $G=\mathrm{GL}$, SL, PGL, PSL, $n,m\geqslant 3$, K and L are fields of characteristic 0) are elementarily equivalent if and only if $m=n$ and the fields K and L are elementarily equivalent.

The investigations were continued in 1992, when using the construction of an ultrapower and the isomorphism theorem Beĭdar and Mikhalev [3] put forward a general approach to the elementary equivalence problem for different algebraic structures and generalized Maltsev's theorem to the case when K and L are division rings or associative rings.

In 1998–2005, this author continued looking at several problems of this kind (see [4]–[9]). Maltsev's results were generalized for linear unitary groups over division rings or associative rings with inclusions, and also for Chevalley groups over fields.

An associative ring R with unit is called *local* if it contains exactly one maximal ideal (which coincides with the radical J of this ring). This is equivalent to the fact that the invertible elements of the ring R form an ideal.

In [8] the following results on elementary equivalence of Chevalley groups over local rings were announced.

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Theorem 1. *Let $G_\pi(\Phi,R)$ and $G_{\pi'}(\Phi',R')$ (or $E_\pi(\Phi,R)$ and $E_{\pi'}(\Phi',R')$) be two (elementary) Chevalley groups over local rings R and R' with invertible 2 (in the case of the root system G_2, with invertible 6), and let Φ and Φ' be indecomposable root systems of rank >1. Then from elementary equivalence of these Chevalley groups it follows that $\Phi\cong\Phi'$ and $R\equiv R'$.*

Theorem 2. *Let $G=G_\pi(\Phi,R)$ and $G'=G_{\pi'}(\Phi,R')$ (or $E_\pi(\Phi,R)$ and $E_{\pi'}(\Phi,R')$) be two (elementary) Chevalley groups over elementarily equivalent local rings R and R', where the representations π and π' have isomorphic weight lattices. Then the groups G and G' are elementarily equivalent.*

This paper is concerned with the detailed proofs of these theorems and also the Main Theorem.

Main Theorem. *Let $G=G_\pi(\Phi,R)$ and $G'=G_{\pi'}(\Phi',R')$ (or $E_\pi(\Phi,R)$ and $E_{\pi'}(\Phi',R')$) be two (elementary) Chevalley groups over infinite local rings R and R' with invertible 2 (in the case of the root system G_2 with invertible 6), with indecomposable root systems Φ,Φ' of ranks >1 and with weight lattices Λ and Λ', respectively. Then the groups G and G' are elementarily equivalent if and only if the root systems Φ and Φ' are isomorphic, the rings R and R' are elementarily equivalent, and the lattices Λ and Λ' coincide.*

§ 1. Necessary information about Chevalley groups

The basic notions about root systems, semisimple Lie algebras, Chevalley groups, which will be used in this paper, can be found in the author's papers [9]–[11]. Detailed information about root systems can be found in the books [12] and [13]. More detailed information about semisimple Lie algebras can be found in the book [12]. More detailed information about elementary Chevalley groups is contained in the book [14], and about Chevalley groups (also over rings) in [15] and [16] (see also later references in these papers).

We fix some arbitrary (indecomposable) root system Φ of rank $l\geqslant 2$; we suppose that in this system there are n positive and n negative roots.

Additionally we fix some infinite local ring R with invertible 2 (in the case of G_2 with invertible 6).

We consider an arbitrary Chevalley group $G_\pi(\Phi,R)$ constructed by the root system Φ, a ring R and a representation π of the corresponding Lie algebra. It is known that a Chevalley group is determined by the root system, the ring R and the weight lattice of the representation π. We shall denote this lattice by Λ or Λ_π. If we consider an elementary Chevalley group, we denote it by $E_\pi(\Phi,R)$.

The subgroup of all diagonal (in a standard basis of weight vectors) matrices of the Chevalley group $G_\pi(\Phi,R)$ is called the *standard maximal torus* of $G_\pi(\Phi,R)$ and is denoted by $T_\pi(\Phi,R)$. This group is isomorphic to $\mathrm{Hom}(\Lambda_\pi,R^*)$.

Let us denote by $h(\chi)$ an element of $T_\pi(\Phi,R)$ corresponding to the homomorphism $\chi\in\mathrm{Hom}(\Lambda(\pi),R^*)$.

In particular, $h_\alpha(u)=h(\chi_{\alpha,u})$, $u\in R^*$, $\alpha\in\Phi$, where

$$\chi_{\alpha,u}\colon \lambda\mapsto u^{\langle\lambda,\alpha\rangle},\qquad \lambda\in\Lambda_\pi.$$

The connection between Chevalley groups and the corresponding elementary Chevalley groups is a considerable problem in the theory of Chevalley groups over rings. If for elementary Chevalley groups there exists a convenient system of generators $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, and all relations between these generators are studied, it is not possible to do a similar thing with the Chevalley groups themselves.

If K is an algebraically closed field, then

$$G_\pi(\Phi, R) = E_\pi(\Phi, R)$$

for any representation π. This equality is not true even in the case of fields that are not algebraically closed.

But if G is a universal group and the ring R is *semilocal* (that is, it contains only finitely many maximal ideals), then we have the condition

$$G_{\mathrm{sc}}(\Phi, R) = E_{\mathrm{sc}}(\Phi, R)$$

(see [17]–[20]).

Let us show the difference between Chevalley groups and their elementary subgroups in the case when a ring R is semilocal and the Chevalley group is not universal. In this case

$$G_\pi(\Phi, R) = E_\pi(\Phi, R)T_\pi(\Phi, R)$$

(see [17], [18], [20]), and the elements $h(\chi)$ are connected with elementary generators by the formula

$$h(\chi)x_\beta(\xi)h(\chi)^{-1} = x_\beta(\chi(\beta)\xi). \tag{1}$$

It is known that the subgroup of elementary matrices $E_2(R) = E_{\mathrm{sc}}(A_1, R)$ is not necessarily normal in the special linear group $\mathrm{SL}_2(R) = G_{\mathrm{sc}}(A_1, R)$ (see [21]–[23]). But if Φ is an irreducible root system of rank $l \geqslant 2$, then $E(\Phi, R)$ is always normal in $G(\Phi, R)$. In the case of semilocal rings with $1/2$ it is easy to show that

$$[G(\Phi, R), G(\Phi, R)] = E(\Phi, R).$$

§ 2. Proof of Theorem 2

In the book [1] it is proved that elementary equivalence is preserved under taking direct products, therefore the result below directly follows.

Proposition 1. *If a semisimple Lie algebra $\mathscr{L} = \mathscr{L}_1 \oplus \dots \oplus \mathscr{L}_k$, where the algebras $\mathscr{L}_1, \dots, \mathscr{L}_k$ are simple, R and R' are rings, then the pairwise equivalence of the* (*elementary*) *Chevalley groups*

$$G_{\pi|_{\mathscr{L}_1}}(\mathscr{L}_1, R) \ \ \text{and} \ \ G_{\pi'|_{\mathscr{L}_1}}(\mathscr{L}_1, R'), \quad \dots, \quad G_{\pi|_{\mathscr{L}_k}}(\mathscr{L}_k, R) \ \ \text{and} \ \ G_{\pi'|_{\mathscr{L}_k}}(\mathscr{L}_k, R')$$

implies the elementary equivalence of the (*elementary*) *Chevalley groups $G_\pi(\mathscr{L}, R)$ and $G_{\pi'}(\mathscr{L}, R')$.*

Therefore we need only prove our theorem for simple Lie algebras.

The following theorem holds for arbitrary commutative rings with unit R and R'.

Theorem 3. *If two Chevalley groups $G=G_\pi(\Phi,R)$ and $G'=G_\pi(\Phi,R')$ are constructed by the same complex Lie algebra of type Φ and the same representation π of it, and also by elementarily equivalent rings R and R', then $G\equiv G'$.*

Proof. As we know from the definition of a Chevalley group,

$$\begin{aligned} G&=\bigl\{(a_{i,j})\in M_N(R)\mid p_1(a_{i,j})=p_2(a_{i,j})=\dots=p_m(a_{i,j})=0\bigr\},\\ G'&=\bigl\{(a_{i,j})\in M_N(R')\mid p_1(a_{i,j})=p_2(a_{i,j})=\dots=p_m(a_{i,j})=0\bigr\},\end{aligned}$$

where $p_1,p_2,\dots,p_m$ are some known polynomials with integer coefficients and N is some known integer number.

Suppose that we have some sentence φ of the group language, which is considered on groups G and G'. We translate it to a sentence $\widetilde{\varphi}$ of the ring language in the following way:

the subformula $\forall\, g\quad \psi(g)$ to the subformula

$$\forall\, a^g_{1,1},\dots,a^g_{N,N}\quad \bigl(p_1(a^g_{1,1},\dots,a^g_{N,N})=0\wedge\dots\wedge p_m(a^g_{1,1},\dots,a^g_{N,N})=0 \implies \widetilde{\psi}(a^g_{1,1},\dots,a^g_{N,N})\bigr);$$

the subformula $\exists\, g\quad \psi(g)$ is translated to the subformula

$$\exists\, a^g_{1,1},\dots,a^g_{N,N}\quad \bigl(p_1(a^g_{1,1},\dots,a^g_{N,N})=0\wedge\dots\wedge p_m(a^g_{1,1},\dots,a^g_{N,N})=0 \wedge \widetilde{\psi}(a^g_{1,1},\dots,a^g_{N,N})\bigr);$$

the subformula $g=h$ is translated to the subformula

$$a^g_{1,1}=a^h_{1,1}\wedge\dots\wedge a^g_{N,N}=a^h_{N,N};$$

the subformula $g=h\cdot f$ is translated to the subformula

$$\bigwedge_{i,j=1}^N\Bigl(a^g_{i,j}=\sum_{k=1}^n a^h_{i,k}\cdot a^f_{k,j}\Bigr).$$

It is clear that $G(R)\vDash\varphi$ if and only if $R\vDash\widetilde{\varphi}$.

Therefore, if rings R and R' are elementarily equivalent, then for any sentence φ of the group language

$$G\vDash\varphi\iff R\vDash\widetilde{\varphi}\iff R'\vDash\widetilde{\varphi}\iff G'\vDash\varphi.$$

So $G\equiv G'$ and the proof is complete.

The following theorem holds for local and semilocal rings R and R' with $1/2$.

Theorem 4. *If two elementary Chevalley groups $E=E_\pi(R,\Phi)$ and $E'=E_\pi(R',\Phi)$ are constructed by the same complex Lie algebra of type Φ and the same representation π of it, and also by elementarily equivalent semilocal rings R and R' with $1/2$, then $E\equiv E'$.*

This theorem clearly follows from the previous one and Proposition 2 in the next section.

§ 3. Transfer to an elementary adjoint group

Here we want to prove that if two (elementary) Chevalley groups are elementarily equivalent, then their root systems coincide, initial rings are elementarily equivalent, weight lattices coincide.

For our convenience we suppose all rings to be infinite. Note that this assumption does not limit the generality of our result, since in the case of two finite rings R and R' the (elementary) Chevalley groups $G(R)$ and $G(R')$ ($E(R)$ and $E(R')$) are finite, that is,

$$G(R) \equiv G(R') \quad \Longrightarrow \quad G(R) \cong G(R').$$

Therefore we can refer to the results proved earlier that show that in this case $\Phi \cong \Phi'$ and $R \cong R'$, that is, $\Phi \cong \Phi'$ and $R \equiv R'$.

First we demonstrate the following result.

Proposition 2. *If two Chevalley groups G and G' are elementarily equivalent, then their elementary subgroups E and E' are also elementarily equivalent.*

Lemma. *Let $G = G_\pi(\Phi, R)$ be a Chevalley group, $E = E_\pi(\Phi, R)$ its elementary subgroup, R a semilocal ring with $1/2$ (and with $1/3$, if $\Phi \cong G_2$). Then $E = [G, G]$ and there exists a number N depending on Φ, but not on R (nor on the representation π) such that every element of the group E is a product of at most N commutators of the group G.*

Proof. Let a root system Φ have rank l.

If R is a (semi)local ring, then for every element $g \in E$ we have the *Gauss decomposition* (see [18])

$$g = uhvu', \qquad u, u' \in U, \quad v \in V, \quad h \in H.$$

It is known (see [14]) that elements u, u' and v are represented as the products of at most n (the number of positive roots of the system Φ) elements $x_\alpha(t)$, and h is a product of at most l elements of the form $h_\alpha(t)$, which, in their turn, are products of at most six elements $x_\alpha(t)$.

Therefore every element of the group $E_\pi(\Phi, R)$ is a product of at most $6l + 3n$ elements $x_\alpha(t)$, where n is the number of positive roots, which depends on l as shown in Table 1.

Table 1

root-system type	rank	n
A_l	l	$(l^2 + l)/2$
B_l	l	l^2
C_l	l	l^2
D_l	l	$l^2 - l$
E_6	6	36
E_7	7	63
E_8	8	120
F_4	4	24
G_2	2	6

Now we only need to show that every $x_\alpha(t)$ is a product of some (bounded above) number of commutators. To do this we must consider root types separately.

Namely, if we consider any of the root systems $A_l, l \geqslant 2$, $D_l, l \geqslant 4$, E_l $(l = 6, 7, 8)$, then every root in it can be included in some root system of type A_2, that is, we can suppose that $\alpha = \alpha_i + \alpha_j$ for some roots α_i and α_j; these three roots form the set of positive roots of the system A_2. In this case

$$[x_{\alpha_i}(t), x_{\alpha_j}(s)] = x_\alpha(\pm ts),$$

therefore $x_\alpha(t)$ is a commutator.

For the root systems B_l $(l \geqslant 2)$, C_l $(l \geqslant 3)$, F_4 every root can be considered as a long or short root of the system B_2. In this system every root has the form

$$\pm e_1, \pm e_2 \quad \text{or} \quad \pm e_1 \pm e_2.$$

Note that

1) $\pm e_1 \pm e_2 = (\pm e_1) + (\pm e_2)$ and no linear combination of the roots $\pm e_1$ and $\pm e_2$ with natural coefficients different from $(\pm e_1) + (\pm e_2)$ is a root, therefore

$$[x_{\pm e_1}(t), x_{\pm e_2}(s)] = x_{\pm e_1 \pm e_2}(\pm 2ts),$$

that is, $x_{\pm e_1 \pm e_2}(t)$ is a commutator (since $1/2 \in R$);

2) we have $\pm e_1 = (\pm e_1 - e_2) + e_2$, so

$$[x_{\pm e_1 - e_2}(t), x_{e_2}(s)] = x_{\pm e_1}(\pm 2ts) x_{\pm e_1 + e_2}(cts^2),$$

therefore $x_{\pm e_1}(t)$ is a product of two commutators.

Thus any element $x_\alpha(t)$ is a product of at most 2 commutators.

For the root system G_2 any root has the form

$$\begin{gathered}\pm(e_1 - e_2), \qquad \pm(e_1 - e_3), \qquad \pm(e_2 - e_3),\\ \pm(2e_1 - e_2 - e_3), \qquad \pm(2e_2 - e_1 - e_3), \qquad \pm(2e_3 - e_1 - e_2).\end{gathered}$$

We have

$$\begin{aligned}\pm(2e_1 - e_2 - e_3) &= \pm(2e_2 - e_1 - e_3) + (\mp(2e_3 - e_1 - e_2)),\\ \pm(2e_2 - e_1 - e_3) &= \pm(2e_3 - e_1 - e_2) + (\pm(2e_1 - e_2 - e_3)),\\ \pm(2e_3 - e_1 - e_2) &= \pm(2e_1 - e_2 - e_3) + (\mp(2e_2 - e_1 - e_3)).\end{aligned}$$

Consequently, for any long root α the element $x_\alpha(t)$ is a commutator.

Then we have

$$e_1 - e_2 = (e_1 - e_3) + (e_3 - e_2),$$

so that

$$[x_{e_1 - e_3}(t), x_{e_3 - e_2}(s)] = x_{e_1 - e_2}(\pm 3ts) x_{-2e_2 + e_1 + e_3}(c_1 ts^2) x_{2e_1 - e_2 - e_3}(c_2 t^2 s).$$

Therefore, for $1/3 \in R$ every element $x_{e_i - e_j}(t)$ is a product of three commutators. Consequently, every element $x_\alpha(t)$ is a product of at most three commutators.

So we see that any element of an elementary Chevalley group $E_\pi(\Phi, R)$ is a product of not more than M commutators of the group $G_\pi(\Phi, R)$, where the number M depends only on the root system Φ. The proof is complete.

Proof of Proposition 2. Consider the set of sentences

$$\mathrm{Define}_M := \forall\, x_1,\dots,x_M, y_1,\dots,y_M, z_1,\dots,z_M, t_1,\dots,t_M\ \exists\, v_1,\dots,v_M, u_1,\dots,u_M \\ \big(([x_1,y_1]\cdots[x_M,y_M])([z_1,t_1]\cdots[z_M,t_M]) = [u_1,v_1]\cdots[u_M,v_M]\big).$$

Every such sentence states that any element of the commutant of a group under consideration is a product of at most M commutators. We know that for the groups G and G' there exists (the same, since they are elementarily equivalent) M such that the sentence Define_M holds in both groups. In this case the formula

$$\mathrm{Commut}_M(x) := \exists\, u_1,\dots,u_M, v_1,\dots,v_M \quad \big(x = [u_1,v_1]\cdots[u_M,v_M]\big)$$

defines in both groups G and G' subgroups E and E', respectively, therefore these subgroups are elementarily equivalent, which completes the proof.

So we see that if we have two elementarily equivalent Chevalley groups E and E', we also have two elementarily equivalent adjoint Chevalley groups E_{ad} and E'_{ad} that are central quotients of the original groups.

§ 4. Factorization for local rings

Since the radical J is the unique maximal (that is, the greatest proper) ideal of a ring R, the subgroup $E_J = E_{\mathrm{ad}}(\Phi, R, J)$ generated by $x_\alpha(t)$, $t \in J$, is the greatest proper normal subgroup of $E = E_{\mathrm{ad}}(\Phi, R)$ (see [24]).

Therefore, if we can show that the subgroup E_J is definable in the group E, then, factorizing E by E_J we obtain the Chevalley group $\widetilde{E} \cong E_{\mathrm{ad}}(R/J)$, that is, the Chevalley group over a field, and after that refer to the results proved in [9] on elementary equivalence of Chevalley groups over fields.

Proposition 3. *The subgroup $E_J = E_{\mathrm{ad}}(\Phi, R, J)$ is definable in $E = E_{\mathrm{ad}}(\Phi, R)$.*

Proof. Consider in E elements A satisfying the formula

$$\mathrm{NoInv}_N(A) = \varphi_1(A) \wedge \varphi_2(A),$$

where the formula

$$\varphi_1(A) := \bigwedge_{i,j=0}^{N} \forall\, X_1,\dots,X_i, Y_1,\dots,Y_j \quad \Big(\bigvee_{k=0}^{N} \exists\, Z_1,\dots,Z_k \\ \big((1\cdot X_1AX_1^{-1}\cdots X_iAX_i^{-1})\cdot(1\cdot Y_1AY_1^{-1}\cdots Y_jAY_j^{-1}) \\ = (1\cdot Z_1AZ_1^{-1}\cdots Z_kAZ_k^{-1})\big)\Big)$$

means that elements $X_1AX_1^{-1}\cdots X_kAX_k^{-1}$ for $k \leqslant N$ form a subgroup of E, and the formula

$$\varphi_2(A) = \bigwedge_{i=0}^{N} \forall\, X_1,\dots,X_i, X \quad \Big(\bigvee_{k=0}^{N} \exists\, Z_1,\dots,Z_k \\ \big(X(X_1AX_1^{-1}\cdots X_iAX_i^{-1})X^{-1} = Z_1AZ_1^{-1}\cdots Z_kAZ_k^{-1}\big)\Big) \\ \wedge\, \exists\, X \quad \Big(\bigwedge_{i=0}^{N} \forall\, X_1,\dots,X_i \quad (X \neq X_1AX_1^{-1}\cdots X_iAX_i^{-1})\Big)$$

means that this subgroup is normal and does not coincide with the whole of E.

If for a given A and some N this formula is true, than it means that the minimal normal subgroup of E containing A is a proper subgroup of E. As we know, every proper normal subgroup of E is contained in E_J, therefore $A \in E_J$.

On the other hand, for any $A = x_\alpha(u), u \in J$, $\alpha \in \Phi$, the formula $\mathrm{NoInv}_N(A)$ is true for some rather big N (which can be chosen unique for a given root system).

Let us fix the minimal natural N such that if for some A the sentence $\mathrm{NoInv}_N(A)$ does not hold, then for this A no sentence $\mathrm{NoInv}_p(A)$, $p > N$, holds.

Now consider M such that in our group the following sentence is true:

$$\mathrm{Norm}_{M,N} = \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4,$$

where

$$\psi_1 := \bigwedge_{i,j=0}^{M} \forall\, X_1, \dots, X_i, Y_1, \dots, Y_j$$
$$\Big(\mathrm{NoInv}_N(X_1) \wedge \dots \wedge \mathrm{NoInv}_N(X_i) \wedge \mathrm{NoInv}_N(Y_1) \wedge \dots \wedge \mathrm{NoInv}_N(Y_j)$$
$$\Longrightarrow \bigvee_{k=0}^{M} \exists\, Z_1, \dots, Z_k$$
$$\big(\mathrm{NoInv}_N(Z_1) \wedge \dots \wedge \mathrm{NoInv}_N(Z_k) \wedge X_1 \cdots X_i Y_1 \cdots Y_j = Z_1 \cdots Z_k\big)\Big)$$

(this sentence means that the products $X_1 \cdots X_k$, $k \leqslant M$, of elements of E satisfying $\mathrm{NoInv}_N(X)$ form a subgroup of E);

$$\psi_2 := \bigwedge_{i=0}^{M} \forall\, X_1, \dots, X_i, X \quad \Big(\mathrm{NoInv}_N(X_1) \wedge \dots \wedge \mathrm{NoInv}_N(X_i)$$
$$\Longrightarrow \bigvee_{j=0}^{M} \exists\, Y_1, \dots, Y_j$$
$$\big(\mathrm{NoInv}_N(Y_1) \wedge \dots \wedge \mathrm{NoInv}_N(Y_j) \wedge X(X_1 \cdots X_i)X^{-1} = Y_1 \cdots Y_j\big)\Big)$$

(this sentence means that the said subgroup is normal);

$$\psi_3 := \exists\, X_1, X_2 \quad \big(\mathrm{NoInv}_N(X_1) \wedge \mathrm{NoInv}_N(X_2) \wedge X_1 \neq X_2\big)$$

(the subgroup is not trivial);

$$\psi_4 := \exists\, X \quad \Big(\bigwedge_{i=0}^{M} \forall\, X_1, \dots, X_i$$
$$\big(\mathrm{NoInv}_N(X_1) \wedge \dots \wedge \mathrm{NoInv}_N(X_i) \implies X \neq X_1 \cdots X_i\big)\Big)$$

(the subgroup does not coincide with the whole group E).

With the help of this formula we find M such that every element of the group E_J is generated by at most M elements $x_\alpha(u)$, $u \in J$.

Now the formula

$$\mathrm{Normal}_{M,N}(X) := \bigwedge_{i=0}^{M} \exists\, X_1, \dots, X_i$$
$$\big(\mathrm{NoInv}_N(X_1) \wedge \dots \wedge \mathrm{NoInv}_N(X_i) \wedge X = X_1 \cdots X_i\big)$$

defines in E the subgroup E_J, which finishes the proof.

For the moment we obtain the following implication:

$$\begin{aligned} &G_\pi(\Phi, R) \equiv G_{\pi'}(\Phi', R') \quad \Longrightarrow \quad E_\pi(\Phi, R) \equiv E_{\pi'}(\Phi', R') \\ &\quad\Longrightarrow \quad E_{\mathrm{ad}}(\Phi, R) \equiv E_{\mathrm{ad}}(\Phi', R') \quad \Longrightarrow \quad E_{\mathrm{ad}}(\Phi, R/J) \equiv E_{\mathrm{ad}}(\Phi', R'/J') \\ &\quad\Longrightarrow \quad \Phi \equiv \Phi'. \end{aligned}$$

The last implication follows from the main theorem of [9] (elementary equivalence of Chevalley groups over fields).

Now we can always suppose that the root system of our Chevalley group is known.

§ 5. Formulae for the Gauss decomposition of Chevalley groups

Recall that we have a root system Φ of rank > 1. The set of simple roots is denoted by Δ, the set of positive roots is denoted by Φ^+. The subgroup $U = U(R)$ of the Chevalley group G (E) is generated by the elements $x_\alpha(t)$, $\alpha \in \Phi^+$, $t \in R$, and the subgroup $V = V(R)$ is generated by the elements $x_{-\alpha}(t)$, $\alpha \in \Phi^+$ $t \in R$.

For invertible $t \in R^*$ we denote $x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ by $w_\alpha(t)$, and we denote $w_\alpha(t)w_\alpha(1)^{-1}$ by $h_\alpha(t)$.

The group $H = H(R)$ is generated by all $h_\alpha(t)$, $\alpha \in \Phi$, $t \in R^*$.

Proposition 4. (i) *Every element x of a Chevalley group G (E) over a local ring R can be represented in the form*

$$x = utvu' \quad (\textit{resp. } x = uhvu'),$$

where $u, u' \in U(R)$, $v \in V(R)$, $t \in T(R)$, $h \in H(R)$.

(ii) *For decompositions $x_1 = u_1t_1v_1u_1'$ and $x_2 = u_2t_2v_2u_2'$, where*

$$u_i = x_{\alpha_1}(t_1^{(i)}) \cdots x_{\alpha_n}(t_n^{(i)}), \qquad u_i' = x_{\alpha_1}(s_1^{(i)}) \cdots x_{\alpha_n}(s_n^{(i)}),$$
$$v_i = x_{-\alpha_1}(r_1^{(i)}) \cdots x_{-\alpha_n}(r_n^{(i)}), \qquad t_i = h_{\alpha_1}(\xi_1^{(i)}) \cdots h_{\alpha_l}(\xi_l^{(i)}),$$
$$i = 1, 2,$$

there exists a first order formula of the ring language

$$\varphi(t_1^{(1)}, \dots, t_n^{(1)}, t_1^{(2)}, \dots, t_n^{(2)}, s_1^{(1)}, \dots, s_n^{(1)}, s_1^{(2)}, \dots, s_n^{(2)},$$
$$r_1^{(1)}, \dots, r_n^{(1)}, r_1^{(2)}, \dots, r_n^{(2)}, \xi_1^{(1)}, \dots, \xi_n^{(1)}, \xi_1^{(2)}, \dots, \xi_n^{(2)}),$$

which is true if and only if

$$x_1 = x_2.$$

(iii) *Similarly, for decompositions* $x_1 = u_1t_1v_1u_1'$, $x_2 = u_2t_2v_2u_2'$ *and* $x_3 = u_3t_3v_3u_3'$, *where*

$$u_i = x_{\alpha_1}(t_1^{(i)})\cdots x_{\alpha_n}(t_n^{(i)}), \qquad u_i' = x_{\alpha_1}(s_1^{(i)})\cdots x_{\alpha_n}(s_n^{(i)}),$$
$$v_i = x_{-\alpha_1}(r_1^{(i)})\cdots x_{-\alpha_n}(r_n^{(i)}), \qquad t_i = h_{\alpha_1}(\xi_1^{(i)})\cdots h_{\alpha_l}(\xi_l^{(i)}),$$
$$i = 1, 2, 3,$$

there exists a first order formula of the ring language

$$\psi(t_1^{(i)}, \dots, t_n^{(i)}, s_1^{(i)}, \dots, s_n^{(i)}, r_1^{(i)}, \dots, r_n^{(i)}, \xi_1^{(i)}, \dots, \xi_n^{(i)}),$$

that is true if and only if

$$x_3 = x_1 \cdot x_2.$$

Proof. (i) We only need to prove that

$$TUVUx_\alpha(t) \in TUVU \qquad \forall\, \alpha \in \Phi^*, \quad \forall\, t \in R, \tag{2}$$
$$TUVUh(\chi) \in TUVU \qquad \forall\, h(\chi) \in T, \tag{3}$$
$$TUVUx_{-\alpha}(\pm 1) \in TUVU \qquad \forall\, \alpha \in \Delta, \tag{4}$$

since

$$\begin{aligned} x_{-\alpha}(t) &= w_\alpha(\pm 1)x_\alpha(t)w_\alpha(\pm 1)^{-1} \\ &= x_\alpha(\pm 1)x_{-\alpha}(\mp 1)x_\alpha(\pm 1)x_\alpha(t)x_\alpha(\mp 1)x_{-\alpha}(\pm 1)x_\alpha(\mp 1), \end{aligned}$$

and $w_\alpha(1) = w_{\alpha_{i_1}}(1)\cdots w_{\alpha_{i_k}}(1)$, where $\alpha_{i_1}, \dots, \alpha_{i_k} \in \Delta$.

Relation (2) is absolutely clear because $Ux_\alpha(t) \subset U \ \ \forall\, \alpha \in \Phi^+, \forall\, t \in R$.

Relation (3) follows from

$$h(\chi)x_\alpha(t)h(\chi)^{-1} = x_\alpha(\chi(\alpha)t).$$

Now we shall prove relation (4).

Without loss of generality we suppose that $\alpha = \alpha_1 \in \Delta = \{\alpha_1, \dots, \alpha_l\}$. Then

$$\begin{aligned} B &= h(\chi)x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n)x_{\alpha_1}(s_1)\cdots x_{\alpha_n}(s_n)x_{-\alpha_1}(1) \\ &= h(\chi)x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n) \\ &\quad \times x_{\alpha_1}(s_1)x_{-\alpha_1}(1)x_{\alpha_2}(s_2')\cdots x_{\alpha_n}(s_n'), \end{aligned} \tag{5}$$

where $s_2', \dots, s_n'$ are polynomials with integer coefficients of $s_2, \dots, s_n$.

If $1 + s_1 \in R^*$, then the equality is continued as follows:

$$\begin{aligned} B &= h(\chi)x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n)x_{-\alpha_1}(\mu)h_{\alpha_1}(\eta)x_{\alpha_1}(s_1')\cdots x_{\alpha_n}(s_n') \\ &= h'(\chi)x_{\alpha_1}(t_1')\cdots x_{\alpha_n}(t_n')x_{-\alpha_1}(r_1')\cdots x_{-\alpha_n}(r_n')x_{\alpha_1}(s_1')\cdots x_{\alpha_n}(s_n'), \end{aligned}$$

where μ, η, $s_1', \dots, s_n'$, $t_1', \dots, t_n'$ $r_1', \dots, r_n'$, $h'(\chi)$ are Laurent polynomials (with integer coefficients) of the old variables.

If $1 + s_1 \notin R^*$, then $s_1 \in R^*$ and

$$\begin{aligned} x_{\alpha_1}(s_1)x_{-\alpha_1}(1) &= x_{\alpha_1}(1 + s_1)x_{\alpha_1}(-1)x_{-\alpha_1}(1)x_{\alpha_1}(-1)x_{\alpha_1}(1) \\ &= x_{\alpha_1}(1 + s_1)w_{\alpha_1}(-1)x_{\alpha_1}(1) = w_{\alpha_1}(-1)x_{-\alpha_1}(-1 - s_1)x_{\alpha_1}(1). \end{aligned}$$

Then equality (5) is continued as follows:

$$\begin{aligned} B &= h(\chi)x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n)\\ &\qquad \times w_{\alpha_1}(-1)x_{-\alpha_1}(-1-s_1)x_{\alpha_1}(1)x_{\alpha_2}(s_2')\cdots x_{\alpha_n}(s_n')\\ &= h(\chi)x_{\alpha_2}(t_2')\cdots x_{\alpha_n}(t_n')x_{\alpha_1}(t_1)x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n)\\ &\qquad \times w_{\alpha_1}(-1)x_{-\alpha_1}(1)x_{\alpha_2}(s_2')\cdots x_{\alpha_n}(s_n'). \end{aligned}$$

Now we use the equality

$$x_{\alpha_1}(t_1)x_{-\alpha_1}(r_1) = x_{\alpha_1}(1+t_1)x_{-\alpha_1}(1)x_{\alpha_1}(1-r_1)w_{\alpha_1}(1).$$

We obtain the following continuation of (5):

$$\begin{aligned} B &= h(\chi)x_{\alpha_2}(t_2')\cdots x_{\alpha_n}(t_n')x_{\alpha_1}(1+t_1)x_{-\alpha_1}(1)x_{\alpha_1}(1-r_1)\\ &\qquad \times w_{\alpha_1}(1)x_{-\alpha_2}(r_2)\cdots x_{-\alpha_n}(r_n)\\ &\qquad \times w_{\alpha_1}(-1)x_{-\alpha_1}(-1-s_1)x_{\alpha_1}(1)x_{\alpha_2}(s_2')\cdots x_{\alpha_n}(s_n')\\ &= h(\chi)x_{\alpha_1}(t_1'')\cdots x_{\alpha_n}(t_n'')x_{-\alpha_1}(1)x_{\alpha_1}(1-r_1)x_{-\alpha_2}(r_2')\cdots x_{-\alpha_n}(r_n')\\ &\qquad \times x_{-\alpha_1}(-1-s_1)x_{\alpha_1}(1)x_{\alpha_2}(s_2')\cdots x_{\alpha_n}(s_n')\\ &= h'(\chi)x_{\alpha_1}(\bar t_1)\cdots x_{\alpha_n}(\bar t_n)x_{-\alpha_1}(\bar r_1)\cdots x_{-\alpha_n}(\bar r_n)x_{\alpha_1}(\bar s_1)\cdots x_{\alpha_n}(\bar s_n), \end{aligned}$$

where all the new parameters are rational functions with integer coefficients of the old parameters.

Therefore assertion (i) is proved.

(ii) and (iii). For $\alpha\in\Phi^+$, $\alpha=\alpha_i$, let

$$\begin{aligned} &\psi^{\alpha,+}(t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_n;\\ &\qquad \bar t_1,\dots,\bar t_n,\bar r_1,\dots,\bar r_n,\bar s_1,\dots,\bar s_n,\bar\xi_1,\dots,\bar\xi_l;t) \end{aligned}$$

be the formula

$$\begin{aligned} &t_1=\bar t_1\wedge\cdots\wedge t_n=\bar t_n\wedge r_1=\bar r_1\wedge\cdots\wedge r_n\\ &\quad =\bar r_n\wedge\xi_1=\bar\xi_1\wedge\cdots\wedge\xi_l=\bar\xi_l\wedge\eta^{\alpha,+}(s_1,\dots,s_n;\bar s_1,\dots,\bar s_n;t), \end{aligned}$$

where

$$\eta^{\alpha,+}(\,\cdot\,)=\eta_1^{\alpha,+}(s_1,\dots,s_n,\bar s_1,t)\wedge\cdots\wedge\eta_n^{\alpha,+}(s_1,\dots,s_n,\bar s_n,t),$$

and the formula $\eta_j^{\alpha_i,+}(s_1,\dots,s_n,\bar s_j,t)$ has the form $s_j=p(s_1,\dots,s_n,t)$ and p is a polynomial of $n+1$ variables with integer coefficients such that this formula is true if and only if

$$\begin{aligned} &x_{\alpha_1}(s_1)\cdots x_{\alpha_n}(s_n)x_{\alpha_i}(t)\\ &\qquad = x_{\alpha_1}(\,\cdot\,)x_{\alpha_2}(\,\cdot\,)\cdots x_{\alpha_{j-1}}(\,\cdot\,)x_{\alpha_j}(\bar s_j)x_{\alpha_{j+1}}(\,\cdot\,)\cdots x_{\alpha_n}(\,\cdot\,). \end{aligned}$$

So we see that the formula $\psi^{\alpha,+}(\,\cdot\,)$ holds if and only if

$$\bar x = h_{\bar\xi_1\ldots\bar\xi_l}(\chi)x_{\alpha_1}(\bar t_1)\cdots x_{\alpha_n}(\bar t_n)x_{-\alpha_1}(\bar r_1)\cdots x_{-\alpha_n}(\bar r_n)x_{\alpha_1}(\bar s_1)\cdots x_{\alpha_n}(\bar s_n)$$

and for

$$x = h_{\xi_1\ldots\xi_l}(\chi)x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n)x_{\alpha_1}(s_1)\cdots x_{\alpha_n}(s_n)$$

we have the equality
$$\overline{x} = x \cdot x_\alpha(t).$$

Now let
$$h(\chi_{\lambda_1\dots\lambda_l}) \in T(R).$$

Suppose the formula
$$\psi^T(t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_n;\\ \overline{t}_1,\dots,\overline{t}_n,\overline{r}_1,\dots,\overline{r}_n,\overline{s}_1,\dots,\overline{s}_n,\overline{\xi}_1,\dots,\overline{\xi}_l;\lambda_1,\dots,\lambda_l)$$
is true for
$$\overline{x} = x \cdot h(\chi_{\lambda_1\dots\lambda_l}).$$

It has the form
$$\overline{\xi}_1 = \xi_1\cdot\lambda_1 \wedge \dots \wedge \overline{\xi}_l = \xi_l\cdot\lambda_l \wedge \overline{t}_1 = \psi^T_{1,1}(t_1,\lambda_1,\dots,\lambda_l) \wedge\dots\wedge \overline{t}_n\\ = \psi^T_{1,n}(t_n,\lambda_1,\dots,\lambda_l)\wedge \overline{r}_1 = \psi^T_{2,1}(r_1,\lambda_1,\dots,\lambda_l)\wedge\dots\wedge\overline{r}_n\\ = \psi^T_{2,n}(r_n,\lambda_1,\dots,\lambda_l)\wedge\overline{s}_1 = \psi^T_{3,1}(s_1,\lambda_1,\dots,\lambda_l)\wedge\dots\wedge\psi^T_{3,n}(s_n,\lambda_1,\dots,\lambda_l),$$
where $\psi^T_{i,j}$ is a Laurent polynomial (a monomial) with integer coefficients of its arguments.

Finally, for $x_{-\alpha_i}(1)$ and $x_{-\alpha_i}(-1)$, where $\alpha_i \in \Delta$ in the proof of (i) we have shown that there exist formulae $\psi^{i,1}(\dots)$ and $\psi^{i,-1}(\dots)$ of the variables
$$t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_l,\overline{t}_1,\dots,\overline{t}_n,\overline{r}_1,\dots,\overline{r}_n,\overline{s}_1,\dots,\overline{s}_n,\overline{\xi}_1,\dots,\overline{\xi}_l,$$
that hold in the cases $\overline{x} = x\cdot x_{-\alpha_i}(1)$ and $\overline{x} = x\cdot x_{-\alpha_i}(-1)$, respectively.

Now let us construct a formula $\psi^{w_i}(\,\cdot\,)$ that holds for $\overline{x} = x\cdot w_{\alpha_i}$, $\alpha_i\in\Delta$.

It can be constructed as follows:
$$\psi^{w_i}(t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_l,\\ \overline{t}_1,\dots,\overline{t}_n,\overline{r}_1,\dots,\overline{r}_n,\overline{s}_1,\dots,\overline{s}_n,\overline{\xi}_1,\dots,\xi_l)\\ = \exists\, t'_1,\dots,t'_n,t''_1,\dots,t''_n,r'_1,\dots,r'_n,r''_1,\dots,r''_n,s'_1,\dots,s'_n,s''_1,\dots,s''_n\in R\\ \exists\,\xi'_1,\dots,\xi'_l,\xi''_1,\dots\xi''_l\in R^*\\ \psi^{\alpha_i,+}(t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_l,\\ t'_1,\dots,t'_n,r'_1,\dots,r'_n,s'_1,\dots,s'_n,\xi'_1,\dots,\xi_l;1)\\ \wedge\,\psi^{i,-1}(t'_1,\dots,t'_n,r'_1,\dots,r'_n,s'_1,\dots,s'_n,\xi'_1,\dots,\xi'_l,\\ t''_1,\dots,t''_n,r'_1,\dots,r''_n,\xi''_1,\dots,\xi''_l)\\ \wedge\,\psi^{\alpha_i,+}(t''_1,\dots,t''_n,r''_1,\dots,r''_n,s''_1,\dots,s''_n,\xi''_1,\dots,\xi''_l,\\ \overline{t}_1,\dots,\overline{t}_n,\overline{r}_1,\dots,\overline{r}_n,\overline{s}_1,\dots,\overline{s}_n,\overline{\xi}_1,\dots,\overline{\xi}_n;1).$$

In the group language it states:
$$\exists\, x',x''\quad x' = x\cdot x_{\alpha_i}(1)\wedge x'' = x'\cdot x_{-\alpha_i}(-1)\wedge\overline{x} = x''\cdot x_{\alpha_i}(1).$$

If $\alpha\in\Phi^+\setminus\Delta$, then to construct a formula $\psi^{w_\alpha}(\,\cdot\,)$ that holds for $\overline{x} = x\cdot w_\alpha$ we decompose w_α into the product of simple reflections $w_\alpha = w_{i_1}\dots w_{i_k}$ and apply consecutively multiplication by $w_{i_1},\dots,w_{i_k}$.

Now let us write a formula

$$\psi^{\alpha,-}(t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_l,\\ \overline{t}_1,\dots,\overline{t}_n,\overline{r}_1,\dots,\overline{r}_n,\overline{s}_1,\dots,\overline{s}_n,\overline{\xi}_1,\dots,\overline{\xi}_l;t)$$

that holds for

$$\overline{x}=x\cdot x_{-\alpha}(t).$$

It has the form

$$\begin{aligned}&\exists\, t_1',\dots,t_n',t_1'',\dots,t_n'',r_1',\dots,r_n',r_1'',\dots,r_n'',s_1',\dots,s_n',s_1'',\dots,s_n''\in R\\ &\quad\exists\,\xi_1',\dots,\xi_l',\xi_1'',\dots,\xi_l''\in R^*\\ &\psi^{w_\alpha}(t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_l,\\ &\qquad t_1',\dots,t_n',r_1',\dots,r_n',s_1',\dots,s_n',\xi_1',\dots,\xi_l')\\ &\quad\wedge\psi^{\alpha,+}(t_1',\dots,t_n',r_1',\dots,r_n',s_1',\dots,s_n',\xi_1',\dots,\xi_l',\\ &\qquad\quad t_1'',\dots,t_n'',r_1'',\dots,r_n'',s_1'',\dots,s_n'',\xi_1'',\dots,\xi_l'';-t)\\ &\quad\wedge\psi^{w_\alpha}(t_1'',\dots,t_n'',r_1'',\dots,r_n'',s_1'',\dots,s_n'',\xi_1'',\dots,\xi_l'',\\ &\qquad\quad \overline{t}_1,\dots,\overline{t}_n,\overline{r}_1,\dots,\overline{r}_n,\overline{s}_1,\dots,\overline{s}_n,\overline{\xi}_1,\dots,\overline{\xi}_l).\end{aligned}$$

Now we can easily write a formula

$$\psi^{\otimes}(t_1,\dots,t_n,r_1,\dots,r_n,s_1,\dots,s_n,\xi_1,\dots,\xi_l,\overline{t}_1,\dots,\overline{t}_n,\\ \overline{r}_1,\dots,\overline{r}_n,\overline{s}_1,\dots,\overline{s}_n,\overline{\xi}_1,\dots,\overline{\xi}_l;t_1',\dots,t_n',r_1',\dots,r_n',s_1',\dots,s_n',\xi_1',\dots,\xi_l')$$

that holds for $\overline{x}=x\cdot x'$ as a composition of the formulae $\psi^{\alpha,+}$, $\psi^{\alpha,-}$, ψ^T obtained above.

But since writing an element $x\in G$ in the form $x=tuvu'$ is not unique, we obtain a formula that holds for $x_3=x_1\cdot x_2$ only for some certain forms for x_1,x_2,x_3.

However, we need these parameters to satisfy the formula $\widetilde{\psi}^{\otimes}(\,\cdot\,)$ if and only if they define elements $x_1,x_2,x_3\in G$ such that $x_3=x_1\cdot x_2$, that is, if there exist parameters defining x_3' such that the forms for x_3 and x_3' define the same elements, and also

$$\psi^{\otimes}(x_1,x_2,x_3).$$

The fact that two elements x_1 and x_2 are equal means that $x_1\cdot x_2^{-1}=1$. We find a formula expressing that an element is equal to 1:

$$x=x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)h(\chi_{\xi_1\dots\xi_l})x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n)x_{\alpha_1}(s_1)\cdots x_{\alpha_n}(s_n)=1.$$

Let $utvu'=1$, where $u,u'\in U$, $t\in T$, $v\in V$. Then $TV\ni tv=u^{-1}u'^{-1}\in U$. Since $TV\cap U=1$, we have $tv=1\wedge u'u=1$.

Let $\psi^U(t_1,\dots,t_n,t_1',\dots,t_n',\overline{t}_1,\dots\overline{t}_n)$ be a formula that holds if and only if

$$x_{\alpha_1}(\overline{t}_1)\cdots x_{\alpha_n}(\overline{t}_n)=x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)x_{\alpha_1}(t_1')\cdots x_{\alpha_n}(t_n').$$

Then a formula expressing that x is equal to the unit has the form

$$\begin{aligned}&\psi^{(1)}(t_1,\dots,t_n,\xi_1,\dots,\xi_l,r_1,\dots,r_n,s_1,\dots,s_n)\\ &\qquad=(\xi_1=1)\wedge\dots\wedge(\xi_l=1)\wedge r_1=1\wedge\dots\wedge r_n\\ &\qquad=1\wedge\psi^u(t_1,\dots,t_n,s_1,\dots,s_n,1,\dots,1).\end{aligned}$$

To find an inverse element to

$$x = x_{\alpha_1}(t_1)\cdots x_{\alpha_n}(t_n)h(\chi_{\xi_1\cdots\xi_l})x_{-\alpha_1}(r_1)\cdots x_{-\alpha_n}(r_n)x_{\alpha_1}(s_1)\cdots x_{\alpha_n}(s_n),$$

we express x^{-1} in the form $UTVU$:

$$\begin{aligned}
x^{-1} &= x_{\alpha_n}(-s_n)\cdots x_{\alpha_1}(-s_1)x_{-\alpha_n}(-r_n)\cdots x_{-\alpha_1}(-r_1)h(\chi_{\xi_1^{-1}\ldots\xi_l^{-1}})\\
&\quad\times x_{\alpha_n}(-t_n)\cdots x_{\alpha_1}(-t_1)\\
&= x_{\alpha_n}(-s_n)\cdots x_{\alpha_1}(-s_1)h(\chi_{\xi_1^{-1}\ldots\xi_l^{-1}})\\
&\quad\times x_{-\alpha_n}(-q_n(\xi_1,\ldots,\xi_l)r_n)\cdots x_{-\alpha_1}(-q_1(\xi_1,\ldots,\xi_l)r_1)x_{\alpha_n}(-t_n)\cdots x_{\alpha_1}(-t_1)\\
&= x_{\alpha_1}(p_1^+(-s_n,\ldots,-s_1))\cdots x_{\alpha_n}(p_n^+(-s_n,\ldots,-s_1))h(\chi_{\xi_1^{-1}\ldots\xi_l^{-1}})\\
&\quad\times x_{-\alpha_1}(p_1^-(-q_n(\xi_1,\ldots,\xi_l)r_n,\ldots,-q_1(\xi_1,\ldots,\xi_l)r_1)\\
&\quad\times\cdots\times x_{-\alpha_n}(p_n^-(-q_n(\xi_1,\ldots,\xi_l)r_n,\ldots,-q_1(\xi_1,\ldots,\xi_l)r_1)\\
&\quad\times x_{\alpha_1}(p_1^+(-t_n,\ldots,t_1))\cdots x_{\alpha_n}(p_n^+(-t_n,\ldots,-t_1)),
\end{aligned}$$

where $p_1^+(\,\cdot\,)$, …, $p_n^+(\,\cdot\,)$, $p_1^-(\,\cdot\,)$, …, $p_n^-(\,\cdot\,)$ are polynomials of n variables with integer coefficients and $q_1(\,\cdot\,)$, …, $q_n(\,\cdot\,)$ are Laurent monomials of l variables.

Therefore the formula

$$\begin{aligned}
&\psi^{(-1)}(t_1,\ldots,t_n,\xi_1,\ldots,\xi_l,r_1,\ldots,r_n,s_1,\ldots,s_n,\\
&\qquad t_1',\ldots,t_n',\xi_1',\ldots,\xi_l',r_1',\ldots,r_n',s_1',\ldots,s_n)\\
&\quad:= (t_1'=p_1^+(-t_n,\ldots,-t_1))\wedge\cdots\wedge(t_n'=p_n^+(-t_n,\ldots,-t_1))\wedge(\xi_1'=\xi_1^{-1})\\
&\qquad\wedge\cdots\wedge(\xi_l'=\xi_l^{-1})\wedge(r_1'=p_1^-(-q_1(\xi_1,\ldots,\xi_l)r_1,\ldots,-q_n(\xi_1,\ldots,\xi_l)r_n)\\
&\qquad\wedge\cdots\wedge(r_n'=p_n^-(-q_1(\xi_1,\ldots,\xi_l)r_1,\ldots,-q_n(\xi_1,\ldots,\xi_l)r_n)\\
&\qquad\wedge(s_1'=p_1^+(-s_n,\ldots,-s_1)\wedge\cdots\wedge s_n'\\
&\quad= p_n^+(-s_n,\ldots,-s_1)
\end{aligned}$$

holds if and only if $x'=x^{-1}$.

Consequently, the required formula $\psi^{(=)}(\,\cdot\,)$ that is true if and only if $x=x'$ has the form

$$\begin{aligned}
&\psi^{(=)}(t_1,\ldots,t_n,\xi_1,\ldots,\xi_l,r_1,\ldots,r_n,s_1,\ldots,s_n,\\
&\qquad t_1',\ldots,t_n',\xi_1',\ldots,\xi_l',r_1',\ldots,r_n',s_1',\ldots,s_n')\\
&\quad= \exists\, t_1'',\ldots,t_n'',\bar t_1,\ldots,\bar t_n,r_1'',\ldots,r_n'',\bar r_1,\ldots,\bar r_n,s_1'',\ldots,s_n'',\bar s_1,\ldots,\bar s_n\in R\\
&\qquad\exists\,\xi_1'',\ldots,\xi_l'',\bar\xi_1,\ldots,\bar\xi_l\in R^*\\
&\psi^{(-1)}(t_1',\ldots,t_n',\xi_1',\ldots,\xi_l',r_1',\ldots,r_n',s_1',\ldots,s_n',\\
&\qquad t_1'',\ldots,t_n'',\xi_1'',\ldots,\xi_l'',r_1'',\ldots,r_n'',s_1'',\ldots,s_n'')\\
&\quad\wedge\psi^{\oplus}(t_1,\ldots,t_n,\xi_1,\ldots,\xi_l,r_1,\ldots,r_n,s_1,\ldots,s_n,t_1',\ldots,t_n',\xi_1',\ldots,\xi_l',\\
&\qquad\quad r_1',\ldots,r_n',s_1',\ldots,s_n',\bar t_1,\ldots,\bar t_n,\bar\xi_1,\ldots,\bar\xi_l,\bar r_1,\ldots,\bar r_n,\bar s_1,\ldots,\bar s_n)\\
&\quad\wedge\psi^{(1)}(\bar t_1,\ldots,\bar t_n,\bar\xi_1,\ldots,\bar\xi_l,\bar r_1,\ldots,\bar r_n,\bar s_1,\ldots,\bar s_n).
\end{aligned}$$

So we now have a formula for (ii).

Finally we can write a formula for (iii):

$$
\begin{aligned}
&\psi^{(\otimes)}(t_1,\dots,t_n,\xi_1,\dots,\xi_l,r_1,\dots,r_n,s_1,\dots,s_n,t_1',\dots,t_n',\xi_1',\dots,\xi_l',\\
&\qquad r_1',\dots,r_n',s_1',\dots,s_n',\bar t_1,\dots,\bar t_n,\bar\xi_1,\dots,\bar\xi_l,\bar r_1,\dots,\bar r_n,\bar s_1,\dots,\bar s_n)\\
&\quad = \exists\, t_1'',\dots,t_n'',r_1'',\dots,r_n'',s_1'',\dots,s_n''\in R\quad \exists\,\xi_1'',\dots,\xi_l''\in R^*\\
&\psi^{\otimes}(t_1,\dots,t_n,\xi_1,\dots,\xi_l,r_1,\dots,r_n,s_1,\dots,s_n,t_1',\dots,t_n',\xi_1',\dots,\xi_l',\\
&\qquad r_1',\dots,r_n',s_1',\dots,s_n',t_1'',\dots,t_n'',\xi_1'',\dots,\xi_l'',r_1'',\dots,r_n'',s_1'',\dots,s_n'')\\
&\quad\wedge \psi^{(=)}(t_1'',\dots,t_n'',\xi_1'',\dots,\xi_l'',r_1'',\dots,r_n'',s_1'',\dots,s_n'',\\
&\qquad\qquad \bar t_1,\dots,\bar t_n,\bar\xi_1,\dots,\bar\xi_l,\bar r_1,\dots,\bar r_n,\bar s_1,\dots,\bar s_n).
\end{aligned}
$$

The proof of Proposition 4 is complete.

§ 6. Elementary equivalence of the initial rings

Suppose that we have two elementarily equivalent elementary adjoint Chevalley groups E and E' of the same type Φ (of rank > 1) over local rings R and R' with invertible 2 (if $\Phi = G_2$, then also with invertible 3).

From [25], [10], [11] it follows that if two such groups are isomorphic, then the rings R and R' are isomorphic.

If the groups E and E' are elementarily equivalent, then by the Keisler-Shelah theorem (see [1]) there exists an ultrafilter D such that

$$\prod_D E \cong \prod_D E'.$$

From Proposition 4 it follows that

$$\prod_D E_{\mathrm{ad}}(\Phi, R) \cong E_{\mathrm{ad}}\Big(\Phi, \prod_D R\Big).$$

Therefore,

$$E_{\mathrm{ad}}\Big(\Phi, \prod_D R\Big) \cong E_{\mathrm{ad}}\Big(\Phi, \prod_D R'\Big).$$

Consequently,

$$\prod_D R \cong \prod_D R',$$

from which it follows that

$$R \equiv R'.$$

§ 7. Conclusion

Now let us collect all the proved facts and finally obtain the proof of the main theorem.

- The reverse implication is completely proved in § 2, even for a more general class of rings.

Suppose now that we have two elementarily equivalent Chevalley groups satisfying all conditions from the theorem.

- In § 4 it was proved that in this case the root systems are isomorphic.
- In § 6 it was proved that the initial rings are elementarily equivalent.
- The isomorphism of weight lattices follows from the fact that with regard to our groups over local rings we can with the help of factorization by the greatest normal subgroup obtain Chevalley groups over residue fields with the same weight lattices as the initial groups. Then the required result follows from the similar result of [9].

The proof of the Main Theorem is complete.

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