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CHARACTERIZATION OF MODEL MIRIMANOV–VON NEUMANN CUMULATIVE SETS

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CONTENTS

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Introduction

The crisis that appeared in the naive set theory at the beginning of the 20th century has led to the construction of several strict axiomatic set theories. The most useful among them is the *set theory in the Zermelo–Fraenkel axiomatics* (ZF) (1908, 1922 [8]) and the *class and set theory in the von Neumann–Burnice–Hödel axiomatics* (NBG) (1928, 1937, 1940 [10]).

In 1917, using transfinite induction Mirimanov [20] constructed the *cumulative collection* (≡*hierarchy*) *of sets* V_α for all order numbers α having the following properties:

(1) $V_0 \equiv \emptyset$;
(2) $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$ ($\mathcal{P}(V_\alpha)$ denotes the set of all subsets of the set V_α);
(3) $V_\alpha = \cup\left(V_\beta | \beta \in \alpha \right)$ for any limit order number α.

It turns out that *cumulative sets* V_α themselves and the collection $\left(V_\alpha | \alpha \in \mathbf{On} \right)$ as a whole have many remarkable properties. In particular, von Neumann proved in [22] that the regularity axiom in ZF is equivalent to the property $\forall x \exists \alpha(\alpha$ is an ordinal number $\wedge x \in V_\alpha)$ and the class $\cup\left(V_\alpha | \alpha \in \mathbf{On} \right)$ is an abstract (≡ class) standard model for the ZF theory in ZF. Models of the ZF and NBG theories of the form $(V_\alpha, =, \in)$ are said to be *natural*.

After the introduction of the concept of a (*strongly*) *inaccessible cardinal number* by Zermelo in [28] and by Sierpinski–Tarski in [24], Zermelo [28] (not strictly) and Shepherdson [23] (strictly) proved that *a set U is a supertransitive standard model for the NBG theory iff it has the form $V_{\varkappa+1}$ for a certain inaccessible cardinal number $\varkappa$.* Thus, the natural model of the NBG theory was explained.

The Zermelo–Shepherdson theorem admits the following equivalent reformulation: *a set U is a supertransitive standard model for the ZF theory with the strong replacement property* ($\forall X \forall f(X \in U \wedge f \in U^X \Rightarrow rng\, f \in U)$) *iff it has the form $V_\varkappa$ for a certain inaccessible cardinal number $\varkappa$.*

Starting from the requirements of category theory, instead of the metaconcept of a supertransitive standard model set with the strong replacement property for the ZF theory, Ehresmann [6], Dedecker [5], Sonner [25], and Grothendieck (see [9]), introduced an equivalent set-theoretic concept of a *universal set* U (see [18], I.6, [7], [12]), which is defined by the following properties:

(1) $x \in U \Rightarrow x \subset U$;
(2) $x \in U \Rightarrow \mathcal{P}(x), \cup x \in U$;
(3) $x, y \in U \Rightarrow x \cup y, \{x, y\}, \langle x, y \rangle, x \times y \in U$;
(4) $x \in U \wedge f \in U^x \Rightarrow rng\, f \in U$ (*strong replacement property*);
(5) $\omega \in U$ (here, $\omega \equiv \{0, 1, 2, \dots\}$ is the set of all finite ordinal numbers).

To deal with categories in the set-theoretic framework, they suggested to strengthen the ZF theory by adding the *universality axiom* AU, according to which *each set is an element of a certain universal set.*

The equivalent form of the Zermelo–Shepherdson theorem states that the universality axiom AU is equivalent to the *inaccessibility axiom* AI according to which *for every ordinal number, there exists an inaccessible cardinal number strictly greater than it.*

For axiomatic construction of inaccessible cardinal numbers, in [26] (see also [17], IX, § 1 and § 5), Tarski introduced the concept of a *Tarski set* U, which is defined by the following properties:

(1) $x \in U \Rightarrow x \subset U$ (*transitivity property*);
(2) $x \in U \Rightarrow \mathcal{P}(x) \in U$ (*exponentiality property*);
(2) $(x \subset U \wedge \forall f(f \in U^x \Rightarrow rng\, f \neq U) \Rightarrow x \in U$ (*Tarski property*).

In [26], Tarskii proved that the set $V_\varkappa$ (≡ *inaccessible cumulative set*) is a Tarski set for each cardinal number $\varkappa$.

Also, in [26], Tarski proved that the inaccessibility axiom AI is equivalent to the *Tarski axiom* AT, according to which *every set is an element of a certain Tarskii set.*

In connection with the Tarski theorem, the following problem remains open unit now: *to what extent is the axiomatic concept of Tarski set is wider than the constructive concept of inaccessible cumulative set*?

In this paper, we give an answer to this question: *the concepts of an inaccessible cumulative set and that of an uncountable Tarski set are equivalent.*

The equivalence of the concepts of an inaccessible cumulative set and an uncountable Tarski set is proved by using the concept of a universal set. More precisely, it is proved that *every uncountable Tarski set is universal.*

As a result, we obtain the following theorem on the characterization of natural models of the NBG theory: *the following properties are equivalent for a set* U:

(1) U *is an inaccessible cumulative set, i.e.,* $U = V_\varkappa$ *for a certain inaccessible cardinal number* $\varkappa$;
(2) $\mathcal{P}(U)$ *is a supertransitive standard model for the NBG theory*;
(3) U *is a supertransitive standard-model with the strong replacement property of the* ZF *theory*;
(4) U *is a universal set*;
(5) U *is an uncountable Tarski set.*

The Zermelo–Shepherdson theorem yields a canonical form of supertransitive standard models of the NBG theory and an (equivalent) canonical form of standard models with the strong replacement property of the ZF theory. However, Montague and Vaught proved in [21] that for any inaccessible cardinal number $\varkappa$, there exists an ordinal number $\theta < \varkappa$ such that it is inaccessible and the cumulative set V_θ is a supertransitive standard model of the ZF theory. Therefore, the problem on the canonical forms of supertransitive standard models of the ZF theory turned out to be more complicated.

Since the concept of model in the ZF theory cannot be defined by a finite set of formulas, in this paper, using the formula schema and its relativization to the set V_θ, we introduce the concept of a (*strongly*) *scheme-inaccessible cardinal number* θ and prove a scheme analog of the Zermelo–Shepherdson theorem.

To prove this theorem, we introduce the concept of a *scheme-universal set*, which is a scheme analog of the concept of a universal set. Moreover, in this paper, we introduce the concept of a *scheme Tarski set*, which is a scheme analog of the concept of a Tarski set.

As a result, we prove the *theorem on the characterization of natural models of the ZF theory*: *the following properties are equivalent for a set* U:

(1) U *is a scheme-inaccessible cumulative set, i.e.,* $U = V_\theta$ *for a certain scheme-inaccessible cardinal number* θ;
(2) U *is a supertransitively standard model for the* ZF *theory*;
(3) U *is a scheme-universal set*;
(4) U *is a scheme Tarski set.*

In this paper, the problems mentioned above are solved for the ZF set theory (with the axiom of choice). For the NBG set theory, all things are equally true. For the reader's convenience, we present all the necessary facts that are not sufficiently reflected in the literature or merely refer to the mathematical folklore, with complete proofs.

1. Some Facts From Zermelo–Fraenkel Set Theory

1.1. Classes in the ZF set theory. We first present a list of proper axioms and axiom schemes of the ZF theory, the Zermelo–Fraenkel theory with the axiom of choice (see [11, 17], and [13]).

This theory is a first-order theory with two binary predicates: the *belonging* predicate symbols $\in$ (we write $A \in B$) and the *equality* $=$ (we write $A = B$).

The equality predicate $=$ satisfies the following axiom and the axiom scheme:

(1) $\forall x(x = x)$ (*reflexivity of equality*);
(2) $(x = y) \Rightarrow (\varphi(x, x) \Rightarrow \varphi(x, y))$ (*interchange of equals*), where x and y are variables, $\varphi(x, x)$ is an arbitrary formula, and $\varphi(x, y)$ is obtained from $\varphi(x, x)$ by replacing (not necessary all) free entrances of x by entrances of y so that the condition that y is free for all x that are replaced holds.

Objects of ZF theory are called *sets.* Further, in the paper, the denoting sign-alternation σ for a sign-alternation ρ will be introdused in the form of sign-alternation $\rho \equiv \sigma$ (σ *is the notation* ρ).

It is convenient to consider the totality $\mathbf{C}$ of all sets A satisfying a given formula $\varphi(x)$. Such a totality $\mathbf{C}$ is called the *class defined by the formula* φ. The totality $\mathbf{C}(\vec{u})$ of all sets A satisfying a formula $\varphi(x, \vec{u})$ is called the *class* (ZF) *defined by the formula* φ *through the parameter* $\vec{u}$. Along with this, we will use the notation

$$A \in \mathbf{C} \equiv \varphi(A), \quad A \in \mathbf{C}(\vec{u}) \equiv \varphi(A, \vec{u}),$$

and

$$\mathbf{C} \equiv \{x|\varphi(x)\}, \quad \mathbf{C}(\vec{u}) \equiv \{x|\varphi(x, \vec{u})\}.$$

If $\mathbf{C} \equiv \{x|\varphi(x)\}$ and φ contains only one variable x, then the class $\mathbf{C}$ is said to be *completely determined by the formula* φ.

Every set A can be considered as the class $\{x|x \in A\}$.

A class $\mathbf{C} \equiv \{x|\varphi(x)\}$ is called a *subclass of a class* $\mathbf{D} \equiv \{x|\psi(x)\}$ (denoted by $\mathbf{C} \subset \mathbf{D}$) if $\forall x(\varphi(x) \Rightarrow \psi(x))$. Two classes $\mathbf{C}$ and $\mathbf{D}$ are said to be *equal* if $(\mathbf{C} \subset \mathbf{D}) \wedge (\mathbf{D} \subset \mathbf{C})$. In what follows, we will use the notation $\{x \in A|\varphi(x)\} \equiv \{x|x \in A \wedge \varphi(x)\}$. If a class $\mathbf{C}$ is not equal to any set, then $\mathbf{C}$ is called a *proper class*. The class of all sets $\mathbf{V} \equiv \{x|x = x\}$ is said to be *universal*

For two classes $\mathbf{C} \equiv \{x|\varphi(x)\}$ and $\mathbf{D} \equiv \{x|\psi(x)\}$, let us define the *binary union* $\mathbf{C} \cup \mathbf{D}$ and the *binary intersection* $\mathbf{C} \cap \mathbf{D}$ as the classes

$$\mathbf{C} \cup \mathbf{D} \equiv \{x|\varphi(x) \vee \psi(x)\} \text{ and} \mathbf{C} \cap \mathbf{D} \equiv \{x|\varphi(x) \wedge \psi(x)\}.$$

A1 (*volume axiom*).

$$\forall X \forall Y (\forall u(u \in X \Leftrightarrow u \in Y) \Rightarrow X = Y).$$

For two sets A and B, define an *unordered pair* $\{A, B\}$ as the class $\{A, B\} \equiv \{z|z = A \vee z = b\}$.

A2 (*pair axiom*).

$$\forall u \forall v \exists x \forall z (z \in x \Leftrightarrow z = u \vee z = v).$$

It follows from **A2** and **A1** that an unordered pair of sets is a set.

For two sets A and B, define the following:

— *a unit set* $\{A\} \equiv \{A, A\}$;
— *an ordered pair* $\langle A, B\rangle \equiv \{\{A\}, \{A, B\}\}$.

It follows from the above that $\{A\}$ and $\langle A, B\rangle$ are sets.

Lemma 1. *$\langle A, B\rangle = \langle A', B'\rangle$ iff $A = A'$ and $B = B'$.*

AS3 (*isolation axiom scheme*).

$$\forall X \exists Y \forall u (u \in Y \Leftrightarrow u \in X \wedge \varphi(u, \vec{p})),$$

where the formula $\varphi(u, \vec{p})$ does not freely contain the variable Y.

The isolation axiom scheme asserts that the class $\{u|u \in X \wedge \varphi(u, \vec{p})\}$ is a set. This set is unique by **A1**.

Consider the class $\mathbf{C}(\vec{p}) = \{u|\varphi(u, \vec{p})\}$. Then Scheme **AS3** can be expressed as $\forall X \exists Y (Y = \mathbf{C}(\vec{p}) \cap X)$.

For two classes $\mathbf{A}$ and $\mathbf{B}$ define the *difference* $\mathbf{A} \setminus \mathbf{B}$ as the class $\mathbf{A} \setminus \mathbf{B} \equiv \{x \in \mathbf{A}|x \notin \mathbf{B}\}$. If A is a set, then the difference $A \setminus \mathbf{B}$ is a set by **AS3**.

Since $A \cap B = \{x \in A|x \in B\} \subset A$, by **AS3**, we have that for any sets A and B, the binary intersection $A \cap B$ is a set.

For a class $\mathbf{C} \equiv \{x|\varphi(x)\}$, define the *union* $\cup\mathbf{C}$ as the class $\cup\mathbf{C} \equiv \{z|\exists x(\varphi(x) \wedge z \in x)\}$.

A4 (*Union axiom*).

$$\forall X \exists Y \forall u (u \in Y \Leftrightarrow \exists z (u \in z \wedge z \in X)).$$

From **A4** and **A1**, it is deduced that for any set A, its union $\cup A$ is a set.

The equality $A \cup B = \cup\{A, B\}$ holds. Hence, for any sets A and B, their binary union $A \cup B$ is a set.

The complete ensemble of a class $\mathbf{C}$ is the class $\mathcal{P}(\mathbf{C}) \equiv \{u|u \subseteq \mathbf{C}\}$.

A5 (*axiom of the set of subsets* ($\equiv$ *of complete ensemble*)).

$$\forall X \exists Y \forall u (u \in Y \Leftrightarrow u \subset X).$$

If A is a set, then by **A5** and **A1**, $\mathcal{P}(A)$ is a set.

For two classes **A** and **B** define the (*coordinatewise*) *product*

$$\mathbf{A} * \mathbf{B} \equiv \{x | \exists u \exists v (u \in \mathbf{A} \wedge v \in \mathbf{B} \wedge x = \langle u, v \rangle)\}.$$

The property that $A * B$ is a set for the sets A and B follows from **AS3**, since $A * B \subseteq \mathcal{P}P(A \cup B))$.

A class (in particular, a set) **C** is called a *correspondence* if $\forall u (u \in \mathbf{C} \Rightarrow \exists x \exists y (u = \langle x, y \rangle)))$. For a correspondence **C**, consider the classes $dom\,\mathbf{C} \equiv \{u | \exists v (\langle u, v \rangle \in \mathbf{C})\}$ and $rng\,\mathbf{C} \equiv \{v | \exists u (\langle u, v \rangle \in \mathbf{C}\}$.

If **C** is a set, then by **A4** and **AS3**, it follows from $dom\,\mathbf{C} \subset \cup \cup \mathbf{C}$ that $dom\,\mathbf{C}$ is also a set.

A correspondence **F** is called a *function* ($\equiv$ *mapping*) if $\forall x \forall y \forall y' (\langle x, y \rangle \in \mathbf{F} \wedge \langle x, y' \rangle \in \mathbf{F} \Rightarrow y = y')$. The formula expressing the property to be a function for a class **F** will be denoted by $func(\mathbf{F})$. For the expression $\langle x, y \rangle \in \mathbf{F}$, we use the following notation: $y = \mathbf{F}(x)$, $\mathbf{F} : x \mapsto y$, etc.

A correspondence **C** is called a *correspondence from a class* **A** *into a class* **B** if $dom\,\mathbf{C} \subset \mathbf{A}$ and $rng\,\mathbf{C} \subset \mathbf{B}$ (denoted by $\mathbf{C} : \mathbf{A} \prec \mathbf{B}$). A function **F** is a *function from a class* **A** *into a class* **B** if $dom\,\mathbf{F} = \mathbf{A}$ and $rng\,\mathbf{F} \subset \mathbf{B}$ (denoted by $\mathbf{F} : \mathbf{A} \to \mathbf{B}$).

The formula expressing the property of the class **F** to be a function from the class **A** into the class **B** will be denoted by $\mathbf{F} \leftrightharpoons \mathbf{A} \to \mathbf{B}$. The formulas $(\mathbf{F} \leftrightharpoons \mathbf{A} \to \mathbf{B}) \wedge \forall x, y \in \mathbf{A}(\mathbf{F}(x) = \mathbf{F}(y) \Rightarrow x = y)$ and $(\mathbf{F} \leftrightharpoons \mathbf{A} \to \mathbf{B}) \wedge rng\,\mathbf{F} = \mathbf{B}$ will be denoted by $\mathbf{F} \leftrightharpoons \mathbf{A} \rightarrowtail \mathbf{B}$ and $\mathbf{F} \leftrightharpoons \mathbf{A} \leftrightarrows \mathbf{B}$, respectively. The conjunction of these formulas will be denoted by $\mathbf{F} \leftrightharpoons \mathbf{A} \leftrightarrows \mathbf{B}$. These formulas define the *injectivity*, the *surjectivity*, and the *bijectivity* of the function $\mathbf{F} : \mathbf{A} \to \mathbf{B}$, respectively.

The class $\{f | f \text{is a function} \wedge dom\,f = \mathbf{A} \wedge rng\,f \subseteq \mathbf{B}\}$ of all functions from the class **A** into the class **B** which are sets is denoted by $\mathbf{B}^{\mathbf{A}}$ or $Map(\mathbf{A}, \mathbf{B})$. Since $B^A \subset \mathcal{P}(A * B)$, the class B^A is a set for all sets A and B.

The *restriction of a function* **F** *to a class* $\mathbf{A}'$ is defined as the class $\mathbf{F}|\mathbf{A}' \equiv \{x | \exists u \exists v (x = \langle u, v \rangle \wedge x \in \mathbf{F} \wedge u \in \mathbf{A}')\}$. The *image* and the *preimage of a class* **D** *under a function* **F** are defined as the classes $\mathbf{F}[\mathbf{D}] \equiv \{v | \exists u \in \mathbf{D}(v = \mathbf{F}(u))\}$ and $\mathbf{F}^{-1}[\mathbf{D}] \equiv \{u | \mathbf{F}(u) \in \mathbf{D}\}$.

A correspondence **C** from a class **A** into a class **B** is also called a (*multivalued) collection of subclasses* $\mathbf{B}_a \equiv \mathbf{C}\langle a \rangle \equiv \{y | y \in \mathbf{B} \wedge \langle a, y \rangle \in \mathbf{C}\}$ *of the class* **B** *indexed by the class* **A**. In this case, the correspondence **C** and the class $rng\,\mathbf{C}$ are also denoted by $\left(\mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \right)$ and $\cup \left(\mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \right)$, respectively. The class $\cup \left(\mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \right)$ is also called the *union of the collection* $\left(\mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \right)$. The class $\{y | \forall x \in \mathbf{A}(y \in \mathbf{B}_x)\}$ is called the *the intersection of the collection* $\left(\mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \right)$ and is denoted by $\cap \left(\mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \right)$. To each class **A**, we canonically put in correspondence the *collection* $\left(a \subset \mathbf{V} | a \in \mathbf{A} \right)$ *of element sets of the class* **A**. The relation $\cup \mathbf{A} = \cup \left(a \subset \mathbf{V} | a \in \mathbf{A} \right)$ holds for this collection.

A function **F** from a class **A** into a class **B** is also called a *simple collections of elements* $b_a \equiv \mathbf{F}(a)$ *of the class* **B** *indexed by the class* **A**. In this case, the function **F** and the class $rng\,\mathbf{F}$ are also denoted by $(b_a \in \mathbf{B} | a \in \mathbf{A})$ and $\{b_a \in \mathbf{B} | a \in \mathbf{A}\}$, respectively. The collection $(b_a \in \mathbf{V} | a \in \mathbf{A})$ is also denoted by $(b_a | a \in \mathbf{A})$. To each class **A**, we canonically put in correspondence the *simple collection* $(a \in \mathbf{A} | a \in \mathbf{A})$ *of elements of the class* **A**. Clearly, $\{a \in \mathbf{A} | a \in \mathbf{A}\} = \mathbf{A}$.

AS6 (*axiom substitution scheme*).

$$\forall x \forall y \forall y' (\varphi(x, y, \vec{p}) \wedge \varphi(x, y', \vec{p}) \Rightarrow y = y') \Rightarrow \forall X \exists Y \forall x \in X \forall y (\varphi(x, y, \vec{p}) \Rightarrow y \in Y),$$

where the formula $\varphi(x, y, \vec{p})$ does not freely contain the variable Y.

Also, Scheme **AS6** can be also expressed as follows: if **F** is a function, then for any set X, the class $\mathbf{F}[X]$ is a set.

If A is a set, then the axiom substitution scheme implies that the class $rng\,\mathbf{F} \equiv \{b_a \in \mathbf{B} | a \in \mathbf{A}\}$ is a set. Then it follows from $\mathbf{F} \subset A \times rng\,\mathbf{F}$ that the class $\mathbf{F} \equiv (b_a \in \mathbf{B} | a \in A)$ is also a set. Therefore, in the case where A is a set, the following notation is used: $F : A \to \mathbf{B}$ and $F \equiv (b_a \in \mathbf{B} | a \in A)$.

A7 (*empty set axiom*).

$$\exists x \forall z (\neg (z \in x)).$$

It follows from **A1** that a set containing no element is unique. It is denoted by $\emptyset$.

A8 (*infinity axiom*).

$$\exists Y(\emptyset \in Y \wedge \forall u(u \in Y \Rightarrow u \cup \{u\} \in Y)).$$

According to this axiom, there exists a set I containing $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$, etc.

A9 (*regularity axiom*).

$$\forall X(X \neq \emptyset \Rightarrow \exists x(x \in X \wedge x \cap X = \emptyset)).$$

The function $f : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ is called the *choice function for the set* A if $f(X) \in X$ for any $X \in \mathcal{P}(A) \setminus \{\emptyset\}$.

The following last axiom postulates the existence of the choice function for any nonempty set.

A10 (*axiom of choice* (AC)).

$$\forall X(X \neq \emptyset \Rightarrow \exists z((z \risingdotseq \mathcal{P}(X) \setminus \{\emptyset\} \to X) \wedge \forall Y(Y \in \mathcal{P}(X) \setminus \{\emptyset\} \Rightarrow z(Y) \in Y))).$$

The axiomatic system **A1**–**A9** is called the *Zermelo–Fraenkel axiomatic set theory* ZF (*with the axiom of choice*).

1.2. Ordinal and cardinals in the ZF set theory. If $n \in \omega$, then a subclass R of the class $\mathbf{A}^n \equiv Map(n, \mathbf{A})$ is called an *n-fold* ($\equiv$ *n-ary*) *relation on the class* $\mathbf{A}$.A mapping $\mathbf{O} : \mathbf{A}^n \to \mathbf{A}$ is called an *n-fold* ($\equiv$ *n-ary*) *operation on the class* $\mathbf{A}$.

A binary relation $\leq$ on a class $\mathbf{P}$ is called an *ordering of the class* $\mathbf{P}$ ($\equiv$ *an order on the class* $\mathbf{P}$) if the following conditions hold:

(1) $\forall p \in \mathbf{P}(p \leq p)$;
(2) $\forall p, q \in \mathbf{P}(p \leq q \wedge q \leq p \Rightarrow p = q)$;
(3) $\forall p, q, r \in \mathbf{P}(p \leq q \wedge q \leq r \Rightarrow p \leq r)$.
If, in addition,
(4) $\forall p, q \in \mathbf{P}(p \leq q \vee q \leq p)$,
then $\leq$ is called a *linear ordering of* $\mathbf{P}$.
A class $\mathbf{P}$ endowed with an order $\leq$ is said to be *ordered*.
An ordered class $\mathbf{P}$ is said to be *completely ordered* (c. o.) if
(5) $\forall Q(\emptyset \neq Q \subseteq \mathbf{P} \Rightarrow \exists x \in Q(\forall y \in Q(x \leq y)))$, i.e., if every nonempty subset of the class $\mathbf{P}$ has a minimal element.

Let a class $\mathbf{P}$ be ordered by a relation $\leq$, and let $\mathbf{A}$ be a nonempty subclass of the class $\mathbf{P}$. An element $p \in \mathbf{P}$ is called the *least upper bound* or *supremum of the subclass* $\mathbf{A}$ if $\forall x \in \mathbf{A}(x \leq p) \wedge \forall y \in \mathbf{P}((\forall x' \in \mathbf{A}(x' \leq y)) \Rightarrow p \leq y)$. This formula is denoted by $p = \sup \mathbf{A}$. An element $a \in \mathbf{A}$ is called a *maximal* [*minimal*] *element of the class* $\mathbf{A}$ if $b \leq a$ [$b \geq a$] for any $b \in \mathbf{A}$. This formula is denoted by $a = gr\, \mathbf{A}$ [$a = sm\, \mathbf{A}$]. If the class $\mathbf{P}$ is linearly ordered, then a maximal [minimal] element is unique.

A class $\mathbf{S}$ is said to be *transitive* if $\forall x(x \in \mathbf{S} \Rightarrow x \subset \mathbf{S})$. The class $\mathbf{S}$ is said to be *quasi-transitive* if $\forall x \forall y(x \in \mathbf{S} \wedge y \subset x \Rightarrow y \in \mathbf{S})$. A transitive and quasi-transitive class is said to be *supertransitive*.

A class [set] $\mathbf{S}$ is called an *ordinal* [*ordinal number*] if $\mathbf{S}$ is transitive and completely ordered by the relation $\in \cup =$ on $\mathbf{S}$. The property of a set S to be an ordinal number will be denoted by $On(S)$.

As usual, ordinal numbers are denoted by Greek letters α, β, γ, etc. The class of all ordinal numbers is denoted by $\mathbf{On}$. The relation $\alpha \leq \beta \equiv \alpha = \beta \vee \alpha \in \beta$ is a natural ordering of the class of ordinal numbers. The class $\mathbf{On}$ is transitive and linearly ordered by the relation $\in \cup =$.

Let us present several simple assertions on ordinal numbers:

(1) if α is an ordinal number, A is a set, and $A \in \alpha$, then A is an ordinal number;
(2) $\alpha = \{\beta | \beta \in \alpha\}$ for every ordinal number α;
(3) $\alpha + 1 \equiv \alpha \cup \{\alpha\}$ is a least ordinal number greater than α;
(4) every nonempty set of ordinal numbers has the least element.

Therefore, the ordered class $\mathbf{On}$ is completely ordered. Thus, $\mathbf{On}$ is an ordinal.

Lemma 1. *Let* $\mathbf{A}$ *be a nonempty subclass of the class* $\mathbf{On}$. *Then* $\mathbf{A}$ *has a minimal element.*

Lemma 2. *If* A *is a nonempty set of ordinal numbers, then*:

(1) *the class* $\cup A$ *is an ordinal number*;
(2) $\cup A = \sup A$ *in the ordered class* **On**.

Corollary. *The class* **On** *is proper.*

An ordinal number α is said to be *consequent* if $\alpha = \beta + 1$ for a certain ordinal number β. Otherwise, α is said to be *limit*. This unique number β will be denoted by $\alpha - 1$. The formula expressing the property to be consequent [limit] for an ordinal number α will be denoted by $Son(\alpha)$ [$Lon(\alpha)$].

Lemma 3. *An ordinal number* α *is limit iff* $\alpha = \sup \alpha$.

The least (in the class **On**) nonzero limit ordinal is denoted by ω. The existence of such an ordinal follows from **A7**, **AS6**, and **AS3**. Ordinals less than ω are called *natural numbers.*

Collections $\left(\mathbf{B}_n \subset \mathbf{B} | n \in N \subset \omega\right)$ and $(b_n \in \mathbf{B} | n \in N \subset \omega)$, where N is an arbitrary subset in ω, are called *sequences.* If $N \subset m \in \omega$, then the sequences are said to be *finite*, and they are said to be *infinite* in the opposite case.

Theorem 1 (transfinite induction principle)**.** *Let* **C** *be a class or ordinal numbers such that*:

(1) $\emptyset \in \mathbf{C}$;
(2) $\alpha \in \mathbf{C} \Rightarrow \alpha + 1 \in \mathbf{C}$;
(3) (α *is a limit ordinal number* $\wedge \alpha \subset \mathbf{C}) \Rightarrow \alpha \in \mathbf{C}$. *Then* $\mathbf{C} = \mathbf{On}$.

Theorem 2 (construction by transfinite induction)**.** *For each function* $\mathbf{G} : \mathbf{V} \to \mathbf{V}$*, there exists a function* $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ *such that for any* $\alpha \in \mathbf{On}$,

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|\alpha).$$

There is the following $\in$-induction principle in ZF.

Lemma 4. *If a class* **C** *is such that*

$$\forall x (x \subset \mathbf{C} \Rightarrow x \in \mathbf{C}),$$

then $\mathbf{C} = \mathbf{V}$.

Two sets A and B are said to be *equivalent* ($A \sim B$) if there exists a one-to-one ($\equiv$ bijective) function $u : A \leftrightarrows B$.

An ordinal number α is said to be *cardinal* if for each ordinal number β, the relations $\beta \leq \alpha$ and $\beta \sim \alpha$ imply $\beta = \alpha$. The class of all cardinal numbers will be denoted by **Cn**. The class **Cn** with the order induced from the class **On** is completely ordered.

Theorem 3. *For any set* A*, there exists an ordinal number* α *such that* $A \sim \alpha$.

Now, for a set A, consider the class $\{x | x \in \mathbf{On} \wedge x \sim A\}$. By Theorem 3, this class is nonempty, and, therefore, it contains a minimal element α. Obviously, α is a cardinal number. Moreover, this class contains only one cardinal number α. This number α is called the *cardinality of the set* A (and is denoted by $|A|$ or $card\, A$). Two sets having the same cardinality are said to be *equicardinal* (denoted by$|A| = |B|$). A set of cardinality ω is said to be *denumerable.* Sets of cardinality $n \in \omega$ are said to be *finite.* A set is said to be *countable* if it is finite or denumerable. A set is said to be *infinite* if it is not finite.

Note that if $\varkappa$ is an infinite cardinal number, then $\varkappa$ is a limit ordinal number.

Let α an ordinal. The *confinality* of α is the ordinal number $cf(\alpha)$ equal to the minimal ordinal number β for which there exists a function f from β into α such that $\cup rng\, f = \alpha$. The number $cf(\alpha)$ is a cardinal number.

A cardinal $\varkappa$ is said to be *regular* if $cf(\varkappa) = \varkappa$, i.e., for any ordinal number β for which there exists a function $f : \beta \to \varkappa$ such that $\cup rng\, f = \varkappa$, we have $\varkappa \leq \beta$, where $\cup rng\, f = \varkappa$ means that for any $y \in \varkappa$, there exists $x \in \beta$ such that $y < f(x)$.

A cardinal $\varkappa > \omega$ is said to be (*strongly*) *inaccessible* if $\varkappa$ is regular and $card\,\mathcal{P}(\lambda) < \varkappa$ for all ordinal numbers $\lambda < \varkappa$. The property of a cardinal number $\varkappa$ to be inaccessible will be denoted by $Icn(\varkappa)$. The class of all inaccessible cardinal numbers will be denoted by **In**.

The existence of inaccessible cardinals cannot be proved in ZF (see [13], 13).

2. Cumulative Sets and Their Properties

2.1. Construction of cumulative sets. Let us apply the construction by transfinite induction to the following situation. Consider the class

$$\mathbf{G} \equiv \{Z | \exists X \exists Y (Z = \langle X, Y\rangle \wedge ((X = \emptyset \Rightarrow Y = \emptyset)$$
$$\vee (X \neq \emptyset \Rightarrow (\neg func(X) \Rightarrow Y = \emptyset) \vee (func(X) \Rightarrow (\neg On(dom\,X) \Rightarrow Y = \emptyset)$$
$$\vee (On(dom\,X) \Rightarrow (Son(dom\,X) \Rightarrow Y = X(dom\,X - 1) \cup \mathcal{P}(X(dom\,X - 1)))$$
$$\vee (Lon(dom\,X) \Rightarrow Y = \cup rng\,X))))))\}.$$

If we express the definition of the class **G** less formally, then **G** consists of all pairs $\langle X, Y\rangle$ for which we have the following five cases which exclude each other:

(1) if $X = \emptyset$, then $Y = \emptyset$;
(2) if $X \neq \emptyset$ and X is not a function, then $Y = \emptyset$;
(3) if $X \neq \emptyset$, X is a function and $dom\,X$ is not an ordinal number, then $Y = \emptyset$;
(4) if $X \neq \emptyset$, X is a function, $dom\,X$ is an ordinal number, and $dom\,X = \alpha + 1$, then $Y = X(\alpha) \cup \mathcal{P}(X(\alpha))$.
(5) if $X \neq \emptyset$, X is a function, and $dom\,X$ is a limit ordinal number, then $Y = \cup rng\,X$.

By definition, **G** is a correspondence. Since any set X has one of the properties listed above, we have $dom\,\mathbf{G} = \mathbf{V}$. Since in each of the above five cases, the set Y is uniquely determined by the set X, using the property of an ordered pair from Lemma 1 (Sec. 1.1), we verify that **G** is a function from **V** into **V**.

According to Theorem 2 (Sec. 1.2), for the function **G**, there exists a function $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ such that the following relation holds for any $\alpha \in \mathbf{On}$:

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|\alpha).$$

It follows from Case (1) for the function **G** that $\mathbf{F}(\emptyset) = \mathbf{G}(\mathbf{F}|\emptyset) = \mathbf{G}(\emptyset) = \emptyset$.

It follows from case (4) that if β is the subsequent cardinal number and $\beta = \alpha + 1$, then $\mathbf{F}(\beta) = \mathbf{G}(\mathbf{F}|\beta) = (\mathbf{F}|\beta)(\alpha) \cup \mathcal{P}((\mathbf{F}|\beta)(\alpha)) = \mathbf{F}(\alpha) \cup \mathcal{P}(\mathbf{F}(\alpha))$.

Finally, it follows from Case (5) that if α is a limit ordinal number, then $\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|\alpha) = \cup rng\,(\mathbf{F}|\alpha) = \cup\big(\mathbf{F}(\beta)|\beta \in \alpha\big)$.

Denote $\mathbf{F}(\alpha)$ by V_α. We have obtained the collection $\big(V_\alpha \subset \mathbf{V}|\alpha \in \mathbf{On}\big)$ satisfying the following relations:

(1) $V_0 = \emptyset$;
(2) $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$;
(3) $V_\alpha = \cup\big(V_\beta|\beta \in \alpha\big)$ if α is a limit ordinal number.

This collection is called the *Mirimanov–von Neumann cumulative collection*, and its elements V_α are called *Mirimanov–von Neumann cumulative sets.*

2.2. Properties of cumulative sets.

Lemma 1. *If α and β are ordinal numbers, then*:

(1) $\alpha < \beta \Leftrightarrow V_\alpha \in V_\beta$;
(2) $\alpha = \beta \Leftrightarrow V_\alpha = V_\beta$;
(3) $\alpha \subset V_\alpha$ *and* $\alpha \in V_{\alpha+1}$.

Proof. (1) and (2). By the transfinite induction, let us prove that for any ordinal number β, $(\alpha \in \beta \Rightarrow V_\alpha \in V_\beta)$.

If $\beta = \emptyset$, then this is obvious, since $\forall\alpha\neg(\alpha \in \beta)$.

If $(\alpha \in \beta \Rightarrow V_\alpha \in V_\beta)$ for a certain ordinal number β, then consider the ordinal number $\beta+1$. It follows from $\alpha \in \beta + 1$ that $\alpha \in \beta \vee \alpha = \beta$. If $\alpha \in \beta$, then $V_\alpha \in V_\beta$ by the induction assumption, and since $V_{\beta+1} = V_\beta \cup \mathcal{P}(V_\beta)$, we have $V_\alpha \in V_{\beta+1}$. If $\alpha = \beta$, then $V_\alpha = V_\beta \in V_{\beta+1}$, since $V_\beta \in \mathcal{P}(V_\beta)$. Therefore, $\alpha \in \beta + 1 \Rightarrow V_\alpha \in V_{\beta+1}$ holds for $\beta + 1$.

Now assume that β is a limit ordinal number and $\forall\gamma \in \beta\forall\alpha(\alpha \in \gamma \Rightarrow V_\alpha \in V_\gamma)$. Let α be such that $\alpha \in \beta$. Since β is a limit ordinal number, we have $\alpha+1 \in \beta$. Since $V_\beta = \cup\big(V_\gamma|\gamma \in \beta\big)$, we have $V_{\alpha+1} \subset V_\beta$. Since $V_\alpha \in V_{\alpha+1}$ in this case, we have $V_\alpha \in V_\beta$.

Clearly, $\alpha = \beta \Rightarrow V_\alpha = V_\beta$. If $V_\alpha = V_\beta$, then either $\alpha < \beta$, or $\alpha = \beta$, or $\beta < \alpha$. If $\alpha < \beta$, then $V_\alpha \in V_\beta$; if $\beta < \alpha$, then $V_\beta \in V_\alpha$; therefore, $\beta = \alpha$.

If $V_\alpha \in V_\beta$, then $\alpha < \beta$, since $V_\alpha = V_\beta$ for $\alpha = \beta$, and for $\beta < \alpha$, $V_\beta \in V_\alpha$.

(3) Consider the class $\mathbf{C} \equiv \{x|x \in \mathbf{On} \wedge x \subset V_x\}$. Since $0 \subset \emptyset = V_0$, we have $0 \in \mathbf{C}$. If $\alpha \in \mathbf{C}$, then $\alpha \subset V_\alpha$ implies $\alpha + 1 \equiv \alpha \cup \{\alpha\} \subset V_\alpha \subset V_{\alpha+1}$. Let α be a limit ordinal number, and let $\alpha \subset \mathbf{C}$. By construction, $V_\alpha = \cup\big(V_\beta|\beta \in \alpha\big)$. If $x \in \alpha$, then $x \in \mathbf{C}$ means that $x \subset V_x$. Hence $x \in \mathcal{P}(V_x) \subset V_{x+1}$. Since α is a limit number $x + 1 \in \alpha$ implies $x \in V_\alpha$. Therefore, $\alpha \subset V_\alpha$ and hence $\alpha \in \mathbf{C}$. By Theorem 1, $\mathbf{C} = \mathbf{On}$.

The lemma is proved. □

Using the $\in$-induction principle, by Lemma 4 (Sec. 1.2), we prove the following von Neumann identity.

Lemma 2. $\mathbf{V} = \cup\big(V_\alpha|\alpha \in \mathbf{On}\big)$.

The proof was given in ([13], 9, Theorem 13). Moreover, von Neumann proved that this identity is equivalent to the regularity axiom (see ibid).

Lemma 3. *For any ordinal number α, $z \subset x \in V_\alpha$ implies $z \in V_\alpha$.*

Proof. Let us prove this assertion by transfinite induction.

Precisely, let $\mathbf{C} = \{\alpha|\alpha \in \mathbf{On} \wedge \forall x\forall z(z \subset x \in V_\alpha \Rightarrow z \in V_\alpha)\}$. We show that $\mathbf{C} = \mathbf{On}$.

If $\alpha = \emptyset$, then, obviously, $\alpha \in \mathbf{C}$.

Assume that $\alpha \in \mathbf{C}$. Then let us prove that $\alpha + 1 \in \mathbf{C}$. Let $z \subset x \in V_{\alpha+1}$. Since $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$, we have $z \subset x \in V_\alpha \cup \mathcal{P}(V_\alpha)$. Therefore, $x \in V_\alpha$ or $x \subset V_\alpha$. If $z \subset x \in V_\alpha$, then $z \in V_\alpha$ by the induction assumption, and hence $z \in V_{\alpha+1}$. If $x \subset V_\alpha$ and $z \subset x$, then $z \subset V_\alpha$, and, therefore, $z \in V_{\alpha+1}$. Thus, $\alpha + 1 \in \mathbf{C}$.

If α is a limit ordinal and $\forall\beta \in \alpha(\beta \in \mathbf{C})$, then $z \subset x \in V_\alpha$ implies $\exists\beta \in \alpha(z \subset x \in V_\beta)$, whence, by the induction assumption, it follows that $\exists\beta \in \alpha(z \in V_\beta)$, i.e., $z \in V_\alpha$.

Therefore, $\mathbf{C} = \mathbf{On}$ by the transfinite induction principle, and the lemma is proved. □

This lemma shows that any cumulative set V_α is quasi-transitive.

Lemma 4. *For any ordinal number α,*

$$\forall x(x \in V_\alpha \Rightarrow x \subset V_\alpha).$$

Proof. We prove this lemma also using the transfinite induction.

The desired formula hods for $\alpha = \emptyset$, since $\forall x\neg(x \in V_\emptyset)$.

For a certain ordinal number α, let

$$\forall x(x \in V_\alpha \Rightarrow x \subset V_\alpha).$$

Consider the ordinal number $\alpha+1$. If $x \in V_{\alpha+1}$, then $x \in V_\alpha \vee x \in \mathcal{P}(V_\alpha)$, or, more precisely, $x \in V_\alpha \vee x \subset V_\alpha$. In the case where $x \in V_\alpha$, we have $x \subset V_\alpha$ by the induction assumption, and since $V_\alpha \subset V_{\alpha+1}$, we have $x \subset V_{\alpha+1}$. If $x \subset V_\alpha$, then we immediately obtain from $V_\alpha \subset V_{\alpha+1}$ that $x \subset V_{\alpha+1}$.

Now let α be a limit ordinal number, and, moreover, let $\forall\beta \in \alpha\forall x(x \in V_\beta \Rightarrow x \subset V_\beta)$. Then $x \in V_\alpha$ implies $\exists\beta \in \alpha(x \in V_\beta)$; by the inductive assumption, $\exists\beta \in \alpha(x \subset V_\beta)$, and, therefore, $x \subset V_\alpha$.

The lemma is proved. □

This lemma shows that any cumulative set is transitive.

Corollary 1. *If α and β are ordinal numbers and $\alpha \leq \beta$, then $V_\alpha \subset V_\beta$.*

Corollary 2. *For any ordinal number α, the inclusion $V_\alpha \subset \mathcal{P}(V_\alpha)$ and the equality $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ hold.*

Proof. If $x \in V_\alpha$, then by this lemma, $x \subset V_\alpha$, i.e., $x \in \mathcal{P}(V_\alpha)$. Therefore, $V_\alpha \subset \mathcal{P}(V_\alpha)$. Whence $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha) = \mathcal{P}(V_\alpha)$. □

Corollary 3. *If α and β are ordinal numbers and $\alpha < \beta$, then $|V_\alpha| < |V_\beta|$.*

Proof. By the previous Corollaries 1 and 2, $V_\alpha \subset \mathcal{P}(V_\alpha) = V_{\alpha+1} \subset V_\beta$. Using the Cantor theorem, we obtain $|V_\alpha| < |\mathcal{P}(V_\alpha)| = |V_{\alpha+1}| \leq |V_\beta|$. □

Lemma 5. *For any ordinal number α, if $x \in V_{\alpha+1}$, then $x \subset V_\alpha$.*

Proof. Assume that $x \in V_{\alpha+1}$. This means that $x \in V_\alpha \vee x \subset V_\alpha$. If $x \subset V_\alpha$, then all the things are proved. If $x \in V_\alpha$, then by the previous lemma, $x \subset V_\alpha$, which is what was required. □

Lemma 6. *For any ordinal number α,*

$$\forall x\forall y(x \in V_\alpha \wedge y \in V_\alpha \Rightarrow x \cup y \in V_\alpha).$$

Proof. We use the transfinite induction principle once again.

If $\alpha = \emptyset$, then the condition of the lemma holds, since $\forall x\neg(x \in V_\emptyset)$.

Now let $\alpha = \beta + 1$ for a certain ordinal number β. Then by Lemma 5, the property $x \in V_\alpha \wedge y \in V_\alpha$ implies $x \subset V_\beta \wedge y \subset V_\beta$, and, therefore, $x \cup y \subset V_\beta$, whence $x \cup y \in V_{\beta+1}$, i.e., $x \cup y \in V_\alpha$.

Now assume that α is a limit ordinal number and $\forall\beta \in \alpha\forall x\forall y(x \in V_\beta \wedge y \in V_\beta \Rightarrow x \cup y \in V_\beta)$. Then $x, y \in V_\alpha$ implies $\exists\beta \in \alpha(x, y \in V_\beta)$, whence, by the induction assumption, we have $\exists\beta \in \alpha(x \cup y \in V_\beta)$, and, therefore, $x \cup y \in V_\alpha$, which is what was required to prove. □

Lemma 7. *For any limit ordinal number α, it follows from $x \in V_\alpha$ that $\mathcal{P}(x) \in V_\alpha$.*

Proof. Assume that α is a certain limit ordinal number and $x \in V_\alpha$. Then $\exists\beta \in \alpha$ such that $x \in V_\beta$. Let us show that $\mathcal{P}(x) \subset V_\beta$ in this case. Indeed, by Lemma 3, $x \in V_\beta$ and $z \subset x$ imply $z \in V_\beta$; therefore, $\forall z(z \in \mathcal{P}(x) \Rightarrow z \in V_\beta)$, which means that $\mathcal{P}(x) \subset V_\beta$. If $\mathcal{P}(x) \subset V_\beta$, then $\mathcal{P}(x) \in V_{\beta+1}$, i.e., $\mathcal{P}(x) \in V_\alpha$, which was required to prove. □

Corollary 1. *For any limit ordinal number α, it follows from $x, y \in V_\alpha$ that $\{x\}, \{x, y\}, \langle x, y\rangle \in V_\alpha$.*

Proof. By Lemma 7, $\mathcal{P}(x) \in V_\alpha$. By Lemma 3, it follows from $\{x\} \subset \mathcal{P}(x)$ that $\{x\} \in V_\alpha$. Now Lemma 6 implies $\{x, y\} \in V_\alpha$. By the property proved above, it follows from this that $\langle x, y\rangle \in V_\alpha$. □

Corollary 2. *For any limit ordinal number α, it follows from $X, Y \in V_\alpha$ that $X * Y \in V_\alpha$.*

Proof. Let $x \in X$ and $y \in Y$. Then $\{x\} \subset X \cup Y$ and $\{y\} \subset X \cup Y$ imply $\{x, y\} \subset X \cup Y$. By Lemma 6, $X \cup Y \in V_\alpha$. Since $\{x\} \in \mathcal{P}(X \cup Y)$ and $\{x, y\} \in \mathcal{P}(X \cup Y)$, it follows that $\langle x, y\rangle \equiv \{\{x\}, \{x, y\}\} \subset \mathcal{P}(X \cup Y)$. Hence $\langle x, y\rangle \in \mathcal{P}(\mathcal{P}(X \cup Y))$. Therefore, $X * Y \subset \mathcal{P}(\mathcal{P}(X \cup Y))$. By Lemmas 6, 7, and 3, $X * Y \in V_\alpha$. □

Lemma 8. *If $\alpha \geq \omega$, then $\omega \subset V_\alpha$. If $\alpha > \omega$, then $\omega \in V_\alpha$.*

Proof. By Lemma 1, $\omega \subset V_\omega \subset V_\alpha$. If $\alpha > \omega$, then $\omega \subset V_\omega \in V_{\omega+1}$ implies $\omega \in V_{\omega+1} \subset V_\alpha$. □

Let λ be an ordinal number. Consider the collection $K(\lambda) \equiv \left(M_\beta | \beta \in \lambda+1 \right)$ of sets $M_\beta \equiv \{x \in Map(\mathcal{P}(|V_\beta|), |\mathcal{P}(|V_\beta|)| \,|\, x \leftrightharpoons \mathcal{P}(|V_\beta|) \leftrightarrows |\mathcal{P}(|V_\beta|)|\}$ of all corresponding bijective mappings $\beta \in \lambda+1$ and the set $M(\lambda) \equiv \cup\left(M_\beta | \beta \in \lambda+1 \right)$. By the axiom of choice, there exists a choice function $ch(\lambda) : \mathcal{P}(M(\lambda)) \setminus \{\emptyset\} \to M(\lambda)$ such that $ch(\lambda)(P) \in P$ for any $P \in \mathcal{P}(M(\lambda)) \setminus \{\emptyset\}$. Since $M_\beta \subset M(\lambda)$ for $\beta \in \lambda+1$, it follows that $c_\beta(\lambda) \equiv ch(\lambda)(M_\beta) \in M_\beta$, i.e., $c_\beta(\lambda)$ is a bijection from $\mathcal{P}(|V_\beta|)$ onto $|\mathcal{P}(|V_\beta|)|$.

The following assertion can be called the *theorem on the initial synchronization of cardinality of cumulative sets*. It is new and belongs to the authors.

Theorem 1. *Let λ be an ordinal number. Then for any ordinal number $\alpha \le \lambda$, there exists a unique collection $u(\alpha) \equiv u(\lambda)(\alpha) \equiv (f_\beta | \beta \in \alpha+1)$ of bijective functions $f_\beta : V_\beta \leftrightarrows |V_\beta|$ such that*:

(1) *$f_0 \equiv \emptyset$;*
(2) *if $\gamma < \beta \in \alpha+1$, then $f_\gamma = f_\beta | V_\gamma$;*
(3) *if $\beta \in \alpha+1$ and $\beta = \gamma+1$, then $f_\beta | V_\gamma = f_\gamma$ and $f_\beta(x) = c_\gamma(\lambda)(f_\gamma[x])$ for each $x \in V_\beta \setminus V_\gamma = \mathcal{P}(V_\gamma) \setminus V_\gamma$;*
(4) *if $\beta \in \alpha+1$ and β is a limit ordinal number, then $f_\beta = \cup\left(f_\gamma | \gamma \in \beta \right)$.*

The uniqueness property implies $u(\alpha)|\delta+1 = u(\delta)$ for each $\delta \le \alpha$, i.e., these collections extend each other.

Proof. We first verify the uniqueness of the collection $u \equiv u(\alpha)$. For α, let there exist a collection $v \equiv (g_\beta | \beta \in \alpha+1)$ of bijective functions $g_\beta : V_\beta \leftrightarrows |V_\beta|$ having Properties (1)–(4). Consider the set $D' \equiv \{\beta \in \alpha+1 | f_\beta = g_\beta\}$, the class $\mathbf{D}'' \equiv \mathbf{On} \setminus (\alpha+1)$, and the class $\mathbf{D} \equiv D' \cup \mathbf{D}''$. Clearly, $0 \in D' \subset \mathbf{D}$.

Let $\beta \in \mathbf{D}$. If $\beta \ge \alpha$, then $\beta+1 \in \mathbf{D}'' \subset \mathbf{D}$. Let $\beta < \alpha$. Then $\beta \in D'$ and $\beta+1 \in \alpha+1$. Therefore, by Property (3), $f_{\beta+1}(x) = f_\beta(x) = g_\beta(x) = g_{\beta+1}(x)$ for any $x \in V_\beta$ and $f_{\beta+1}(x) = c_\beta(\lambda)(f_\beta[x]) = c_\beta(\lambda)(g_\beta[x]) = g_{\beta+1}(x)$ for any $x \in V_{\beta+1} \setminus V_\beta$, i.e., $f_{\beta+1} = g_{\beta+1}$. Thus, $\beta+1 \in D' \subset \mathbf{D}$. Therefore, $\beta \in \mathbf{D}$ implies $\beta+1 \in \mathbf{D}$.

Let β be a limit ordinal number, and let $\beta \subset \mathbf{D}$. If $\beta \cap \mathbf{D}'' \ne \emptyset$, then there exists $\gamma \in \beta$ such that $\gamma \ge \alpha+1$. Hence $\beta > \gamma \ge \alpha+1$ implies $\beta \in \mathbf{D}'' \subset \mathbf{D}$. Let $\beta \cap \mathbf{D}'' = \emptyset$, i.e., let $\beta \subset D'$. Then for any $\gamma \in \beta$, $f_\gamma = g_\gamma$ holds. Since $\beta \subset \alpha+1$, it follows that $\beta \le \alpha+1$. If $\beta = \alpha+1$, then $\beta \in \mathbf{D}'' \subset \mathbf{D}$. Let $\beta \in \alpha+1$. If $x \in V_\beta = \cup\left(V_\gamma | \gamma \in \beta \right)$, then $x \in V_\gamma$ for a certain $\gamma \in \beta$. Then by Property (2), $f_\beta(x) = f_\gamma(x) = g_\gamma(x) = g_\beta(x)$ for any $x \in V_\beta$, i.e., $f_\beta = g_\beta$. Thus, $\beta \in D' \subset \mathbf{D}$. Therefore, the properties that β is a limit ordinal number and that $\beta \subset \mathbf{D}$ imply $\beta \in \mathbf{D}$.

By the transfinite induction principle, $\mathbf{D} = \mathbf{On}$. Hence $D' = \alpha+1$. Therefore, $u = v$.

In what follows, instead of $c_\gamma(\lambda)$, we will merely write c_γ.

Consider the set C' consisting of all ordinal numbers $\alpha \le \lambda$ for which there exists a collection $u(\alpha)$ having Properties (1)–(4). Also, consider the classes $\mathbf{C}'' \equiv \mathbf{On} \setminus (\lambda+1)$ and $\mathbf{C} \equiv C' \cup \mathbf{C}''$. Since $V_0 = \emptyset$ and $|V_0| = 0$, the collection $u(0) \equiv (f_\beta | \beta \in 1)$ with the bijective function $f_0 = \emptyset : V_0 \leftrightarrows |V_0|$ has Properties (1)–(4), and, therefore, $0 \in \mathbf{C}$.

Let $\alpha \in \mathbf{C}$. If $\alpha \ge \lambda$, then $\alpha+1 \in \mathbf{C}'' \subset \mathbf{C}$. Now let $\alpha < \lambda$. Then $\alpha+1 \in \lambda+1$ means that we can use the function c_α. Since $\alpha \in C'$, for α, there exists a unique collection $u \equiv (f_\beta | \beta \in \alpha+1)$. Define the collection $v \equiv (g_\beta | \beta \in \alpha+2)$ of bijective functions $g_\beta : V_\beta \leftrightarrows |V_\beta|$ setting $g_\beta \equiv f_\beta$ for any $x \in V_\alpha$ and $g_{\alpha+1}(x) \equiv c_\alpha(f_\alpha[x])$ for any $x \in V_{\alpha+1} \setminus V_\alpha = \mathcal{P}(V_\alpha) \setminus V_\alpha$.

Let us verify that v has Properties (1)–(4). Let $\beta \in \alpha+2$. If $\beta \in \alpha+1$, then Properties (1)–(4) obviously hold. Let $\beta = \alpha+1$. Then $g_\beta(x) = g_{\alpha+1}(x) = f_\alpha(x) = g_\alpha(x)$ for any $x \in V_\alpha$ and $g_\beta(x) = g_{\alpha+1}(x) = c_\alpha(f_\alpha[x]) = c_\alpha(g_\alpha[x])$ for any $x \in V_\beta \setminus V_\alpha$. Moreover, $g_\beta | V_\alpha = f_\alpha = g_\alpha$. Therefore, $\gamma < \beta$ implies $g_\beta | V_\gamma = g_\alpha | V_\gamma = f_\alpha | V_\gamma = f_\gamma = g_\gamma$. Thus, $\alpha+1 \in C' \subset \mathbf{C}$.

Let α be a limit ordinal number, and let $\alpha \subset \mathbf{C}$. If $\alpha \cap \mathbf{C}'' \ne \emptyset$, then there exists $\beta \in \alpha$ such that $\beta \ge \alpha+1$. Hence $\alpha > \beta \le \lambda+1$ implies $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha \cap \mathbf{C}'' = \emptyset$, i.e., let $\alpha \subset C'$. Then for any $\beta \in \alpha$, there exists a unique collection $u_\beta \equiv (f_\gamma^\beta | \gamma \in \beta+1)$ of bijective functions $f_\gamma^\beta : V_\gamma \leftrightarrows |V_\gamma|$ with Properties (1) – (4). Since $\alpha \subset \lambda+1$, it follows that $\alpha \le \lambda+1$. If $\alpha = \lambda+1$, then $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Further, let $\alpha \in \lambda+1$. For any $\delta \le \beta \in \alpha$, consider the collection $w \equiv u_\beta | \delta+1 \equiv (f_\gamma^\beta | \gamma \in \delta+1)$. The collection w

has Properties (1)–(4). By the uniqueness proved above, $w = u_\delta$. Therefore, $u_\delta = u_\beta|\delta+1$, i.e., $f_\gamma^\delta = f_\gamma^\beta$ for any $\gamma \in \delta+1$. In particular, $f_\delta^\delta = f_\delta^\beta$ for any $\delta \le \beta$.

Define the collection $v \equiv (g_\beta|\beta \in \alpha+1)$ of functions g_β setting $g_\beta \equiv f_\beta^\beta$ for any $\beta \in \alpha$ and $g_\alpha(x) \equiv f_\gamma^\beta(x)$ for any $x \in V_\alpha = \cup\left(V_\gamma|\gamma \in \alpha\right)$ and any $\gamma \le \beta \in \alpha$ such that $x \in V_\gamma$. Clearly, $g_\beta \leftrightharpoons V_\beta \leftrightarrows |V_\beta|$ for any $\beta \in \alpha$. Let us verify that $g_\alpha \leftrightharpoons V_\alpha \to |V_\alpha|$. By Corollary 1 of Lemma 4 (Sec. 2.2), $V_\gamma \subset V_\alpha$. Hence $|V_\gamma| \subset |V_\alpha|$. Therefore, for any $x \in V_\alpha$, we have $g_\alpha(x) \equiv f_\gamma^\beta(x) \in |V_\gamma| \subset \varkappa \equiv \cup\left(|V_\gamma| \subset |V_\alpha|\,|\gamma \in \alpha\right) \subset |V_\alpha|$. Let $x, y \in V_\alpha$ and $g_\alpha(x) = g_\alpha(y)$. Then $x \in V_\gamma$ and $y \in V_\delta$ for certain $\gamma, \delta \in \alpha$. Consider the number β being maximal among the numbers γ and δ. By definition, $f_\beta^\beta(x) = g_\alpha(x) = g_\alpha(y) = f_\beta^\beta(y)$. From the injectivity of this function, we conclude that $x = y$. Hence the function g_α is injective. Let $z \in \varkappa$. Then $z \in |V_\gamma|$ for a certain $\gamma \in \alpha$. Since the function $f_\gamma^\gamma : V_\gamma \leftrightarrows |V_\gamma|$ is injective, it follows that $z = f_\gamma^\gamma(x)$ for a certain $x \in V_\gamma \subset V_\alpha$. Hence $z = g_\alpha(x)$. Therefore, g_α is a bijective function from V_α onto $\varkappa$, i.e., $V_\alpha \sim \varkappa$.

By Corollary 3 of Lemma 4 (Sec. 2.2), $|V_\gamma| \in |V_\alpha|$. Therefore, there exists a set $A \equiv \{x \in |V_\alpha|\,|\exists y \in \alpha(x = |V_y|)\} = \{|V_\gamma|\,|\gamma \in \alpha\}$ of ordinal numbers $|V_\gamma|$. Since α is a limit ordinal number, it follows that $A \ne \emptyset$. Therefore, by Lemma 2 (Sec. 1.2), the set $\cup A = \sup A$ is an ordinal number. If $z \in \cup A = \{z|\exists x \in A(z \in x)\}$, then $z \in |V_\gamma| \subset \varkappa$ for a certain $\gamma \in \alpha$. Conversely, if $z \in \varkappa$, then $z \in |V_\gamma| \in A$ for a certain $\gamma \in \alpha$. Therefore, $z \in \cup A$. Thus, $\varkappa = \cup A$, i.e., $\varkappa$ is an ordinal number.

Let us prove that $\varkappa$ is a cardinal number. Let β be an ordinal number, $\beta \le \varkappa$, and let $\beta \sim \varkappa$. Assume that $\beta < \varkappa$. Then $\beta \in \varkappa$ implies $\beta \in |V_\gamma|$ for a certain $\gamma \in \alpha$. Hence $\beta < |V_\gamma| = card\,|V_\gamma| \le |\varkappa| = |\beta|$. Since β is an ordinal number, it follows that $|\beta| \le \beta$. As a result, we arrive at the inequality $\beta < \beta$, which is impossible. The obtained contradiction implies $\beta = \varkappa$. This means that $\varkappa$ is a cardinal number.

Since $\varkappa$ is a cardinal number and $\varkappa \sim V_\alpha$, it follows that $\varkappa = |V_\alpha|$. Therefore, $g_\alpha \leftrightharpoons V_\alpha \leftrightarrows |V_\alpha|$.

Let us verify that the collection v has Properties (1)–(4). By the definition of this collection, $g_0 \equiv f_0^0 = \emptyset$. Let $\gamma < \beta \in \alpha+1$. If $\beta \in \alpha$, the equality $f_\gamma^\gamma = f_\gamma^\beta$ proved above implies $g_\beta|V_\gamma = f_\beta^\beta|V_\gamma = f_\gamma^\beta = f_\gamma^\gamma \equiv g_\gamma$. If $\beta = \alpha$, then by construction, $g_\beta|V_\gamma = g_\alpha|V_\gamma = f_\gamma^\gamma \equiv g_\gamma$. Therefore, Property (2) holds for v.

Let $\beta \in \alpha+1$, $\beta = \gamma+1$, and let $x \in V_\beta = \mathcal{P}(V_\gamma)$. If $\beta \in \alpha$, then the equality $f_\gamma^\gamma = f_\gamma^\beta$ proved above implies $g_\beta(x) = f_\beta^\beta(x) = f_\gamma^\beta(x) = f_\gamma^\gamma(x) = g_\gamma(x)$ for any $x \in V_\gamma$ and $g_\beta(x) = f_\beta^\beta(x) = c_\gamma(f_\gamma^\beta[x]) = c_\gamma(f_\gamma^\gamma[x]) = c_\gamma(g_\gamma[x])$ for any $x \in V_\beta \setminus V_\gamma$. Therefore, Property (3) holds for v.

Property (4) follows from Property (2). It follows from the verified properties that $\alpha \in C' \subset \mathbf{C}$.

By the transfinite induction principle, $\mathbf{C} = \mathbf{On}$, and, therefore, $C' = \lambda+1$. ☐

Note that since the functions $c_\gamma(\lambda)$ depend on the number λ, we cannot compose the collections $u(\lambda)(\alpha)$ extending each other into a global collection indexed by all order numbers.

Corollary. *For any limit ordinal number α, we have* $|V_\alpha| = \cup\left(|V_\beta|\,|\beta \in \alpha\right) = \cup\{|V_\beta|\,|\beta \in \alpha\} = \sup\{|V_\beta|\,|\beta \in \alpha\}$.

Proof. Consider the number $\lambda \equiv \alpha$. By Theorem 1, there exists the corresponding collection $u(\alpha) \equiv (f_\beta|\beta \in \alpha+1)$. Since α is a limit ordinal number and $\alpha \in \alpha+1$, by Property (4), it follows that $f_\alpha = \cup\left(f_\beta|\beta \in \alpha\right)$. Therefore, $|V_\alpha| = rng\,f_\alpha = \cup\left(rng\,f_\beta|\beta \in \alpha\right) = \cup\left(|V_\beta|\,|\beta \in \alpha\right) = \cup\{|V_\beta|\,|\beta \in \alpha\} = \sup\{|V_\beta|\,|\beta \in \alpha\}$, where the latter equality follows from Lemma 2 (Sec. 1.2). ☐

2.3. Properties of inaccessible cumulative sets. The sets $V_\varkappa$ for inaccessible cardinal numbers $\varkappa$s will be called *inaccessible cumulative sets*. They have a number of specific properties. We present these properties with complete proofs, since their proofs are practically absent in ([13], 13) and are not obvious.

Lemma 1. *For any inaccessible cardinal number $\varkappa$ and any ordinal number $\alpha \in \varkappa$, we have $|V_\alpha| < \varkappa$.*

Proof. Consider the set $C' \equiv \{x \in \varkappa|\,|V_x| < \varkappa\}$ and the classes $\mathbf{C}'' \equiv \mathbf{On} \setminus \varkappa$ and $\mathbf{C} \equiv C' \cup \mathbf{C}''$. Since $V_0 = \emptyset$, it follows that $|V_0| = 0 < \varkappa$. Therefore, $0 \in \mathbf{C}$.

Let $\alpha \in \mathbf{C}$. If $\alpha \ge \varkappa$, then $\alpha+1 \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha < \varkappa$. Then $\alpha \in C'$. If $\alpha+1 = \varkappa$, then $\alpha+1 \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha+1 < \varkappa$. Since $V_\alpha \sim |V_\alpha|$, it follows that $\mathcal{P}(V_\alpha) \sim \mathcal{P}(|V_\alpha|)$. Hence $|\mathcal{P}(V_\alpha)| = |\mathcal{P}(|V_\alpha|)|$. By Corollary

2 of Lemma 4 (Sec. 2.2), $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = |\mathcal{P}(|V_\alpha|)|$. Since $|V_\alpha| < \varkappa$ and the cardinal number $\varkappa$ is inaccessible, it follows that $|\mathcal{P}(|V_\alpha|)| < \varkappa$. Hence $|V_{\alpha+1}| < \varkappa$. Therefore, $\alpha + 1 \in C' \subset \mathbf{C}$.

Let α be a limit ordinal number, and let $\alpha \subset \mathbf{C}$. If $\alpha \cap \mathbf{C}'' \neq \emptyset$, then there exists $\beta \in \alpha$ such that $\beta \geq \varkappa$. Hence $\alpha > \beta \geq \varkappa$ implies $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha \cap \mathbf{C}'' = \emptyset$, i.e., $\alpha \subset C' \subset \varkappa$. If $\alpha = \varkappa$, then $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha < \varkappa$. By $\alpha \subset C'$, for any $\beta \in \alpha$, we have $|V_\beta| < \varkappa$. Hence, $\sup\{|V_\beta| \,|\beta \in \alpha\} \leq \varkappa$.

Using the properties of $|V_\beta| \in \varkappa$, we can correctly define the function $f : \alpha \to \varkappa$ setting $f(\beta) \equiv |V_\beta|$. Clearly, $rng\, f = \{|V_\beta| \,|\beta \in \alpha\}$. By the corollary of Theorem 1, $\cup rng\, f = \cup\{|V_\beta| \,|\beta \in \alpha\} = \sup\{|V_\beta| \,|\beta \in \alpha\} = |V_\alpha|$. By the inequality proved above, we obtain $|V_\alpha| \leq \varkappa$. Assume that $|V_\alpha| = \varkappa$. Then by the regularity of the number $\varkappa$, $\varkappa = \cup rng\, f$ implies $\varkappa \leq \alpha$, which contradicts the initial inequality $\alpha < \varkappa$. Thus, $|V_\alpha| < \varkappa$. Therefore, $\alpha \in C' \subset \mathbf{C}$.

By the transfinite induction principle, $\mathbf{C} = \mathbf{On}$. Therefore, $C' = \varkappa$. □

Lemma 2. *If $\varkappa$ is an inaccessible cardinal, then $\varkappa = |V_\varkappa|$.*

Proof. By Lemma 1 (Sec. 2.2), $\varkappa \subset V_\varkappa$. Hence $\varkappa = |\varkappa| \leq |V_\varkappa|$. By the corollary of Theorem 1 (Sec. 2.2), $|V_\varkappa| = \sup(|V_\beta| \,|\beta \in \varkappa)$. Since $|V_\beta| < \varkappa$ by Lemma 1, it follows that $|V_\varkappa| \leq \varkappa$. As a result, we obtain $\varkappa = |V_\varkappa|$. □

Lemma 3. *If $\varkappa$ is an inaccessible cardinal number, α is an ordinal number such that $\alpha < \varkappa$, and f is a correspondence from V_α into $V_\varkappa$ such that $dom\, f = V_\alpha$ and $f\langle x\rangle \in V_\varkappa$ for any $x \in V_\alpha$, then $rng\, f \in V_\varkappa$.*

Proof. Since $\varkappa$ is a limit ordinal number, it follows that $V_\varkappa = \cup\big(V_\delta|\delta \in \varkappa\big)$. For $x \in V_\alpha$, there exists $\delta \in \varkappa$ such that $f\langle x\rangle \in V_\delta$. Therefore, the nonempty set $\{y \in \varkappa | f\langle x\rangle \in V_y\}$ has a minimal element z. By the uniqueness of z, we can correctly define the function $g : V_\alpha \to \varkappa$ setting $g(x) \equiv z$. Consider the ordinal number $\beta \equiv |V_\alpha|$ and take a certain bijective mapping $h : \beta \leftrightarrows V_\alpha$. Consider the mapping $\varphi \equiv g \circ h : \beta \to \varkappa$ and the ordinal number $\gamma \equiv \cup rng\, \varphi = \sup rng\, \varphi \leq \varkappa$.

Assume that $\gamma = \varkappa$. Since the cardinal $\varkappa$ is regular, the assumption $\cup rng\, \varphi = \varkappa$ implies $\varkappa \leq \beta \equiv |V_\alpha|$. However, by Lemma 1, $|V_\alpha| < \varkappa$. The obtained contradiction implies $\gamma < \varkappa$.

Since h is bijective, then $rng\, \varphi = rng\, g$. Hence $\gamma = \sup rng\, g$. If $x \in V_\alpha$, then $f\langle x\rangle \in V_z = V_{g(x)}$. By Lemma 1 (Sec. 2.2), it follows from $g(x) \leq \gamma$ that $V_{g(x)} \subset V_\gamma$. Hence, by Lemma 1 (Sec. 2.2), $f\langle x\rangle \in V_\gamma$ implies $f\langle x\rangle \subset V_\gamma$. Therefore, $rng\, f \subset V_\gamma$. By Lemma 1 (Sec. 2.2), $rng\, f \in V_{\gamma+1} \subset V_\varkappa$. □

Lemma 4. *If $\varkappa$ is an inaccessible cardinal number, $A \in V_\varkappa$, and f is a correspondence from A into $V_\varkappa$ such that $f\langle x\rangle \in V_\varkappa$ for any $x \in A$, then $rng\, f \in V_\varkappa$.*

Proof. Since $\varkappa$ is a limit ordinal number, it follows that $V_\varkappa = \cup\big(V_\alpha|\alpha \in \varkappa\big)$. Therefore, $A \in V_\alpha$ for a certain $\alpha \in \varkappa$. By Lemma 4 (sec. 2.2), $A \subset V_\alpha$. Define the correspondence g from V_α into $V_\varkappa$ setting $g|A \equiv f$ and $g\langle x\rangle \equiv \emptyset \subset V_\varkappa$ for any $x \in V_\alpha \setminus A$. Then $dom\, g = V_\alpha$ and $rng\, g = rng\, f$. If $x \in A$, then $g\langle x\rangle = f\langle x\rangle \in V_\varkappa$, and if $x \in V_\alpha \setminus A$, then $g\langle x\rangle = \emptyset \in V_\varkappa$. Therefore, by Lemma 3, $rng\, f = rng\, g \in V_\varkappa$. □

Corollary 1. *If $\varkappa$ is an inaccessible cardinal number and $\big(B_a|a \in A\big)$ is a collection of sets such that $A \in V_\varkappa$ and $B_a \in V_\varkappa$ for any $a \in A$, then $\cup\big(B_a|a \in A\big) \in V_\varkappa$.*

Corollary 2. *If $\varkappa$ is an inaccessible cardinal number and $A \in V_\varkappa$, then $\cup A \in V_\varkappa$.*

The following assertion is due to Tarski [26] (see also [17], IX, § 1, Theorem 6). Here, we present another proof of this assertion.

Lemma 5. *If $\varkappa$ is an inaccessible cardinal number, $A \subset V_\varkappa$, and $|A| < |V_\varkappa|$, then $A \in V_\varkappa$.*

Proof. By Lemma 2, $|A| \in |V_\varkappa| = \varkappa \subseteq V_\varkappa$. Consider the bijection $b : |A| \leftrightarrows A \subset V_\varkappa$. By Lemma 4, $A = rng\, b \in V_\varkappa$. □

Lemma 6. *If $\varkappa$ is an inaccessible cardinal number, ε is an ordinal number, and $\varepsilon \in V_\varkappa$, then $\varepsilon \in \varkappa$.*

Proof. Since $V_\varkappa = \cup\big(V_\alpha|\alpha \in \varkappa\big)$, it follows that $\varepsilon \in V_\alpha$ for a certain $\alpha \in \varkappa$. By Lemma 4 from Sec. 2.2, $\varepsilon \subset V_\alpha$. By Lemma 1, $|\varepsilon| \leq |V_\alpha| < \varkappa$. Assume that $\varepsilon \geq \varkappa$. Then $\varkappa \subset \varepsilon$ implies $\varkappa = |\varkappa| \leq |\varepsilon|$, which contradicts the previous inequality. Therefore, $\varepsilon < \varkappa$. □

3. Universal Sets and Their Connection with Inaccessible Cumulative Sets

3.1. Universal sets and their properties. A set U in ZF theory is said to be *universal* (see [18], I, 6, [7], [12]) if it has the following properties:

(1) $x \in U \Rightarrow x \subset U$ (*transitivity property*);

(2) $x \in U \Rightarrow \mathcal{P}(x), \cup x \in U$;

(3) $x \in U \wedge y \in U \Rightarrow x \cup y, \{x, y\}, \langle x, y\rangle, x * y \in U$;

(4) $x \in U \wedge (f \in U^x) \Rightarrow rng\, f \in U$ (*strong substitution property*);

(5) $\omega \in U$.

Clearly, not all of these properties are independent.

The property that a set U is universal will be denoted by $U \bowtie$. Denote by $\mathbf{U}$ the class (possibly, empty) of all universal sets. It immediately follows from the definition of a universal set that the intersection $\cap\mathbf{A} \equiv \{x|\forall U \in \mathbf{A}(x \in U)\}$ of any nonempty subclass $\mathbf{A}$ of the class of universal sets is a universal set.

Let us deduce several properties of universal sets from these conditions.

Lemma 1. *If a set U is universal, then $x \in U \wedge y \subset x \Rightarrow y \in U$.*

Proof. If $x \in U$, then by Property (2), $\mathcal{P}(x) \in U$, and, by Property (1), $\mathcal{P}(x) \subset U$. Since $y \in \mathcal{P}(x)$, it follows that $y \in U$, which is what was required to prove. □

This lemma shows that a universal set is quasi-transitive. This and the transitivity property imply that a universal set is supertransitive.

Lemma 2. *If a set U is universal, then $\emptyset \in U$.*

Proof. This obviously follows from the property that $\omega \in U$ and Property (1). □

Lemma 3. *Let $\left(A_i|i \in I\right)$ be a collection such that $I \in U$ and $A_i \in U$ for any $i \in I$. Then $\cup\left(A_i|i \in I\right) \in U$.*

Proof. Consider a function $f : I \to U$ such that $f(i) \equiv A_i$. By Property (4), $rng\, f \in U$, and, by Property (2), $\cup\left(A_i|i \in I\right) = \cup rng\, f \in U$. □

Lemma 4. *If U is a universal set, then $x \in U \Rightarrow |x| \in U$.*

Proof. Consider the class $\mathbf{C} \equiv \{\alpha \in \mathbf{On}|\alpha \notin U)\}$. This class is nonempty, since otherwise the class $\mathbf{On}$ is a set. Denote its minimal element by $\varkappa$. Assume that there exists $x \in U$ such that $\alpha \equiv |x| \notin U$. Then there exists a one-to-one mapping $f : \alpha \to x$. It follows from $\alpha \in \mathbf{C}$ that $\varkappa \leq \alpha$. Since $\varkappa \subset \alpha$, we can consider the mapping $g \equiv f|\varkappa$. In this case, g is a one-to-one mapping from $\varkappa$ onto $y \equiv rng\, g \subset x$. Since $y \subset x$, it follows that $y \in U$. Then $h \equiv g^{-1}$ is a function from $y \in U$ onto $\varkappa \notin U$. Since $\varkappa$ is a minimal element in the class $\mathbf{C}$, it follows that $\forall \beta \in \varkappa(\beta \in U)$. Therefore, $h(z) \in \varkappa$ implies $h(z) \in U$ for any $z \in y$. By Property (4) it follows from this that $\varkappa \in U$, which contradicts the definition of $\varkappa$. We conclude from the obtained contradiction that $\forall x \in U(|x| \in U)$. □

Let us prove that in a universal set, there exists a $\in$-induction principle analogous to the $\in$-induction principle in ZF (see Lemma 4 (Sec. 1.2)).

Lemma 5. *Let U be a universal set, $C \subset U$, and $\forall x \in U$, let $(x \subset C \Rightarrow x \in C)$. Then $C = U$.*

Proof. Assume that $C \neq U$, i.e., $D \equiv U \setminus C \neq \emptyset$. Then there exists $P \in D$. Clearly, $P \in U$. If $P \cap D = \emptyset$, then we set $X \equiv P$.

Let $P \cap D \neq \emptyset$. Consider the set N consisting of all $n \in \omega$ for which there exists a unique sequence $u \equiv u(n) \equiv (R_k \in U|k \in n+1)$ of sets $R_k \in U$ such that $R_0 = P$ and $R_{k+1} = \cup R_k$ for any $k \in n$. Since the sequence $(R_k|k \in 1)$ such that $R_0 \equiv P$ has this property, it follows that $0 \in N$. Let $n \in N$, i.e., for n, there exists a unique sequence $u \equiv (R_k \in U|k \in n+1)$. Define the sequence $v \equiv (S_k \in U|k \in n+2)$ setting $S_k \equiv R_k \in U$ for any $k \in n+1$ and $S_{n+1} \equiv \cup R_n = \cup S_n$, i.e., $v = u \cup \{\langle n+1, \cup R_n\rangle\}$. Since U

is a universal set, $R_n \in U$ implies $S_{n+1} \in U$. Clearly, the sequence v has the necessary properties. Let us verify its uniqueness. Assume that there exists a sequence $w \equiv (T_k \in U | k \in n+2)$ such that $T_0 = P$ and $\forall k \in n+1 (T_{k+1} = \cup T_k)$. Consider the set M' consisting of all $m \in n+2$ for which $S_m = T_m$. Let $M'' \equiv \omega \setminus (n+2)$, and let $M \equiv M' \cup M''$. Since $S_0 = P = T_0$, it follows that $0 \in M' \subset M$.

Let $m \in M'$. If $m = n+1$, then $m+1 = n+2 \in M'' \subset M$. If $m < n+1$, then $m+1 \in n+2$, and $S_{m+1} = \cup S_m = \cup T_m = T_{m+1}$ implies $m+1 \in M' \subset M$. If $m \in M''$, then $m+1 \in M'' \subset M$. Therefore, $m \in M$ implies $m+1 \in M$. By the natural induction principle, $M = \omega$. Hence $M' = n+2$, and, therefore, $v = w$, i.e., the sequence v is unique. Therefore, $n+1 \in N$. By the natural induction principle, $N = \omega$. Therefore, for any $n \in \omega$, there exists a unique sequence $u(n)$. Because of its uniqueness, it will be denoted by $(R_k^n | k \in n+1)$.

Consider the following formula of the ZF theory: $\varphi(x,y) \equiv (x \in \omega \Rightarrow y = R_x^x) \wedge (x \notin \omega \Rightarrow y = \emptyset)$. According to the axiom substitution scheme, there exists a set Y such that $\forall x \in \omega (\forall y (\varphi(x,y) \Rightarrow y \in Y)$. If $n \in \omega$, then $\varphi(n, R_n^n)$ implies $R_n^n \in Y$. Therefore, in the set $\omega \times Y$, we can define the infinite sequence $u \equiv (R_n \in Y | n \in \omega)$ setting $u \equiv \{z \in \omega \times Y | \exists x \in \omega (z = \langle x, R_x^x \rangle)\}$. It immediately follows from the uniqueness property mentioned above that $u(m) = u(n)|(m+1)$ for all $m \leq n$. Therefore, $u|(n+1) = u(n)$. Consequently, the sequence u has the following properties $R_0 = P$ and $R_{k+1} = \cup R_k$ for any $k \in \omega$. Having the function $u : \omega \to U$, Properties (5), (4), and (2) from the definition of a universal set, we can take the set $A \equiv rng\, u \equiv \{R_n \in U | n \in \omega\} \in U$ and the set $Q \equiv \cup A = \{y | \exists x \in \omega (y \in R_x)\} = \cup \left(R_n | n \in \omega \right) \in U$. Clearly, $R_n \subset Q$ for any $n \in \omega$, and hence $P = R_0 \subset Q$.

Since $P \cap D \neq \emptyset$, it follows that $R \equiv Q \cap D \neq \emptyset$. By the regularity axiom **A8**, there exists $X \in R$ such that $X \cap R = \emptyset$. Clearly, $X \in U$ and $X \subset U$. Let us verify that $X \cap D = \emptyset$. Indeed, assume that there exists $x \in X \cap D$. Since $X \in Q$, it follows that $X \in R_n$ for a certain $n \in \omega$. Hence $x \in X \in R_n$ implies $x \in \cup R_n = R_{n+1} \subset Q$. Therefore, $x \in R$. As a result, we obtain $x \in X \cap R = \emptyset$, which is impossible. The obtained contradiction implies $X \in D$ and $X \cap D = \emptyset$.

Therefore, $X \in U$ and $X \subset C$ in both cases. By the condition, we then obtain $X \in C$, which is impossible, since $X \in D$. This contradiction implies $C = U$. □

For a universal set, the following analog of the von Neumann identity from Lemma 2 (Sec. 2.2) holds.

Lemma 6. *Let U be a universal set. Then*:

1. $V_\alpha \in U$ *for any* $\alpha \in \mathbf{On} \cap U$;
2. $U = \cup \left(V_\alpha \subset U | \alpha \in \mathbf{On} \cap U \right)$.

Proof. 1. Consider the sets $A \equiv \mathbf{On} \cap U$ and $C' \equiv \{\alpha \in A | V_\alpha \in U\}$ and also the classes $\mathbf{C}'' \equiv \mathbf{On} \setminus U$ and $\mathbf{C} \equiv C' \cup \mathbf{C}''$. By Lemma 2, $0 = V_0 = \emptyset \in U$. Let $0 \in \mathbf{C}$. Let $\alpha \in \mathbf{C}$. Assume that $\alpha + 1 \in A$. Since $\alpha \in \alpha + 1 \in U$, by Property (1), it follows that $\alpha \in U$, and hence $\alpha \in A \cap \mathbf{C} = C'$. Then by Properties (2) and (3), the condition $V_\alpha \in U$ implies $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha) \in U$. Therefore, $\alpha + 1 \in C' \subset \mathbf{C}$. In the case $\alpha + 1 \notin A$, we immediately obtain $\alpha + 1 \in \mathbf{C}'' \subset \mathbf{C}$.

Let α be a limit ordinal number, and let $\alpha \subset \mathbf{C}$. Assume that $\alpha \in A$. If $\beta \in \alpha$, then $\beta \in \alpha \in U$ implies $\beta \in A \cap \mathbf{C} = C'$. Then by Lemma 3, the condition $V_\beta \in U$ implies $V_\alpha = \cup \left(V_\beta | \beta \in \alpha \right) \in U$. Therefore, $\alpha \in C' \subset \mathbf{C}$. In the case $\alpha \notin A$, we immediately obtain $\alpha \in \mathbf{C}'' \subset \mathbf{C}$.

By the transfinite induction principle, $\mathbf{C} = \mathbf{On}$, and hence $C' = A$.

2. It follows from what was proved above that $V_\alpha \subset U$ for any $\alpha \in A$. Therefore,

$$P \equiv \cup \left(V_\alpha | \alpha \in A \right) \subset U.$$

Let us show that P satisfies the $\in$-induction principle from Lemma 5. Define the function $r : P \to A$ setting $r(p) \equiv sm\,\{\alpha \in A | p \in V_\alpha\}$ for any $p \in P \subset U$.

Let $x \in U$, and let $x \subset P$. If $x = \emptyset$, then $x \in P$. In what follows, we assume that $x \neq \emptyset$. If $y \in x \subset P$, then $y \in V_\alpha$ for a certain $\alpha \in A$. Consequently, by Lemma 1, $r(y) \leq \alpha \in U$ implies $r(y) \in A$. Therefore, we can consider the function $s \equiv r|x$ from x into A. By Property (4), $R \equiv rng\, s \in U$, and $\rho \equiv \cup R \in U$ by Property (2). Since $\emptyset \neq R \subset \mathbf{On}$, ρ is an ordinal number by Lemma 2 (Sec. 1.2) Hence $\rho \in A$.

If $y \in x$, then by Lemma 1 (Sec. 2.2), $s(y) \subset \rho$ implies $y \in V_{s(y)} \subset V_\rho$. Therefore, by Lemma 3 (Sec. 2.2) $x \subset V_\rho \in V_{\rho+1}$ implies $x \in V_{\rho+1}$. By Property (3), $\rho + 1 = \rho \cup \{\rho\} \in U$ implies $\rho + 1 \in A$. Hence $x \in P$.

Now, Lemma 5 implies $P = U$. □

3.2. Description of the class of all universal sets. The following theorem is implied by the Zermelo–Shepherdson theorem (see [28] (incomplete proof) and [23] (complete proof)) on the canonical form of standard supertransitive model sets for the NBF theory in the ZF set theory (see Sec. 5 below). Here, we give another proof.

Theorem 1. *Let U be an arbitrary universal set. Then*:

(1) $\varkappa \equiv \sup(\mathbf{On} \cap U) = \cup(\mathbf{On} \cap U) \subset U$ *is an inaccessible cardinal number*;

(2) $U = V_\varkappa$;

(3) *the correspondence $q : U \mapsto \varkappa$ such that $U = V_\varkappa$ is an injective isotone mapping from the class* $\mathbf{U}$ *of all universal sets into the class* $\mathbf{In}$ *of all inaccessible cardinal numbers.*

Proof. (1) Since $A \equiv \mathbf{On} \cap U$ is a nonempty set, because it contains the element ω by Property (5), by Lemma 2 (Sec. 1.2), it follows that $\varkappa$ is an ordinal number.

Assume that $\varkappa$ is not a cardinal number. In this case, there exist an ordinal number $\alpha < \varkappa$ and a bijective function $f : \alpha \to \varkappa$. Since $\alpha \in \varkappa \subset U$, It follows that $\alpha \in U$. If $\beta \in \alpha$, then $f(\beta) \in U$. Therefore, by Property (4), $\varkappa = rng\, f \in U$. In this case, by Property (3), $\{\varkappa\} \in U$, and also, by Property (3) $\varkappa^+ \equiv \varkappa \cup \{\varkappa\} \in U$. Since $\varkappa^+ \in \mathbf{On}$, it follows that $\varkappa^+ \in A$, i.e., $\varkappa^+ \leq \varkappa$, which is impossible. Therefore, we conclude from the obtained contradiction that $\varkappa$ is a cardinal number.

Now assume that the cardinal number $\varkappa$ is not regular. Then $\alpha \equiv cf(\varkappa) < \varkappa$. By definition, there exists a function $f : \alpha \to \varkappa$ such that $\cup rng\, f = \varkappa$. As before, $\alpha \in U$ and $f(\beta) \in U$ for all $\beta \in \alpha$, whence, by Property (4), $rng\, f \in U$. Since $\cup rng\, f \in U$ by Property (2), it follows that $\varkappa \in U$. Repeating the arguments of the previous paragraph, we arrive at a contradiction. Therefore, $\varkappa$ is a regular cardinal.

Let λ be a cardinal number such that $\lambda < \varkappa$. Since $\lambda \in \varkappa \subset U$, by Property (2), we have $\mathcal{P}(\lambda) \in U$. By Lemma 4 (Sec. 3.1), $|\mathcal{P}(\lambda)| \in U$. Consequently, $|\mathcal{P}(\lambda)| \leq \varkappa$. Assuming that $\varkappa = |\mathcal{P}(\lambda)| \in U$, as before, we arrive at a contradiction. Therefore, $|\mathcal{P}(\lambda)| < \varkappa$.

Moreover, since $\omega \in U$ by Property (5), it follows that $\omega + 1 = \omega \cup \{\omega\} \in U$. Therefore, $\omega \in \omega + 1 \in A$ implies $\omega \in \cup A = \varkappa$.

Assertion (1) is proved.

(2) It follows from (1) that $\varkappa$ is a limit ordinal number. Therefore, $V_\varkappa = \cup\left(V_\beta | \beta \in \varkappa \right)$. By Lemma 6 (Sec. 3.1), $U = \cup\left(V_\alpha | \alpha \in A \right)$. If $\alpha \in A$, then $\alpha \leq \varkappa$ implies $V_\alpha \subset V_\varkappa$. Therefore, $U \subset V_\varkappa$. If $\beta \in \varkappa = \cup A$, then $\beta \in \alpha \in A$ for a certain α. By Property (1), $\beta \in A$. Therefore, $V_\varkappa \subset U$.

Thus, $U = V_\varkappa$.

(3) It follows from Lemma 1 (Sec. 2.2) that $\varkappa$ is unique. Therefore, we can define the mapping $q : \mathbf{U} \to \mathbf{In}$ such that $q(U) = \varkappa$, where $U = V_\varkappa$. Also, Lemma 1 (Sec. 2.2) implies that q is isotone. □

Corollary 1. *If U is a universal set, then $|U|$ is an inaccessible cardinal number, $|U| = \sup(\mathbf{On} \cap U)$, and $U = V_{|U|}$.*

Proof. By Theorem 1, $U = V_\varkappa$ for the inaccessible cardinal number $\varkappa \equiv \sup(\mathbf{On} \cap U)$. By Lemma 2 (Sec. 2.3), $\varkappa = |V_\varkappa| = |U|$. □

Corollary 2. *If U is a universal set, then $|U| = \sup(|V_\alpha| \mid \alpha \in \mathbf{On} \cap U)$.*

Proof. By Theorem 1, $U = V_\varkappa$ for the inaccessible cardinal $\varkappa \equiv \sup A = \cup A$, where $A \equiv \mathbf{On} \cap U$. Since $\varkappa$ is a limit ordinal number, by the corollary of Theorem 1 (Sec. 2.2), $|V_\alpha| = \sup(|V_\alpha| \mid \alpha \in \varkappa)$. If $\alpha \in \varkappa$, then $\alpha \in a$ for a certain $a \in A$. By the transitivity, $\alpha \in A$. Conversely, if $\alpha \in A$, then $\alpha \leq \varkappa$. Assume that $\alpha = \varkappa$. Then $\varkappa \in U$. However, in proving Theorem 1, we have proved that the condition $\varkappa \in U$ leads to a contradiction. Therefore, $\alpha \in \varkappa$. □

Theorem 2. *For any set U, the following assertions are equivalent*:

(1) *U is an inaccessible cumulative set*;
(2) *U is a universal set.*

Proof. (1) $\vdash$ (2). Let $U = V_\varkappa$ for a certain inaccessible cardinal number $\varkappa$. Let us show that the set U is universal.

The property $x \in U \Rightarrow x \subset U$ follows from Lemma 4 (Sec. 2.2).

The property $x \in U \Rightarrow \mathcal{P}(x) \in U$ follows from Lemma 7 (Sec. 2.2).

The property $x \in U \wedge y \in U \Rightarrow x \cup y \in U$ follows from Lemma 6 (Sec. 2.2).

The properties $x \in U \wedge y \in U \Rightarrow \{x, y\}, \langle x, y\rangle, x \times y \in U$ follow from Corollaries 1 and 2 of Lemma 7 (Sec. 2.2).

The property $\omega \in U$ follows from Lemma 8 (Sec. 2.2).

The properties $x \in U \Rightarrow \cup x \in U$ and $x \in U \wedge (f \in U^x) \Rightarrow rng\, f \in U$ follow from Lemma 4 (Sec. 2.3) and its corollaries.

Therefore, the set U is universal.

(2) $\vdash$ (1). This implication obviously follows from the previous theorem. □

Corollary. *The correspondence $q : U \mapsto \varkappa$ from Theorem 1 such that $U = V_\varkappa$ and $\varkappa = |U|$ is a bijective isotone mapping from* **U** *onto* **In**.

Therefore, the cardinalities of universal sets exhaust all inaccessible cardinal numbers.

This theorem allows us to make the following conclusions on the structure of the class $\mathbf{U} \equiv \{U | U \bowtie\}$ of all universal sets.

The relation $\in \cup =$ is an order relation on the class $\mathbf{U}$. It will be denoted by $\leq$, i.e., $U \leq V$ if $U \in V$ or $U = V$. By Lemma 4 (Sec. 2.2), the class $\mathbf{U}$ is transitive. Therefore, $U \in V$ implies $U \subset V$. Therefore, $U \leq V$ implies $U \subset V$. Let us prove that these relations are equivalent.

Proposition 1. *Let U and V be universal sets. Then the relation $U \leq V$ is equivalent to the relation $U \subset V$.*

Proof. We need to only verify that $U \subset V$ implies $U \leq V$. By Theorem 1, $U = V_\pi$ and $V = V_\varkappa$ for certain inaccessible cardinals π and $\varkappa$. If $\pi = \varkappa$, then $U = V_\pi = V_\varkappa = V$. If $\pi < \varkappa$, then by Lemma 1 (Sec. 2.2), $U = V_\pi \in V_\varkappa = V$. Finally, if $\pi > \varkappa$, then by the same lemma, $V = V_\varkappa \in V_\pi = U \subset V$, which is impossible. Therefore, $U \leq V$. □

The following theorem is of conditional character in the ZF theory. In ZF+AU, the condition of the theorem holds.

Theorem 3. *If the class* **U** *of all universal sets in the ZF theory is nonempty, then it is completely ordered with respect to the order $\subset$. Moreover, any nonempty subclass of the class* **U** *has a minimal element.*

Proof. Let $\emptyset \neq \mathbf{A} \subset \mathbf{U}$, i.e., let $\forall U \in \mathbf{A}(U \bowtie)$. To the class $\mathbf{A}$, the injective and strictly monotone mapping $q : U \mapsto \varkappa$ from the class $\mathbf{U}$ into the class $\mathbf{On}$ of the form $U = V_\varkappa$ from Theorem 1 puts in correspondence a certain subclass $\mathbf{B} \equiv q[\mathbf{A}] \equiv \{x | x \in \mathbf{On} \wedge \exists U \in \mathbf{A}(z = q(\mathbf{U}))\}$ of the class $\mathbf{On}$. By Lemma 1 (Sec. 1.2), it has the minimal element π, which is an inaccessible cardinal. Since $\pi \in \mathbf{B}$, it follows that $\pi = q(U)$ for a certain $U \in \mathbf{A}$, i.e., $U = V_\pi$. Since the mapping q is injective and strictly monotone, U is a minimal element in the class $\mathbf{A}$. □

3.3. Enumeration of the class of all universal sets in the ZF+AU theory and the structural form of the universality axiom. In the ZF+AU set theory, let us consider the class

$$\mathbf{G} \equiv \{Z | \exists X \exists Y (Z = \langle X, Y\rangle \wedge ((X = \emptyset \Rightarrow Y = \cap\{U | U \bowtie\})$$
$$\vee (X \neq \emptyset \Rightarrow (\neg func(X) \Rightarrow Y = \emptyset)(func(X) \Rightarrow (\neg On(dom\, X) \Rightarrow Y = \emptyset)$$
$$\vee (On(dom\, X) \Rightarrow (Son(dom\, X) \Rightarrow Y = \cap\{U | U \bowtie \wedge X(dom\, X - 1) \in U\})$$
$$\vee (Lon(dom\, X) \Rightarrow Y = \cap\{U | U \bowtie \wedge \cap\, rng\, X \subset U\}))))))\}.$$

If we express the definition of the class $\mathbf{G}$ less formally, then $\mathbf{G}$ consists of all pairs $\langle X, Y\rangle$ for which there are the following five possibilities excluding each other:

(1) if $X = \emptyset$, then Y is the intersection of all universal sets (the existence of a nonempty intersection follows from the universality axiom);
(2) if $X \neq \emptyset$ and X is not a function, then $Y = \emptyset$;
(3) if $X \neq \emptyset$, X is a function, and $dom\, X$ is not an ordinal number, then $Y = \emptyset$;
(4) if $X \neq \emptyset$, X is a function, $dom\, X$ is an ordinal number, and $dom\, X = \alpha + 1$, then Y is the intersection of all universal sets U such that $X(\alpha) \in U$ (the existence of this nonempty intersection follows from the universality axiom);
(5) if $X \neq \emptyset$, X is a function, and $dom\, X$ is a limit ordinal number, then Y is the intersection of all universal sets U such that $\cup rng\, X \subset U$ (the existence of this nonempty intersection follows from the universality axiom).

As in Sec. 2.1, we can verify that the class $\mathbf{G}$ is a function from $\mathbf{V}$ into $\mathbf{V}$.

According to Theorem 2 from Sec. 2, for the function $\mathbf{G}$, there exists a function $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ such that the following relation holds for any $\alpha \in \mathbf{On}$:

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|\alpha).$$

It follows from Case (1) for the function $\mathbf{G}$ that $\mathbf{F}(\emptyset) = \mathbf{G}(\mathbf{F}|\emptyset) = \mathbf{G}(\emptyset) = \cap\{U|U \bowtie\}$.

It follows from Case (4) that if β is a subsequent cardinal number and $\beta = \alpha+1$, then $\mathbf{F}(\beta) = \mathbf{G}(\mathbf{F}|\beta) = \cap\{U|U \bowtie \wedge \mathbf{F}(\alpha) \in U\}$.

Finally, it follows from Case (5) that if α is a limit ordinal number, then $\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|\alpha) = \cap\{U|U \bowtie \wedge \cup \left(\mathbf{F}(\beta)|\beta \in \alpha \right) \subset U\}$.

Denote $\mathbf{F}(\alpha)$ by U_α. We have obtained the collection $(U_\alpha \in \mathbf{U}|\alpha \in \mathbf{On})$ satisfying the following relations:

(1) $U_0 = \cap\{U|U \bowtie\}$;
(1) $U_{\alpha+1} = \cap\{U|U \bowtie \wedge U_\alpha \in U\}$.
(1) $U_\alpha = \cap\{U|U \bowtie \wedge \cup \left(U_\beta|\beta \in \alpha \right) \subset U\}$ if α is a limit ordinal number.

Let us prove several properties of this collection.

Lemma 1. *In the ZF+AU set theory, the collection* $U_\alpha \in \mathbf{U}|\alpha \in \mathbf{On})$ *has the following properties*:

(1) $\alpha \in \beta \Leftrightarrow U_\alpha \in U_\beta$ (*strict increase*);
(2) U_0 *is the minimal universal set* (*originality*);
(3) *if* V *is a universal set and* $U_0 \subset V \in U_\alpha$, *then* $V = U_\beta$ *for a certain* $\beta \in \alpha$ (*noncondensibility*);
(4) *if* V *is a universal set, then* $V = U_\alpha$ *for a certain* α (*surjectivity*);
(5) $\alpha \subset U_\alpha$ (*absorbability*).

Proof. (1) By the transfinite induction, let us prove that for any ordinal number β, $(\alpha \in \beta \Rightarrow U_\alpha \in U_\beta)$.

If $\beta = 0$, then this is obvious, since $\forall\alpha\neg(\alpha \in \beta)$.

For a certain ordinal number β, if $\forall\alpha(\alpha \in \beta \Rightarrow U_\alpha \in U_\beta)$, then let us consider the ordinal number $\beta+1$. It follows from $\alpha \in \beta+1$ that $\alpha \in \beta \vee \alpha = \beta$. If $\alpha \in \beta$, then by the inductive assumption, $U_\alpha \in U_\beta$, and since $U_\beta \subset U_{\beta+1}$, it follows that $U_\alpha \in U_{\beta+1}$. If $\alpha = \beta$, then $U_\alpha = U_\beta \in U_{\beta+1}$. Therefore, for $\beta + 1$, we have $\alpha \in \beta + 1 \Rightarrow U_\alpha \in U_{\beta+1}$.

Now assume that β is a limit ordinal number and $\forall\gamma \in \beta\forall\alpha(\alpha \in \gamma \Rightarrow U_\alpha \in U_\gamma)$. Let α be such that $\alpha \in \beta$. Since β is a limit ordinal number, $\alpha+1 \in \beta$. Since $\cup\left(U_\gamma|\gamma \in \beta \right) \subset U_\beta$, it follows that $U_{\alpha+1} \subset U_\beta$. Since $U_\alpha \in U_{\alpha+1}$ in this case, we have $U_\alpha \in U_\beta$.

Clearly, $\alpha = \beta \Rightarrow U_\alpha = U_\beta$. If $U_\alpha \in U_\beta$, then $\alpha \in \beta$, since for $\alpha = \beta$, we have $U_\alpha = U_\beta$, and for $\beta \in \alpha$, we have $U_\beta \in U_\alpha$.

Property (2) holds by construction.

(4) Let V be an arbitrary universal set. If $V = U_0$, then the property is proved. Therefore, assume that $V \neq U_0$.

Consider the class $\mathbf{A} \equiv \{\alpha \in \mathbf{On} | U_\alpha \in V\}$. By construction, $U_0 \subset V$. By Proposition 1 of Sec. 3.2, $U_0 \in V$. Therefore, $0 \in \mathbf{A}$.

Consider the class $\mathbf{F} \equiv \{z | \exists x \in \mathbf{A} \exists y \in \mathbf{In}(z = \langle x, y\rangle \wedge y = q(U_x))\}$, where q is the mapping from Theorem 1 of Sec. 3.2. Clearly, $\mathbf{F}$ is a mapping from $\mathbf{A}$ into $\mathbf{In}$. If $\alpha \in \mathbf{A}$, then $\mathbf{F}(\alpha) = q(U_\alpha) \in q(V)$. Consequently, $rng\,\mathbf{F} \subset q(V)$. Therefore, $B \equiv rng\,\mathbf{F}$ is a set. Let $\alpha, \beta \in \mathbf{A}$, and let $\alpha \neq \beta$. If $\alpha \in \beta$, then by Property (1) proved above, $U_\alpha \in U_\beta$. Since q is isotone, $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$. If $\beta \in \alpha$, then analogously $\mathbf{F}(\beta) < \mathbf{F}(\alpha)$. Therefore, the mapping $\mathbf{F} : \mathbf{A} \rightarrow B$ is bijective. Therefore, we can consider the mapping $\mathbf{F}^{-1} : B \leftrightarrows \mathbf{A}$. Since B is a set, by the substitution axiom scheme, $\mathbf{A} = rng\,\mathbf{F}^{-1}$ is a set. Therefore, in what follows, instead of $\mathbf{A}$, we will write A.

Consider the nonempty class $\mathbf{C} \equiv \mathbf{On} \setminus A$ and the minimal element β in this class. Clearly, $U_\beta \notin V$. Therefore, by Proposition 1 of Sec. 3.2, $V \subset U_\beta$. If $V = U_\beta$, then the property is proved. Let $V \in U_\beta$. Assume that $\beta = \gamma + 1$. Then $\gamma \in A$ implies $U_\gamma \in V$. By Property (2), for the collection of universes, it follows from this that $U_\beta \subset V$. Therefore, in this case, $V = V_\beta$.

Assume that β is limit. If $\gamma \in \beta$, then $\gamma \in A$ implies $U_\gamma \in V$, and by the transitivity property, $U_\gamma \subset V$. Hence $\cup\left(U_\gamma | \gamma \in \beta\right) \subset V$. By Property (3), for the collection of universes, it follows from this that $U_\beta \subset V$. Therefore, in this case, $V = V_\beta$.

(3) This property follows from Properties (1) and (4).

(5) Using Property (1), by induction, we prove that $\alpha \subset U_\alpha$ for any α. Clearly, $\alpha = 0 = \emptyset \subset U_0$.

Let $\alpha \subset U_\alpha$. Since $\alpha + 1 \equiv \alpha \cup \{\alpha\}$, it follows that $\alpha \subset U_\alpha \in U_{\alpha+1}$ implies $\alpha \in U_{\alpha+1}$, and, therefore, $\{\alpha\} \in U_{\alpha+1}$. By the transitivity property, $\alpha \subset U_{\alpha+1}$ and $\{\alpha\} \subset U_{\alpha+1}$ implies $\alpha + 1 \subset U_{\alpha+1}$.

Let α be a limit ordinal number, and let $\beta \subset U_\beta$ for any $\beta \in \alpha$. By Lemmas 2 and 3 (Sec. 1.2) , $\alpha = \sup \alpha = \cup\alpha = \cup\left(\beta | \beta \in \alpha\right)$. Since $\beta \subset U_\beta \subset U_\alpha$, it follows that $\alpha = \cup\alpha \subset U_\alpha$. □

This lemma implies that the collection $(U_\alpha \in \mathbf{U} | \alpha \in \mathbf{On})$ is a natural enumeration of the class of all universal sets in the ZF+AU theory. The following lemma shows that this enumeration is unique.

Lemma 2. *In the ZF+AU set theory, the collection* $(U_\alpha \in \mathbf{U} | \alpha \in \mathbf{On})$ *with Properties* (1)–(3) *from Lemma* 1 *is unique.*

Proof. Assume that there exists a collection $(W_\alpha \in \mathbf{U} | \alpha \in \mathbf{On})$ having Properties(1)–(3) of Lemma 1. Consider the classes $\mathbf{A} \equiv \{\alpha \in \mathbf{On} | U_\alpha = W_\alpha\}$ and $\mathbf{B} = \mathbf{On} \setminus \mathbf{A}$. Since $U_0 = W_0$, it follows that $0 \in \mathbf{A}$.

Assume that $\mathbf{B} \neq \emptyset$. Then by lemma 3 of Sec. 3.2, there exists $\beta = sm\,\mathbf{B}$. If $U_\beta \in W_\beta$, then by Property (3), the condition $W_0 = U_0 \subset U_\beta \in W_\beta$ implies $_\beta = W_\gamma$ for a certain $\gamma \in \beta$.

Since $\gamma \in \mathbf{A}$, it follows that $W_\gamma = U_\gamma$. Therefore, $U_\beta = U_\gamma$ and $\gamma \in \beta$, which contradicts Property (1). If $W_\beta \in U_\beta$, then analogously we arrive at a contradiction. Therefore, $U_\beta = W_\beta$ by Theorem 3 and Proposition 1 of Sec. 3.2. However, this contradicts the definition of the class $\mathbf{B}$. Therefore, we arrive at a contradiction. Therefore, $\mathbf{B} = \emptyset$ and $\mathbf{A} = \mathbf{On}$. □

The following theorem yields the structural form of the universality axiom.

Theorem 1. *In the ZF set theory, the following assertions are equivalent*:

(1) *the universality axiom* $AU \equiv \forall X \exists U (U \bowtie \wedge X \in U)$, *which means that for any set* X, *there exists a universal set* U *such that* $X \in U$;

(2) *there exists a collection* $(U_\alpha \in \mathbf{U} | \alpha \in \mathbf{On})$ *of universal sets having Properties* (1)–(5) *from Lemma* 1;

(3) *the inaccessibility axiom* $AI \equiv \forall \alpha (On(\alpha) \Rightarrow \exists \varkappa (Icn(\varkappa) \wedge \alpha \in \varkappa))$, *which means that for any ordinal number* α, *there exists an inaccessible cardinal number* $\varkappa$ *such that* $\alpha < \varkappa$.

Proof. The deducibility (1) ⊢ (2) was proved in Lemma 1.

(2) ⊢ (3). Take an arbitrary order number α. According to ([11], 4, Lemma 8), there exists a cardinal number β such that $\alpha < \beta$. By condition, $\beta \subset U_\beta$. Consider the cardinal number $\varkappa \equiv |U_{\beta+1}|$. By the universality, $\beta \in U_{\beta+1}$, and, therefore, $\beta \subset U_{\beta+1}$. Therefore, $\beta = |\beta| \leq |U_{\beta+1}| \equiv \varkappa$. Assume that $\beta = \varkappa$. Then $\varkappa \in U_{\beta+1}$ implies $\mathcal{P}(\varkappa) \in U_{\beta+1}$, and, therefore, $\mathcal{P}(\varkappa) \subset U_{\beta+1}$. Using the Cantor theorem, we obtain

$\varkappa = |\varkappa| < |\mathcal{P}(\varkappa)| \le |U_{\beta+1}| \equiv \varkappa$. This contradiction implies $\beta < \varkappa$. Therefore, $\alpha < \varkappa$. By the corollary to Theorem 1 of Sec. 3.2, $\varkappa$ is an inaccessible cardinal number.

(3) $\vdash$ (1). By Lemma 2 of (Sec. 2.2), $X \in V_\alpha$ for a certain ordinal number α. By Condition (3), $\alpha < \varkappa$ for a certain inaccessible cardinal number $\varkappa$. By Lemma 1 (Sec. 2.2), $V_\alpha \in V_\varkappa$. By Theorem 2 (Sec. 3.2), the set $V_\varkappa$ is universal. By Corollary 1 of Lemma 4 (Sec. 2.2), $X \in V_\alpha \subset V_\varkappa$ implies $X \in V_\varkappa$. □

Note that the equivalence of the universality and inaccessibility axioms was proved in [4] by using another method.

Theorem 1 shows the structure of the class of all universal sets in the ZF+AU set theory. The number of universal sets and the number of ordinal numbers in the ZF theory are the same.

3.4. Enumeration of the class of all inaccessible cardinals in the ZF+AI theory and the structural form of the inaccessibility axiom. Now let us enumerate all inaccessible cardinal numbers in the ZF+AI set theory. For this purpose, consider the class

$$\begin{gathered}\mathbf{G} \equiv \{Z | \exists X \exists Y (Z = \langle X, Y\rangle \wedge ((X = \emptyset \Rightarrow Y = sm\,\{\varkappa | Icn(\varkappa)\})(X \neq \emptyset \Rightarrow (\neg func(X) \Rightarrow Y = \emptyset) \\ \vee (func(X) \Rightarrow (\neg On(dom\,X) \Rightarrow Y = \emptyset) \vee (On(dom\,X) \Rightarrow (rng\,X \not\subset \mathbf{On} \Rightarrow Y = \emptyset) \\ \vee (rng\,X \subset \mathbf{On} \Rightarrow (Son(dom\,X) \Rightarrow Y = sm\,\{\varkappa | Icn(\varkappa) \wedge X(dom\,X - 1) \in \varkappa\}) \\ \vee (Lon(dom\,X) \Rightarrow Y = sm\,\{\varkappa | Icn(\varkappa) \wedge \cup rng\,X \subset U\}))))))\}.\end{gathered}$$

If we express the definition of the class $\mathbf{G}$ less formally, then $\mathbf{G}$ consists of all pairs $\langle X, Y\rangle$ for which the following six cases, which are mutually exclusive, hold:

(1) if $X = \emptyset$, then Y is a minimal inaccessible cardinal number (its existence follows from the inaccessibility axiom);
(2) if $X \neq \emptyset$ and X is not a function, then $Y = \emptyset$;
(3) if $X \neq \emptyset$, X is a function, and $dom\,X$ is not an ordinal number, then $Y = \emptyset$;
(4) if $X \neq \emptyset$, X is a function, $dom\,X$ is an ordinal number, and $rng\,X \not\subset \mathbf{On}$, then $Y = \emptyset$;
(5) if $X \neq \emptyset$, X is a function, $dom\,X$ is an ordinal number, $rng\,X \subset \mathbf{On}$, and $dom\,X = \alpha + 1$, then Y is minimal among all inaccessible cardinals $\varkappa$ such that $X(\alpha) \in \varkappa$ (its existence follows from the inaccessibility axiom);
(6) if $X \neq \emptyset$, X is a function, $rng\,X \subset \mathbf{On}$, and $dom\,X$ is a limit ordinal number, then Y is minimal among all inaccessible cardinals $\varkappa$ such that $\cup rng\,X \subset \varkappa$ (its existence follows from Lemmas 1 and 2 (Sec. 1.2), Axiom AI, and the transitivity of $\varkappa$).

As in Sec. 1.2, we can verify that the class $\mathbf{G}$ is a function from $\mathbf{V}$ into $\mathbf{V}$.

According to Theorem 2 of Sec. 1.2, for the function $\mathbf{G}$, there exists a function $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ such that the following relation holds for any $\alpha \in \mathbf{On}$:

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|\alpha).$$

It follows from Case (1) that for the function $\mathbf{G}$, $\mathbf{F}(\emptyset) = \mathbf{G}(\mathbf{F}|\emptyset) = \mathbf{G}(\emptyset) = sm\{\varkappa | Icn(\varkappa)\}$.

Case (5) implies that if β is a subsequent ordinal number and $\beta = \alpha + 1$, then $\mathbf{F}(\beta) = \mathbf{G}(\mathbf{F}|\beta) = sm\,\{\varkappa | Icn(\varkappa) \wedge \mathbf{F}(\alpha) \in \varkappa\}$.

Finally, Case (6) implies that if α is a limit ordinal number, then $\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|\alpha) = sm\{\varkappa | Icn(\varkappa) \wedge \cup\big(\mathbf{F}(\beta) | \beta \in \alpha\big) \subset \varkappa\}$.

Denote $\mathbf{F}(\alpha)$ by q_α. We have obtained the collection $(q_\alpha \in \mathbf{In} | \alpha \in \mathbf{On})$ of inaccessible cardinal numbers satisfying the following relations:

(1) $q_0 = sm\,\{\varkappa | Icn(\varkappa)\}$;
(2) $q_{\alpha+1} = sm\,\{\varkappa | Icn(\varkappa) \wedge q_\alpha \in \varkappa\}$.
(3) $q_\alpha = sm\,\{\varkappa | Icn(\varkappa) \wedge \cup\big(q_\beta | \beta \in \alpha\big) \subset \varkappa\}$ if α is a limit ordinal number.

Lemma 1. *In the ZF+AI set theory, the collection* $(q_\alpha \in \mathbf{In} | \alpha \in \mathbf{On})$ *has the following properties*:

(1) $\alpha \in \beta \Leftrightarrow q_\alpha \in q_\beta$ (*strict inaccessibility*);

(2) *q_0 is a minimal inaccessible cardinal number (originality)*;
(3) *if p is an inaccessible cardinal and $q_0 \subset p \in q_\alpha$, then $p = q_\beta$ for a certain $\beta \in \alpha$ (noncondensibility)*;
(4) *if p is an inaccessible cardinal, then $p = q_\alpha$ for a certain α (surjectivity)*;
(5) *$\alpha \subset q_\alpha$ (absorbability).*

The proof is analogous to the proof of Lemma 1 (Sec. 3.3). However, it can be obtained from Lemma 1 (Sec. 3.3) by using the isotone bijection $q : \mathbf{U} \to \mathbf{In}$ from the corollary to Theorem 2 (Sec. 3.2).

Lemma 2. *In the ZF+AI set theory, the collection $(q_\alpha \in \mathbf{In} | \alpha \in \mathbf{On})$ with Properties (1)–(3) from Lemma 1 is unique.*

The proof is analogous to the proof of Lemma 2 (Sec. 3.3).

The following assertion yields the structural form of the inaccessibility axiom.

Theorem 1. *In the ZF set theory, the following assertions are equivalent*:

(1) *the inaccessibility axiom AI of Theorem 1*;
(2) *there exists a collection $(q_\alpha \in \mathbf{In} | \alpha \in \mathbf{On})$ of inaccessible cardinal numbers having Properties (1)–(5) of Lemma 1.*

Proof. The deducibility (1) $\vdash$ (2) was proved in Lemma 1.

(2) $\vdash$ (1). Take an arbitrary order number α. By the condition, $\alpha \subset q_\alpha \in q_{\alpha+1} \equiv \beta$. Clearly, $\alpha \neq \beta$ by Property (1). By the transitivity, $\alpha \subset q_\alpha \subset \beta$. Therefore, the nonempty set $\beta \setminus \alpha$ has a minimal element y. Let us verify that $\alpha = y$. Let $x \in y$. Then $x \in y \in \beta$ implies $x \in \beta$. Since $x < y$, we have $x \in \alpha$. This means that $y \subset \alpha$. Conversely, let $x \in \alpha$. It follows from $y \notin \alpha$ that $y \neq x$. Assume that $y \in x$. Then $y \in x \in \alpha$ implies $y \in \alpha$. Consequently, $y \in \alpha \cap (\beta \setminus \alpha) = \emptyset$. This contradiction implies $x \in y$. As a result, $\alpha \subset y$, whence $\alpha = y \in \beta$. □

Theorem 1 shows the structure of the class of all inaccessible cardinal numbers in the ZF+AI set theory. The amount of all inaccessible cardinal numbers is the same as that of ordinal numbers in the ZF theory.

Now let us connect the collections $(V_\alpha \in \mathbf{V} | \alpha \in \mathbf{On})$, $(U_\alpha \in \mathbf{U} | \alpha \in \mathbf{On})$, and $(q_\alpha \in \mathbf{In} | \alpha \in \mathbf{On})$ with each other.

Theorem 2. *In the ZF+AU set theory and the ZF+AI theory equivalent to it, the relation $V_{q_\alpha} = U_\alpha$ holds for any order number α.*

Proof. Since V_{q_0} is a universal set by Theorem 2 (Sec. 3.2), it follows that $U_0 \subset V_{q_0}$. Let U be an arbitrary universal set. By Theorem 1 (Sec. 3.2), $U = V_\varkappa$ for a certain inaccessible cardinal number $\varkappa$. By Lemma 1, $\varkappa = q_\alpha$ for a certain α. Since $q_0 \subset q_\alpha$, it follows that $V_{q_0} \subset V_{q_\alpha} = V_\varkappa = U$. Hence $V_{q_0} \subset \cap\{U | U \bowtie\} = U_0$. As a result, we have proved that $V_{q_0} = U_0$.

Consider the nonempty class $\mathbf{A} \equiv \{\alpha \in \mathbf{On} | V_{q_\alpha} = U_\alpha\}$ and the class $\mathbf{B} \equiv \mathbf{On} \setminus \mathbf{A}$. Assume that $\mathbf{B} \neq \emptyset$. Then there exists a number $\beta \equiv sm\,\mathbf{B} > 0$. Consider the universal sets V_{q_β} and U_β.

Assume that $V_{q_\beta} \in U_\beta$. Then by Lemma 1 (Sec. 3.3), the condition $U_0 = V_{q_0} \subset V_{q_\beta} \in U_\beta$ implies $V_{q_\beta} = U_\gamma$ for a certain $\gamma \in \beta$. It follows from $\gamma < \beta$ that $\gamma \in \mathbf{A}$, and, therefore, $V_{q_\gamma} = U_\gamma$. As a result, we arrive at the relation $V_{q_\beta} = V_{q_\gamma}$. By Lemma 1 (Sec. 2.2), we conclude that $q_\beta = q_\gamma$, and, moreover, $\gamma \in \beta$, which contradicts Lemma 1.

On the other hand, assume that $U_\beta \in V_{q_\beta}$. Since U_β is a universal set, by Theorem 1 (Sec. 3.2), it follows that $U_\beta = V_\varkappa$ for a certain inaccessible cardinal $\varkappa$. Then by Lemma 1 (Sec. 2.2), the chain $V_{q_0} = U_0 \subset U_\beta = V_\varkappa \in V_{q_\beta}$ implies the chain $q_0 \subset \varkappa \in q_\beta$. By Lemma 1, it follows from this that $\varkappa = q_\gamma$ for a certain $\gamma \in \beta$. Since $\gamma \in \mathbf{A}$, it follows that $V_{q_\gamma} = U_\gamma$. As a result, we arrive at the equality $U_\beta = V_\varkappa = V_{q_\gamma} = U_\gamma$ under the condition $\gamma \in \beta$, which contradicts Lemma 1.

By Theorem 3 (Sec. 3.2) and Proposition 1 (Sec. 3.2), we conclude that $V_{q_\beta} = U_\beta$. However, this contradicts the definition of the class $\mathbf{B}$. Therefore, we arrive at a contradiction. Therefore, $\mathbf{B} = \emptyset$ and $\mathbf{A} = \mathbf{On}$. □

Corollary. *In the ZF+AU set theory and the ZF+AI theory equivalent to it, the equality* $|U_\alpha| = q_\alpha$ *holds for any order number* α.

Proof. By Theorem 2 (Sec. 2.3) and Lemma 2 (Sec. 2.3), $|U_\alpha| = |V_{q_\alpha}| = q_\alpha$. ☐

4. Weak Forms of the Axioms of Universality and Inaccessibility

4.1. Axioms of ω-universality and ω-inaccessibility. Along with the universality axiom AU, the following weaker ω-*universality axiom* is considered in the ZF set theory:

$$AU(\omega) \equiv \exists X(\forall U \in X(U \bowtie) \wedge X \neq \emptyset \wedge \forall U \in X \exists V \in X(U \in V)).$$

The explanation of such a name of this axiom is given by the following theorem, which is proved by using Theorem 1 of Sec. 3.2.

Theorem 1. *The following assertions are equivalent in the ZF theory*;

1. $AU(\omega)$;
2. *for any* $n \in \omega$, *there exists a finite set of universal sets which has the cardinality* $n+1$;
3. *for any* $n \in \omega$, *there exists a finite sequence* $u \equiv (U_k | k \in n+1)$ *of universal sets such that* $U_k \in U_l$ *for any* $k \in l \in n+1$, *i.e., the sequence* u *is strictly increasing*;
4. *there exists a universal set* U^*, *and for any* $n \in \omega$, *there exists a unique finite strictly increasing subsequence* $u(n) \equiv (U_k^n | k \in n+1)$ *of universal sets such that* $U_0^n = U^*$ *and the facts that* V *is a universal set* $U_0^n \leq V \leq U_n^n$ *imply* $V = U_k^n$ *for a certain* $k \in n+1$ (*noncondensibility property*);
5. *There exists a denumerable set of universal sets*;
6. *there exists an infinite sequence* $u \equiv (U_n | n \in \omega)$ *of universal sets such that* $U_k < U_l$ *for any* $k \in l \in \omega$, *i.e., the sequence* u *is strictly increasing*;
7. *there exists an infinite strictly increasing sequence* $u \equiv (U_n | n \in \omega)$ *of universal sets such that the properties that* $n \in \omega$, V *is a universal set, and* $U_0 \leq V \leq U_n$ *imply* $V = U_k$ *for a certain* $k \in n+1$ (*noncondensibility property*);
8. *there exists an infinite set of universal sets.*

Proof. (1) $\vdash$ (4). Let W be an empty set whose existence is ensured by Axiom $AU(\omega)$. Consider the nonempty class $\mathbf{W} \equiv \{x | x \bowtie \wedge \exists y \in W(x \leq y)\}$. If $x \in \mathbf{W}$, then $x \leq y$ for a certain $y \in W$. By $AU(\omega)$, for $y \in W$, there exists $z \in W$ such that $y < z$. Hence $x < z \in \mathbf{W}$. Therefore, the class $\mathbf{W}$ has all the properties listed in formula $AU(\omega)$.

Since $\emptyset \neq \mathbf{W} \subset \mathbf{U}$, by Theorem 3 (Sec. 3.2), in $\mathbf{W}$, there exists a minimal element U^*. For any $y \in W$, it follows from $U^* \leq y$ that $W^* \in \mathbf{W}$. The class $\mathbf{W}$ has the following properties: if $z \in \mathbf{U}$ and $z \leq y$ for a certain $y \in \mathbf{W}$, then $z \in \mathbf{W}$.

Consider the set N consisting of all $n \in \omega$ for which there exists a unique sequence $u \equiv u(n) \equiv (U_k \in \mathbf{W} | k \in n+1)$ such that $U_0 = U^*$, $U_k < U_l$ for any $k \in l \in n+1$ and $V \in \mathbf{U}$ and $U_0 \leq V < U_n$ imply $V = U_k$ for a certain $k \in n$.

Since the sequence $(U_k \in \mathbf{W} | k \in 1)$ such that $U_0 \equiv U^*$ has all the properties listed above, then $0 \in N$. Let $n \in N$. By the property of the class $\mathbf{W}$, for $U_n \in \mathbf{W}$, there exists $z \in \mathbf{W}$ such that $U_n < z$. Therefore, the class $\mathbf{J} \equiv \{x \in \mathbf{W} | U_n < x\}$ is nonempty. Therefore, by Theorem 3 (Sec. 3.2), it contains a minimal element A.

Therefore, we can define the sequence $v \equiv (P_k \in \mathbf{W} | k \in n+2)$ setting $P_k \equiv U_k$ for any $k \in n+1$ and $P_{n+1} \equiv A$, i.e., $v = u \cup \{\langle n+1, A \rangle\}$. Clearly, $P_0 = U^*$ and $P_k < P_l$ for any $k \in l \in n+2$. Let $V \in \mathbf{U}$ and $P_0 \leq V < P_{n+1}$. Then $V \in \mathbf{W}$ and $U_0 \leq V < A$. If $V = U_n$, then $V = P_n$. If $V < U_n$, then $U_0 \leq V < U_n$ implies $V = U_k = P_k$ for a certain $k \in n$. Finally, if $V > U_n$, then $V \in \mathbf{J}$. Therefore, $A \leq V$, which contradicts the property $V < A$. Therefore, this case is impossible. It follows from the two previous cases that $V = P_k$ for a certain $k \in n+1$. This means that the sequence v has the necessary properties. Let us prove that it is unique. Assume that there exists a sequence $w \equiv (V_k \in \mathbf{W} | k \in n+2)$ such that $V_0 = U^*$, $V_k < V_l$ for any $k \in l \in n+2$ and $V \in \mathbf{U}$ and $V_0 \leq V < V_{n+1}$ imply $V = V_k$ for a certain $k \in n+1$. Since the sequence $w|n+1 \equiv (V_k \in \mathbf{W}|k \in n+1)$ has all the properties listed above for n, by the uniqueness of the sequence u, we conclude that $u = w|(n+1)$, i.e., $V_k = U_k \equiv P_k$ for all $k \in n+1$. If $V_{n+1} < P_{n+1}$, then by what was proved above, $P_0 = V_0 \leq V_{n+1} < P_{n+1}$ implies $V_{n+1} = P_k = V_k$ for a certain $k \in n+1$, which is impossible. If $P_{n+1} < V_{n+1}$, then analogously $V_0 = P_0 \leq P_{n+1} < V_{n+1}$ implies $P_{n+1} = V_k = P_k$ for a certain $k \in n+1$, which is also impossible. Therefore, $V_{n+1} = P_{n+1}$. Thus, the uniqueness of the sequence v is proved. Therefore, $n+1 \in N$. By the natural induction principle, $N = \omega$. Therefore, for any $n \in \omega$, there exists the unique sequence $u(n)$ indicated above. By its uniqueness, we can denote it by $(U_k^n|k \in n+1)$.

(4) $\vdash$ (7). Consider the following formula of the ZF theory: $\varphi(x,y) \equiv (x \in \omega \Rightarrow y = U_x^x) \wedge (x \notin \omega \Rightarrow y = \emptyset)$. By the axiom substitution scheme **AS6**, for ω, there exists a set Y such that $\forall x \in \omega(\forall y(\varphi(x,y) \Rightarrow y \in Y))$. If $n \in \omega$, then $\varphi(n, U_n^n)$ implies $U_n^n \in Y$. Therefore, in the set $\omega \times Y$, we can define the infinite sequence $u \equiv (U_n \in Y|n \in \omega)$ setting $u \equiv \{z \in \omega \times Y|\exists x \in \omega(z = \langle x, U_x^x\rangle)\}$. The uniqueness property mentioned above implies $u(m) = u(n)|m+1$ for all $m \leq n$. Therefore, $u|n+1 = u(n)$. Clearly, the sequence u has all the necessary properties.

(6) $\vdash$ (1). Consider the following formula of the ZF theory: $\varphi(x,y) \equiv (x \in \omega \Rightarrow y = U_x) \wedge (x \notin \omega \Rightarrow y = \emptyset)$. By the substitution axiom scheme **AS6**, for ω, there exists a set Y such that $\forall x \in \omega(\forall y(\varphi(x,y) \Rightarrow y \in Y))$. If $n \in \omega$, then $\varphi(n, U_n)$ implies $U_n \in Y$. By the axiom isolation scheme **AS3**, the class $X \equiv \{U_n|n \in \omega\} \equiv \{y|\exists x \in \omega(y = U_x)\} = \{y|y \in Y \wedge \exists x \in \omega(y = U_x)\}$ is a set. Since the sequence u strictly increases, the set X satisfies Axiom $AU(\omega)$.

The deducibilities (7) $\vdash$ (6) $\vdash$ (5) $\vdash$ (2) are obvious.

The deducibilities (4) $\vdash$ (3) $\vdash$ (2) are also obvious.

(2) $\vdash$ (3) and (2) $\vdash$ (6). Consider the nonempty class $\mathbf{A}$ of all finite sets consisting of universal sets. Then the class $\mathbf{W} \equiv \cup\mathbf{A}$ is also nonempty, and, therefore, by Theorem 3 (Sec. 3.2), there exists a minimal element U^* in $\mathbf{W}$.

Consider the set N consisting of all $n \in \omega$ for which there exists a unique sequence $u \equiv u(n) \equiv (U_k \in \mathbf{W}|k \in n+1)$ such that $U_0 = U^*$, $U_k < U_l$ for any $k \in l \in n+1$ and such that $V \in \mathbf{W}$ in $U_0 \leq V < U_n$ imply $V = U_k$ for a certain $k \in n$ (**W**-*noncondensibility property)*. Since the sequence $(U_k \in \mathbf{W}|k \in 1)$ such that $U_0 \equiv U^*$ has all the properties listed above, it follows that $0 \in N$. Let $n \in N$, i.e., for n, the sequence $u \equiv (U_k \in \mathbf{W}|k \in n+1)$ is constructed. Consider the finite set $A \equiv \{U_k \in \mathbf{W}|k \in n+1\}$ of cardinality $n+1$. By Condition (2), for $n+2$, there exists a finite set $B \in \mathbf{A}$ of cardinality $n+2$. In B, take a minimal element a and a maximal element b. By definition, $a \geq U^*$. Assume that $b \leq U_n$. Then the inequality $U_0 = U^* \leq a \leq c \leq b \leq U_n$ holds for any $c \in B$. If $c < U_n$, then by the **W**-noncondensibility property, we conclude from $c \in \mathbf{W}$ that $c = U_k$ for a certain $k \in n$, i.e., $c \in A$. If $c = U_n$, then $c \in A$ once again. As a result, we arrive at the inclusion $B \subset A$, which is impossible. The obtained contradiction implies $U_n < b$. Since $b \in \mathbf{W}$, the class $\mathbf{J} \equiv \{x \in \mathbf{W}|U_n < x\}$ is nonempty. Therefore, it contains a minimal element Λ.

Therefore, we can define the sequence $v \equiv (P_k \in \mathbf{W}|k \in n+2)$ setting $P_k \equiv U_k$ for any $k \in n+1$ and $P_{n+1} \equiv \Lambda$, i.e., $v = u \cup \{\langle n+1, \Lambda\rangle\}$. Further, almost in the same way as in deducing (1) $\vdash$ (4) with the replacement of $\mathbf{U}$ by $\mathbf{W}$, we verify that the sequence v has all necessary properties and is unique. Therefore, $n+1 \in N$. By the natural induction principle, $N = \omega$. Therefore, for any $n \in \omega$ there exists the above unique sequence $u(n)$. By the uniqueness, we can denote it by $(U_k^n|k \in n+1)$. This completes the deduction (2) $\vdash$ (3). Further, as in deducing (4) $\vdash$ (7), according to the sequence $(U_k^n|k \in n+1)$, we construct an infinite strictly increasing sequence $u \equiv (U_n|n \in \omega)$ of universal sets. This yields the deduction 2) $\vdash$ (6).

Therefore, we have proved the following deducibility and equivalence criteria: (1) $\vdash$ (4) $\vdash$ (7) $\vdash$ (6) $\vdash$ (1) and (6) $\vdash$ (5) $\vdash$ (2) $\vdash$ (6) and (2) $\sim$ (3). This implies the equivalence of all assertions (1)–(7).

(8) $\vdash$ (6). Let W be an infinite set of universal sets. By Theorem 3 (Sec. 3.2), there exists a minimal element U^* in W.

Consider the set N consisting of all $n \in \omega$ for which there exists a unique sequence $u \equiv u(n) \equiv (U_k \in W | k \in n+1)$ such that $U_0 = U^*$, $U_k < U_l$ for any $k \in l \in n+1$ and such that $V \in W$ and $U_0 \leq V < U_n$ implies $V = U_k$ for a certain $k \in n$ (*W-noncondensibity property*).

Since the sequence $(U_k \in I | k \in 1)$ such that $U_0 = U^*$ has all the properties listed above, it follows that $0 \in N$. Let $n \in N$. Consider the set $J \equiv W \setminus \{U_k | k \in n+1\}$. It is nonempty, since otherwise the set W is finite and, therefore, contains a minimal element Λ. Clearly, $\Lambda \neq U_n$ and $\Lambda \geq U^* = U_0$. Assume that $\Lambda < U_n$. Then by assumption, $U_0 \leq \Lambda < U_n$ implies $\Lambda = U_k$ for a certain $k \in n$, which is impossible. Therefore, $U_n < \Lambda$.

Therefore, we can define the sequence $v \equiv (P_k \in W | k \in n+2)$ setting $P_k \equiv U_k$ for any $k \in n+1$ and $P_{n+1} \equiv \Lambda$, i.e., $v = u \cup \{\langle n+1, \Lambda \rangle\}$. Clearly, $P_0 = U^*$ and $P_k < P_l$ for any $k \in l \in n+2$. Let $V \in W$ and $P_0 \leq V < P_{n+1}$. Then $U_0 \leq V < \Lambda$. If $V = U_n$, then $V = P_n$. If $V < U_n$, then $U_0 \leq V < U_n$ implies $V = U_k = P_k$ for a certain $k \in n$. Finally, if $V > U_n$, then $V > U_k$ for all $k \in n+1$. Hence $V \in J$. Therefore, $\Lambda \leq V$, which contradicts the property $V < \Lambda$. Therefore, this case is impossible. The two previous cases imply $V = P_k$ for a certain $k \in n+1$. This means that the sequence v has the necessary properties. Let us verify its uniqueness. Assume that there exists a sequence $w \equiv (V_k \in W | k \in n+2)$ such that $V_0 = U^*$, $V_k \in V_l$ for any $k \in l \in n+2$ and such that $V \in W$ and $V_0 \leq V < V_{n+1}$ imply $V = V_k$ for a certain $k \in n+1$. Since the sequence $w|n+1 \equiv (V_k \in W | k \in n+1)$ has all the properties listed above for n, by the uniqueness of the sequence u, we conclude that $u = w|n+1$, i.e., $V_k = U_k \equiv P_k$ for all $k \in n+1$. If $V_{n+1} < P_{n+1}$, then by what was proved above, $P_0 = V_0 \leq V_{n+1} < P_{n+1}$ implies $V_{n+1} = P_k = V_k$ for a certain $k \in n+1$, which is impossible. If $P_{n+1} < V_{n+1}$, then in a similar way, $V_0 = P_0 \leq P_{n+1} < V_{n+1}$ implies $P_{n+1} = V_k = P_k$ for a certain $k \in n+1$, which is also impossible. Therefore, $V_{n+1} = P_{n+1}$. Thus, the uniqueness of the sequence v is proved. Therefore, $n+1 \in N$. By the natural induction principle, $N = \omega$. Therefore, for any $n \in \omega$, there exists the unique sequence $u(n)$ indicated above. By the uniqueness, we can denote it by $(U_k^n | k \in n+1)$. Further, as in deducing (4) $\vdash$ (7), according to the sequence $(U_k^n | k \in n+1)$, we construct an infinite strictly increasing sequence $u \equiv (U_n | n \in \omega)$ of universal sets.

(6) $\vdash$ (8). As in proving the deducibility (6) $\vdash$ (1), for the sequence u, consider the set $X \equiv \{U_n | n \in \omega\}$ of its members. Assume that the set X is finite. Then X has the maximal element V, which contradicts the fact that the sequence strictly increases u. □

The fact that the ω-universality axiom is weaker than the universality axiom follows from the following proposition.

Proposition 1. *In the ZF theory, the ω-universality axiom is deduced from the universality axiom.*

Proof. Let us prove that Property (2) of Theorem 1 is deduced from AU. Indeed, let us prove by induction that for any $n \in \omega$, there exists a finite set of universal sets having the cardinality $n+1$.

For $n = 0$, the assertion means that there exists at least one universal set, which obviously holds.

Assume that for a certain $n \in \omega$, there exists a set of cardinality $n+1$ consisting of universal sets. Denote this set by A. By the universality axiom, there exists a universal set U such that $A \in U$, and hence $A \subset U$. If $V \in A$, then $V \neq U$, since otherwise $U \in U$, which is impossible. Consider the set $B \equiv A \cup \{U\}$. Obviously, this set is of cardinality $n+2$. Therefore, by induction, we have proved the desired deducibility. □

Along with the inaccessibility axiom AI, in the ZF theory the following weaker *ω-inaccessibility axiom* is considered:

$$AI(\omega) \equiv \exists X(\forall x \in X(Icn(x)) \wedge X \neq \emptyset \wedge \forall x \in X \exists y \in X(x \in y)).$$

The following theorem yields an explanation of such a name of this axiom.

Theorem 2. *The following assertions are equivalent in the ZF theory*:

(1) $AI(\omega)$;

(2) *for any $n \in \omega$, there exists a finite set of inaccessible cardinals having the cardinality $n+1$;*

(3) *for any* $n \in \omega$, *there exists a finite sequence* $u \equiv (\iota_k | k \in n+1)$ *of inaccessible cardinals such that* $\iota_k < \iota_l$ *for any* $k \in l \in n+1$, *i.e., the sequence* u *strictly increases*;
(4) *there exists an inaccessible cardinal* $\varkappa^*$, *and for any* $n \in \omega$, *there exists a unique finite strictly increasing sequence* $u(n) \equiv (\iota_k^n | k \in n+1)$ *of inaccessible cardinals such that* $\iota_0^n = \varkappa^*$, *and the properties that* $\varkappa$ *is a inaccessible cardinal and* $\iota_0^n \leq \varkappa \leq \iota_n^n$ *imply* $\varkappa = \iota_k^n$ *for a certain* $k \in n+1$ (*noncondensibility property*);
(5) *there exists a denumerable set of inaccessible cardinals*:
(6) *there exists an infinite sequence* $u \equiv (\iota_n | n \in \omega)$ *of inaccessible cardinals such that* $\iota_k < \iota_l$ *for any* $k \in l \in \omega$, *i.e., the sequence* u *strictly increases*;
(7) *there exists an infinite strictly increasing sequence* $u \equiv (\iota_n | n \in \omega)$ *of inaccessible cardinals such that* $n \in \omega$, $\varkappa$ *is an inaccessible cardinal and such that* $\iota_0 \leq \varkappa \leq \iota_n$ *implies* $\varkappa = \iota_k$ *for a certain* $k \in n+1$ (*noncondensibility property*);
(8) *there exists an infinite set of inaccessible cardinals.*

The proof of this theorem is completely analogous to the proof of Theorem 1. However, it can be also obtained from Theorem 1 by using the isotone bijection $q : \mathbf{U} \to \mathbf{In}$ from the corollary of Theorem 2 (Sec. 3.2).

The following proposition is an ω-analog of Theorem 1 (Sec. 3.3).

Proposition 2. *The following axioms are equivalent in the ZF theory*:

(1) *the* ω*-universality axiom* $AU(\omega)$;
(2) *the* ω*-inaccessibility axiom* $AI(\omega)$.

Proof. To prove the equivalence, it suffices to apply the isotone bijection $q : \mathbf{U} \to \mathbf{In}$ from the corollary to Theorem 2 (Sec. 3.2). □

4.2. Comparison of various forms of the universality and inaccessibility axioms.

Lemma 1. *The following assertions are equivalent in the ZF theory*:

(1) *the* ω*-universality axiom* $AU(\omega)$;
(2) $ATU(\omega)$ (*transitive* ω*-universality axiom*) $\equiv$ *there exists a set* Y *such that*:
  (a) $\forall U \in Y (U \bowtie)$;
  (b) $Y \neq \emptyset$;
  (c) $\forall U \forall V (U \bowtie \wedge U \in V \wedge V \in Y \Rightarrow U \in Y)$ (*transitivity property with respect to universal sets*);
  (d) $\forall V \in Y \exists W \in Y (V \in W)$ (*unboundedness property*).

Proof. (1) $\vdash$ (2). Denote by D the set whose existence is asserted in $AU(\omega)$. Consider the set $E \equiv \{U \in \cup D | U \bowtie\}$. If $U \in D$, then $\exists V \in D (U \in V)$ by $AU(\omega)$. Therefore, $D \subset E$. The set E is universally transitive. Indeed, if $U \bowtie$ and $U \in V \in E$, then $U \in V \in W \in D$ for a certain $W \in D$. By the transitivity of the set W, we obtain $U \in W \in D$, i.e., $U \in E$.

If $V \in E$, then by definition, $V \in W \in D \subset E$ for a certain W. Therefore, E satisfies Property (2).

The deducibility (2) $\vdash$ (1) is obvious. □

An analogous lemma holds for inaccessible cardinals with the replacements of $AU(\omega)$ by $AI(\omega)$ and $ATU(\omega)$ by $ATI(\omega)$ (*transitive* ω*-inaccessibility axiom*).

Lemma 2. *Let* E *be a nonempty set of universal sets with the transitivity property with respect to universal sets, i.e.,* E *has Properties* (a)–(c) *of Lemma* 1. *Then* E *contains a minimal universal set* $\mathfrak{a}_0 \equiv U_0 \equiv \cap \mathbf{U}$.

Proof. Let $V \in E$. By Proposition 1 (Sec. 3.2), $V = \mathfrak{a}$ or $\mathfrak{a} \in V$. In the first case, $\mathfrak{a} \in E$. In the second case, by Property (c), $\mathfrak{a} \in V \in E$ implies $\mathfrak{a} \in E$. □

An analogous lemma holds for inaccessible cardinals with the replacement of $\mathfrak{a} \equiv U_0$ by $\mathfrak{q} \equiv q_0 \equiv sm\,\mathbf{In}$.

Along with Axioms AU and $AU(\omega)$, consider one more weaker 1-*universality axiom* $AU(1) \equiv AUS$ (*axiom of universal set*) $\equiv \exists U(U \bowtie)$, which asserts the existence of at least one universal set. In the $ZF + AU(1)$ set theory, the class $\mathbf{U}$ of all universal sets is nonempty, and therefore, contains a minimal element $\mathfrak{a} \equiv U_0 \equiv \cap\mathbf{U}$.

Analogously, along with Axioms AI and $AI(\omega)$, consider one more weaker 1-*inaccessibility axiom* $AI(1) \equiv AIC$ (*axiom of inaccessible cardinal*) $\equiv \exists\varkappa(Icn(\varkappa))$, which asserts the existence of at least one inaccessible cardinal number. In the ZF+AI(1) set theory, the class $\mathbf{In}$ of all inaccessible cardinal numbers is nonempty and, therefore, contains a minimal element $\mathfrak{q} \equiv q_0 \equiv sm\,\mathbf{In}$.

The following proposition is a 1-analog of Theorem 1 (Sec. 3.3) and Proposition 2 (Sec. 4.1).

Proposition 1. *In the ZF theory, the following axioms are equivalent*:

(1) *the* 1-*universality axiom* $AU(1)$;

(2) *the* 1-*inaccessibility axiom* $AI(1)$.

Proof. To prove the equivalence, it suffices to apply the isotone bijection $q : \mathbf{U} \to \mathbf{In}$ from the corollary of Theorem 2 (Sec. 3.2). □

The following relations between these axioms hold:

$$AU \vdash AU(\omega) \vdash AU(1) \text{ and } AI \vdash AI(\omega) \vdash AI(1).$$

Let us show that these axioms are indeed different.

Assertion 1. (1) *If the* $ZF + AU(1)$ *theory is consistent, then the* $ZF + AU(1) + \neg AU(\omega)$ *theory is also consistent.*

(2 *If the* $ZF + AU(1)$ *theory is consistent, then Axiom* $AU(\omega)$ *is not deducible in* $ZF + AU(1)$.

Proof. (1) Let U_0 be a minimal universal set whose existence follows from Axiom $AU(1)$. Consider the classes $\mathbf{W} \equiv \{W | W \bowtie \wedge U_0 \in W\}$ and $\mathbf{D} \equiv \{X | \forall W(W \bowtie \wedge U_0 \in W \Rightarrow X \in W)\}$.

The following two cases are possible. If the class $\mathbf{W}$ is not empty, then it contains a minimal element U_1. Clearly, $\mathbf{D} \subset U_1$. If $X \in U_1$ and $W \in \mathbf{W}$, then $X \in U_1 \subset W$ implies $X \in W$. Hence $X \in \mathbf{D}$. Thus, $\mathbf{D} = U_1$. If the class $\mathbf{W}$ is empty, then $\mathbf{D} = \mathbf{V}$.

By Lemma 1, Axiom $AU(\omega)$ is equivalent to Axiom $ATU(\omega)$. Therefore, we consider the equivalent $T \equiv ZF + AUS + \neg ATU(\omega)$ theory. Consider the class of standard interpretation $\mathbf{M} \equiv (\mathbf{D}, I)$ of the T theory in the $S \equiv ZF + AUS$ set theory in which the correspondence I puts in correspondence the binary relations $\mathbf{E} \equiv \{z | \exists x \exists y(x \in \mathbf{D} \wedge y \in \mathbf{D} \wedge z = (x, y) \wedge x = y)\}$ and $\mathbf{B} \equiv \{z | \exists x \exists y(x \in \mathbf{D} \wedge y \in \mathbf{D} \wedge z = (x, y) \wedge x \in y)\}$ on $\mathbf{D}$ to the predicate symbols $=$ and $\in$ in T

If $\mathbf{D} = U_1$, then by Proposition 1 (Sec. 5.1), the interpretation $M \equiv \mathbf{M} = (U_1, I)$ is a model of the ZF theory in the S theory. If $\mathbf{D} = \mathbf{V}$, then, obviously, the interpretation $\mathbf{M}$ is a class model of ZF in the S theory.

Let us verify that Axiom AUS holds in $\mathbf{M}$. Axiom AUS can be written as follows:

$$AUS \equiv \exists X(\forall x(x \in X \Rightarrow x \subset X \wedge \mathcal{P}(x) \in X \wedge \cup x \in X) \wedge \forall x \forall y(x \in X \wedge y \in X \Rightarrow \{x, y\} \in X)$$
$$\wedge \forall x \forall f(x \in X \wedge f \fallingdotseq x \to X \Rightarrow rng\, f \in X) \wedge \omega \in X).$$

Let us consider the first case. Let s be a certain sequence $x_0, \ldots, x_q, \ldots$ of elements of the domain U_1. Taking the three equivalences proved in Proposition 1 (Sec. 5.1) and the notation from its proof, we obtain

$\widetilde{AUS^t} = \exists X \in U_1(\forall x \in U_1(x \in X \Rightarrow x \subset X \wedge \mathcal{P}(x)^\tau \in X \wedge (\cup x)^\tau \in X) \wedge \forall x \in U_1 \forall y \in U_1(x \in X \wedge y \in X \Rightarrow \{x, y\}^\sigma \in X) \wedge \forall x \in U_1 \forall f \in U_1(x \in X \wedge (f \fallingdotseq x \to X)^\rho \Rightarrow (rng\, f)^\rho \in X) \wedge \omega^\pi \in X)$.

In proving Proposition 1 (Sec. 5.1), it was proved that $\mathcal{P}(x)^\tau = \mathcal{P}(x)$, $(\cup x)^\tau = \cup x$, $\{x, y\}^\sigma = \{x, y\}$, $(f \fallingdotseq x \to X)^\rho \Leftrightarrow (f \fallingdotseq x \to X)$, and $(rng\, f)^\rho = rng\, f$. In a similar way, we can prove that $\omega^\pi = \omega$. Therefore, $\widetilde{AUS^t} \Leftrightarrow \exists X \in U_1 \chi(X)$, where the formula $\chi(X) \equiv \forall x \in U_1(x \in X \Rightarrow x \subset X \wedge \mathcal{P}(x) \in X \wedge \cup x \in X) \wedge \forall x \in U_1 \forall y \in U_1(x \in X \wedge y \in X \Rightarrow \{x, y\} \in X) \wedge \forall x \in U_1 \forall f \in U_1(x \in X \wedge f \fallingdotseq x \to X \Rightarrow rng\, f \in X) \wedge \omega \in X$ is obtained by deleting the indices τ, σ, and ρ in the conjunction kernel of the formula $\widetilde{AUS^t}$. Since U_0 is a universal set, the formula $\chi(U_0)$ holds for it. This means that the formula $\chi(U_0)$ is deduced from Axiom AUS in the S set theory. Hence the formula $\exists X \in U_1 \chi(X)$ is deduced, and, therefore, AUS^t is also deduced.

In the second case, on a sequence s of elements $x_0, \ldots, x_q, \ldots$ of the domain $\mathbf{V}$, the formula AUS is obviously transformed into the formula AUS once again, and, therefore, Axiom AUS holds in $\mathbf{M}$.

It remains to verify the fulfillment of the formula $\neg ATU(\omega)$. By Lemma 2, we can also insert the formula $U_0 \in Y$ in the conjunctive kernel of Axiom $ATU(\omega)$. Therefore, consider the formulas

$\varphi \equiv ATU(\omega) \equiv \exists Y(\forall U(U \in Y \Rightarrow U \bowtie) \wedge U_0 \in Y \wedge \forall U \forall V(U \bowtie \wedge U \in V \wedge V \in Y \Rightarrow U \in Y) \wedge \forall V(V \in Y \Rightarrow \exists W(W \in Y \wedge V \in W)))$ and $\varphi^t \equiv \mathbf{M} \vDash \varphi[s]$.

Let us consider the first case. Taking into account the remarks made after Axiom AUS, we obtain

$\varphi^t \Leftrightarrow \widetilde{\varphi^t} = \exists Y \in U_1(\forall U \in U_1(U \in Y \Rightarrow (U \bowtie)^\sigma) \wedge U_0^\tau \in Y \wedge \forall U \in U_1 \forall V \in U_1((U \bowtie)^\rho \wedge U \in V \wedge V \in Y \Rightarrow U \in Y) \wedge \forall V \in U_1(V \in Y \Rightarrow \exists W \in U_1(W \in Y \wedge V \in W)))$.

In considering the transformation of the previous axiom, we have proved that $(U \bowtie)^\sigma \Leftrightarrow \chi(U)$ and that the same is true for the index ρ.

Since the set U_0 can be defined from the formula $\exists! Z(Z \bowtie \wedge \forall U(U \bowtie \Rightarrow Z \subset U))$, the set U_0^τ is defined from the formula $\exists! Z \in U_1((Z \bowtie)^* \wedge \forall U \in U_1((U \bowtie)^{**} \Rightarrow Z \subset U))$.

As above, $(Z \bowtie)^* \Leftrightarrow \chi(Z)$ and $(U \bowtie)^{**} \Leftrightarrow \chi(U)$. Therefore, U_0^τ is defined from the formula $\exists! Z \in U_1(\chi(Z) \wedge \forall U \in U_1(\chi(U) \Rightarrow Z \subset U))$. It is clear from this that $U_0^\tau = U_0$.

Therefore, $\varphi^t \Leftrightarrow \exists Y \in U_1(\forall U \in U_1(U \in Y \Rightarrow \chi(U)) \wedge U_0 \in Y \wedge \forall U \in U_1 \forall V \in U_1(\chi(U) \wedge U \in V \wedge V \in Y \Rightarrow U \in Y) \wedge \forall V \in U_1(V \in Y \Rightarrow \exists W \in U_1(W \in Y \wedge V \in W)))$.

Assume that the condition φ^t is fixed and consider the set $E \in U_1 = \mathbf{D}$ whose existence follows from this condition. By the condition, $U_0 \in E$. Therefore, φ^t implies that for $U_0 \in U_1$, there exists $W \in U_1$ such that $W \in E$ and $U_0 \in W$. Let us deduce from this that the set W is universal.

Since $W \in E$, it follows that $\chi(W)$. Let $x \in W$. By the transitivity of U_1, it follows from $W \in U_1$ that $x \in U_1$. Therefore, $\chi(W)$ implies $x \subset W$, $\mathcal{P}(x) \in W$, and $\cup x \in W$. Analogously, if $x, y \in W$, then $x, y \in U_1$ and $\chi(W)$ implies $\{x, y\} \in W$. Finally, let $x \in W$ and $f \leftrightharpoons x \to W$. Then $x \in U_1$ and $W \in U_1$ imply $f \subset x * W \in U_1$. Lemma 1 (Sec. 3.1) implies $f \in U_1$. Therefore, $\chi(W)$ implies $rng\, f \in W$. The property $y \subset x \wedge x \in W \Rightarrow y \in W$ and the property $x, y \in W \Rightarrow (\langle x, y \rangle \in W \wedge x \cup y \in W)$ are easily deduced from the above. From $x * y \subset \mathcal{P}(\mathcal{P}(x \cup y))$, the property $x, y \in W \Rightarrow x * y \in W$ is deduced. Finally, $\chi(W)$ directly implies $\omega \in W$. Therefore, W is universal.

Moreover, $U_0 \in W$. Hence $W \in \mathbf{W}$. This implies $U_1 \subset W$. Taking into account Proposition 1 (Sec. 3.2), we conclude that $W \notin U_1$. On the other, hand, from φ^t, we have deduced that $W \in U_1$.

Therefore, in the S theory, from the formula φ^t, we have deduced the formula $\eta \equiv W \in U_1$ and the formula $\neg\eta = W \notin U_1$. Ny the deduction theorem in the S theory, we have deduced the formulas $(\varphi^t \Rightarrow \chi)$ and $(\varphi^t \Rightarrow \neg\chi)$.

Now, applying the classical implicit logical axiom $(\varphi^t \Rightarrow \chi) \Rightarrow ((\varphi^t \Rightarrow \neg\chi) \Rightarrow \neg\varphi^t)$, we sequentially deduce the formulas $(\varphi^t \Rightarrow \neg\chi) \Rightarrow \neg\varphi^t$ and $\neg\varphi^t$. Therefore, from the condition $\mathbf{W} \neq \emptyset$, we have deduced the formula $\neg\varphi^t$. By the deduction theorem, in the S theory, the formula $\mathbf{W} \neq \emptyset \Rightarrow \neg\varphi^t$ is deduced.

In the second case, on a sequence s of elements $x_0, \ldots, x_q, \ldots$ of the sequence $\mathbf{V}$, the formula φ is obviously translated to the formula φ once again, i.e., $\varphi^t = \varphi$.

Assume that the condition $\varphi^t = \varphi$ is fixed and consider the set $E \in \mathbf{V} = \mathbf{D}$ whose existence follows from this condition. By the condition, $U_0 \in E$. Therefore, φ^t implies that for U_0, there exists a universal set $W \in E$ such that $U_0 \in W$. Therefore, $\mathbf{W} \neq \emptyset$. By the deduction theorem in S, the formula $\varphi^t \Rightarrow \mathbf{W} = \emptyset$ is deduced. Applying the logical formula $(\varphi^t \Rightarrow \neg(\mathbf{W} = \emptyset)) \Rightarrow (\mathbf{W} = \emptyset \Rightarrow \neg\varphi^t)$, we deduce the formula $\mathbf{W} = \emptyset \Rightarrow \neg\varphi^t$. Therefore, from the condition $\mathbf{W} = \emptyset$, we have deduced the formula $\mathbf{W} = \emptyset \Rightarrow \neg\varphi^t$. Therefore, the formula $\neg\varphi^t$ is deduced from the condition $\mathbf{W} = \emptyset$. By the deduction theorem in S theory, the formula $\mathbf{W} = \emptyset \Rightarrow \neg\varphi^t$ is deduced. Denote the formula $\mathbf{W} = \emptyset$ by ξ.

Now applying the logical formula $(\xi \Rightarrow \neg\varphi^t) \Rightarrow ((\neg\xi \Rightarrow \neg\varphi^t) \Rightarrow ((\xi \vee \neg\xi) \Rightarrow \neg\varphi^t))$, in the S theory, we sequentially deduce the formulas $(\xi \Rightarrow \neg\varphi^t) \Rightarrow (\xi \vee \neg\xi \Rightarrow \neg\varphi^t)$ and $\xi \vee \neg\xi \Rightarrow \neg\varphi^t$. Since, in the first-order theory, for any formula ξ, the formula $\xi \vee \neg\xi$ is deduced, as a result, the formula $\neg\varphi^t$ is deduced in S.

The latter formula is equal to the formula $\mathbf{M} \vDash (\neg\varphi)[s]$. Therefore, $\mathbf{M}$ is a model of T in S.

(2) We will proceed in the naive propositional logic with the implication symbol $\supset$.

Denote by Φ_a and Ξ_a the totalities of axioms of the T and S theories, respectively.

Consider the propositions $A \equiv cons(S) \supset \neg(\Xi_a \vdash AU(\omega))$ and $B \equiv cons(S) \wedge (\Xi_a \vdash AU(\omega))$. Then $\neg A = cons(S) \wedge \neg\neg(\Xi_a \vdash AU(\omega))$. Using the axiom $\neg\neg C \supset C$, we obtain $\neg A \supset B$.

Clearly, $B \supset (\Phi_a \vdash AU(\omega))$ and $\Phi_a \vdash \neg AU(\omega)$. Therefore, the proposition $B \supset (\Phi_a \vdash AU(\omega)) \wedge (\Phi_a \vdash \neg AU(\omega))$, i.e., the proposition $B \supset \neg cons(T)$, is true. By the deduction rule, $\neg A \supset \neg cons(T)$.

According to the first part of our assertion, the proposition $cons(S) \supset cons(T)$ is true. Therefore, $B \supset cons(T)$ is true. By the deduction rule, we deduce $\neg A \supset cons(T)$.

Therefore, the proposition $(\neg A \supset cons(T)) \wedge (\neg A \supset \neg cons(T))$ is deduced. Applying the tautology $(\neg A \supset C) \wedge (\neg A \supset \neg C) \supset A$ (see [16], I, § 7), we deduce the proposition A. ☐

Corollary. *If the $ZF + AU(1)$ theory is consistent, then Axiom AU is not deducible in $ZF + AU(1)$.*

Remark. In fact, in the ZF+AU(1) theory, we have proved the nondeductibility of the second universal set U_1, i.e., a set such that $U_0 \in U_1$ and $U_1 = \cap\{U | U \bowtie \wedge U_0 \in U\}$.

Analogous assertions hold for inaccessible cardinals with the replacement of AU(1), $AU(\omega)$, and AU by AI(1), $AI(\omega)$, and AI, respectively.

Assertion 2. (1) *If the ZF+$AU(\omega)$ is consistent, then the $ZF+AU(\omega)+\neg AU$ theory is also consistent.*

(2) *If the $ZF + AU(\omega)$ theory is consistent, then Axiom AU is nondeducible in $ZF + AU(\omega)$.*

Proof. (1) Let D be a set whose existence follows from Axiom $AU(\omega)$. Consider the classes $\mathbf{W} \equiv \{W | W \bowtie \wedge D \in W\}$ and $\mathbf{D} \equiv \{X | \forall W(W \bowtie \wedge D \in W \Rightarrow X \in W)\}$.

The following two cases are possible. If the class $\mathbf{W}$ is nonempty, then it contains a minimal element U^*. Clearly, $\mathbf{D} = U^*$. If the class $\mathbf{W}$ is nonempty, then$\mathbf{D} = \mathbf{V}$.

Consider the class of standard interpretation $\mathbf{M} \equiv (\mathbf{D}, I)$ of the $T \equiv ZF + AU(\omega) + \neg AU$ theory in the $S \equiv ZF + AU(\omega)$ set theory for which I is the same as in the proof of the previous assertion. According to this proof, $\mathbf{M}$ is a class model of the ZF theory in the S theory.

Let us verify that Axiom $AU(\omega)$ of the T theory holds in $\mathbf{M}$. This axiom has the form

$AU(\omega) \equiv \exists X(\forall U(U \in X \Rightarrow U \bowtie) \wedge X \neq \emptyset \wedge \forall V(V \in X \Rightarrow \exists W(W \in X \wedge V \in W)))$.

We first consider the first case. Exactly in the same way as was done in proving the previous assertion, it is proved that

$AU(\omega)^t \Leftrightarrow \exists X \in U^*(\forall U \in U^*(U \in X \Rightarrow \chi(U)) \wedge X \neq \emptyset \wedge \forall V \in U^*(V \in X \Rightarrow \exists W \in U^*(W \in X \wedge V \in W)))$.

Consider the set $D \neq \emptyset$. If $U \in D$, then U is a universal set, and, therefore, the formula $\chi(U)$ holds for it. Let $V \in D$. It follows from $AU(\omega)$ that there exists $W \in D$ such that $V \in W$. By the transitivity of U^*, $W \in D \in U^*$ implies $W \in U^*$. This means that the formula $AU(\omega)^t$ is deduced from $AU(\omega)$.

In the second case, the formula $AU(\omega)$ obviously translates into the formula $AU(\omega)$ once again, and, therefore, Axiom $AU(\omega)$ holds in $\mathbf{M}$.

It remains to verify the fulfillment of the formula $\neg AU$.

Consider the formula

$$\varphi \equiv AU \equiv \forall X \exists V(V \bowtie \wedge X \in V).$$

We first consider the first case. Then

$$\varphi^t \Leftrightarrow \forall X \in U^* \exists V \in U^*(\chi(V) \wedge X \in V).$$

Assume that the condition φ^t is fixed. Since $D \in U^*$, according to this condition, for D, there exists a set $W \in U^*$ such that $\chi(W)$ and $D \in W$. In the same way as was done in proving the previous assertion, we deduce from $W \in U^*$ and $\chi(W)$ that W is universal. Moreover, $D \in W$. Hence $W \in \mathbf{W}$. This implies

$U^* \subset W$. Taking into account Proposition 1 (Sec. 3.2), we conclude that $W \notin U^*$. On the other hand, we have deduced from φ^t that $W \in U^*$.

As in the proof of the previous assertion, we conclude from this that the formula $\mathbf{W} \neq \emptyset \Rightarrow \neg\varphi^t$ is deduced in the S theory.

In the second case, the formula φ obviously transforms into the formula φ once again, i.e., $\varphi^t = \varphi$.

Assume that the condition $\varphi^t = \varphi$ is fixed. According to this condition, for the set D, there exists a universal set W such that $D \in W$. This implies $W \in \mathbf{W}$, and, therefore, $\mathbf{W} \neq \emptyset$. By the deduction theorem in S, we deduce the formula $\varphi^t \Rightarrow \mathbf{W} \neq \emptyset$. In the same way as in proving Assertion 1, we deduce from this the formula $\mathbf{W} = \emptyset \Rightarrow \neg\varphi^t$.

Further, as in the proof of Assertion 1, from the deduced formulas $\xi \Rightarrow \neg\varphi^t$ and $\neg\xi \Rightarrow \varphi^t$, we deduce the formula $\neg\varphi^t$, which is equal to the formula $\mathbf{M} \vDash (\neg\varphi)[s]$. Therefore, $\mathbf{M}$ is a model of T in S.

(2) The proof is the same as the proof of item (2) of Assertion 1. □

Therefore, Axiom AU is in fact stronger than Axiom $AU(\omega)$, and Axiom $AU(\omega)$ is strictly stronger than Axiom $AU(1)$. An analogous relation holds for Axioms AI, $AI(\omega)$, and $AI(1)$ equivalent to them.

Note that Axiom $AI(1)$ is not deducible in the ZF theory. Moreover, by the methods formalized in the ZF theory, it is not possible to show that Axiom $AI(1)$ is consistent with the ZF theory (see [14], 12, Theorem 12.12). Analogous assertions hold for Axioms $AI(\omega)$ and AI and the universality axioms equivalent to them.

5. Description of the Class of all Supertransitive Standard Models of the NBG Theory in the ZF Theory

5.1. Supertransitive standard models of the ZF theory having the strong subtitution property. Let U be a certain set in the ZF theory. On U, let us consider the binary *equality relation* $E \equiv \{z \in U * U | \exists x, y \in U(z = (x, y) \wedge x = y)\}$ and the *membership relation* $B \equiv \{z \in U * U | \exists x, y \in U(z = (x, y) \wedge x \in y)\}$. The interpretation $M \equiv (U, I)$ of the ZF or NBG theory in which to predicate symbols = and ∈ the correspondence I puts in correspondence the binary relations E and B on the set U is said to be *standard*.

According to ([3], II, § 7), a set U is said to be a *standard model for the ZF* [*NBG*] theory if the standard interpretation $M \equiv (U, I)$ is a model of the ZF theory [resp. NBG theory].

Recall that for the formula $\varphi(x, y, \dots)$, by $\varphi^U(x, y, \dots)$ we usually denote the *relativization of the formula* φ *to the set* U, i.e., the formula obtained by the replacement of all quantor prefixes $\forall t$ and $\exists t$ by the quantor prefixes $\forall t \in U$ and $\exists t \in U$ in φ, respectively.

Proposition 1. *In the ZF theory, the following assertions are equivalent for a set* U:

(1) *U is a hypermodel for the ZF theory*;

(2) *U is a supertransitive standard model for the ZF theory and U has the strong subtitution property*

$$\forall x \forall f(x \in U \wedge f \in U^x \Rightarrow rng\, f \in U).$$

Proof. (1) ⊢ (2). Consider an arbitrary sequence $s \equiv x_0, \dots, x_q, \dots$ of elements of the set U and transformations of certain axioms and axiom schemes of the ZF theory under the standard interpretation of $M \equiv (U, I)$ on the sequence s (see [19], Chap. 2, § 2).

Instead of $\theta_M[s]$ and $M \vDash \varphi[s]$, we will write θ^t and φ^t for the terms θ and the formulas φ, respectively.

To simplify the further presentation, we first consider the translations of certain simple formulas. Let u and v be certain sets.

The formula $u \in v$ translates into the formula $(u \in v)^t = (\langle u^t, v^t\rangle \in B)$. Denote the latter formula by γ. By definition, this formula is equivalent to the formula $(\exists x \exists y(x \in U \wedge y \in U \wedge \langle u^t, v^t\rangle = \langle x, y\rangle \wedge x \in y))$. Using the property of an ordered pair, we conclude that $u^t = x$ and $v^t = y$. Hence the formula $\delta \equiv (u^t \in v^t)$ is deduced from γ. By the deduction theorem, $\gamma \Rightarrow \delta$. Conversely, consider the formula δ. In the ZF theory, it is proved that for sets u^t and v^t, there exists a set z such that $z = \langle u^t, v^t\rangle$. By the logical axiom scheme, **LAS3** from ([16], III, § 1), from the formula δ, we deduce the formula $(z = \langle u^t, v^t\rangle \Rightarrow u^t \in$

$U \wedge v^t \in U \wedge z = \langle u^t, v^t\rangle \wedge u^t \in v^t$). Since the formula $z = \langle u^t, v^t\rangle$ is deduced from the axioms, the formula $(u^t \in U \wedge v^t \in U \wedge z = \langle u^t, v^t\rangle \wedge u^t \in v^t)$ ia also deduced from the axioms. By **LAS13**, we deduce the formula $\exists x\exists y(x \in U \wedge y \in U \wedge z = \langle x, y\rangle \wedge x \in y)$, which is equivalent to the formula $z \in B$ and, therefore, to the formula γ. By the deduction theorem, $\delta \Rightarrow \gamma$. Therefore, the first equivalence $(u \in v)^t \Leftrightarrow u^t \in v^t$ holds.

The formula $v \subset w$ translates into the formula $(v \subset w)^t$. Denote the latter formula by ε. By the first equivalence proved above, it is equivalent to the formula $\varepsilon' \equiv \forall u \in U(u \in v^t \Rightarrow u \in w^t)$. According to **LAS11**, from the formula ε', we deduce the formula $\varepsilon'' \equiv (x \in U \Rightarrow (x \in v^t \Rightarrow x \in w^t))$. If $x \in v^t$, then $v^t \in U$ and the transitivity of the set U imply $x \in U$. Then it follows from the formula ε'' that $x \in v^t \Rightarrow x \in w^t$. Hence, by the deduction theorem, we deduce the formula $(\varepsilon \Rightarrow (x \in v^t \Rightarrow x \in w^t))$. By the generalization rule, we deduce the formula $\forall x(\varepsilon \Rightarrow (x \in v^t \Rightarrow x \in w^t))$. By **LAS12**, we deduce the formula $(\varepsilon \Rightarrow \forall x(x \in v^t \Rightarrow x \in w^t))$, i.e., the formula $(\varepsilon \Rightarrow v^t \subset w^t)$.

Conversely, let the formula $v^t \subset w^t$ be given. Using the logical axioms, we sequentially deduce from it the formulas $(u \in v^t \Rightarrow u \in w^t)$ and $(u \in U \Rightarrow (u \in v^t \Rightarrow u \in w^t))$. By the generalization rule, we deduce the formula ε'. Hence, by the deduction theorem, we deduce the formula $(v^t \subset w^t \Rightarrow \varepsilon)$. Therefore, the second equivalence $(v \subset w)^t \Leftrightarrow v^t \subset w^t$ holds.

Exactly in the same way as in deducing the first equivalence, we deduce the third equivalence $(u = v)^t \Leftrightarrow u^t = v^t$.

In what follows, we will write not literal transformations of axioms but their equivalent variants obtained by using the mentioned equivalencies.

The volume axiom **A1** translates into the formula $\mathbf{A1}^t \Leftrightarrow \mathbf{A1}^U = \forall X \in U\forall Y \in U(\forall u \in U(u \in X \Leftrightarrow u \in Y) \Rightarrow X = Y)$.

The axiom of pair **A2** translates into the formula $\mathbf{A2}^t \Leftrightarrow \mathbf{A2}^U = \forall u \in U\forall v \in U\exists x \in U\forall z \in U(z \in x \Leftrightarrow z = u \vee z = v)$.

The union axiom **A4** translates into the formula $\mathbf{A4}^t \Leftrightarrow \mathbf{A4}^U = \forall X \in U\exists Y \in U\forall u \in U(u \in X \Leftrightarrow \exists z \in U(u \in z \wedge z \in X))$.

The axiom of the set of subsets **A5** translates into the formula $\mathbf{A5}^t \Leftrightarrow \mathbf{A5}^U = \forall X \in U\exists Y \in U\forall u \in U(u \subset X \Leftrightarrow u \in Y)$.

The axiom substitution scheme **AS6** translates into the formula scheme $\mathbf{AS6}^t \Leftrightarrow \forall x \in U\forall y \in U\forall y' \in U(\varphi^\tau(x, y) \wedge \varphi^\tau(x, y') \Rightarrow y = y') \Rightarrow \forall X \in U\exists Y \in U\forall x \in U(x \in X \Rightarrow \forall y \in U(\varphi^\sigma(x, y) \Rightarrow y \in Y))$, where φ^τ and φ^σ denote the formulas $M \vDash \varphi[s^\tau]$ and $M \vDash \varphi[s^\sigma]$ in which s^τ and s^σ denote the corresponding changes of the sequence s in translating the quantor subformulas indicated above. Denote by $\alpha \Rightarrow \beta$ the latter formula scheme.

The empty set axiom **A7** translates into the formula $\mathbf{A7}^t \Leftrightarrow \mathbf{A7}^U = \exists x \in U\forall z \in U(z \notin x)$.

The infinity axiom **A8** translates into the formula $\mathbf{A8}^t \Leftrightarrow \mathbf{A8}^\tau \equiv \exists Y \in U(\emptyset^t \in Y \wedge \forall y \in U(y \in Y \Rightarrow (y \cup \{y\})^\tau \in Y))$, where the set $\emptyset^t$ is defined from the formula $A7^U$, the set $Z_1 \equiv Z_1(y) \equiv (y \cup \{y\})^\tau$ is defined from the formula $\exists Z_1 \in U\forall u \in U(u \in Z_1 \Leftrightarrow \exists z \in U(u \in z \wedge z \in \{y, \{y\}\}^\sigma))$, the set $Z_2 \equiv Z_2(y) \equiv \{y, \{y\}\}^\sigma$ is defined from the formula $\exists Z_2 \in U\forall u \in U(u \in Z_2 \Leftrightarrow u = y \vee u = \{y\}^\rho)$, and the set $Z_3 \equiv Z_3(y) \equiv \{y\}^\rho$ is defined from the formula $\exists Z_3 \in U\forall u \in U(u \in Z_3 \Leftrightarrow u = y)$.

Since M is a model of the ZF theory, all the transformations written above are deducible formulas in the ZF theory.

Therefore, the formula $\mathbf{A7}^U$ asserts the existence of a certain $x \in U$ denoted by $\emptyset^t$. If $z \in U$, then $\mathbf{A7}^U$ implies $z \notin x$. Now let $z \notin U$; assume that $z \in x$. Then by the transitivity of the set U, we obtain $z \in U$, which contradicts the condition. Hence, $z \notin x$. Therefore, $z \notin x$ is deduced. By the generalization rule, we deduce the formula $\forall z(z \notin x)$ meaning that $x = \emptyset$. Therefore, $\emptyset^t = \emptyset$ and $\emptyset \in U$.

Now let us verify that if $y \in U$, then $Z_3 = \{y\}$. Let $u \in Z_3$. Since $Z_3 \in U$ and U is transitive, it follows that $u \in U$. If $u \in U$, then the formula for Z_3 presented above implies $u = y$. Hence $u \in \{y\}$. Therefore, $Z_3 \subset \{y\}$. Conversely, let $u \in \{y\}$. Then $u = y$. Since $y \in U$, it follows that $u \in U$. Therefore, by the same formula, $u \in Z_3$. Therefore, $\{y\} \subset Z_3$, which implies the required equality. This equality leads to the disappearance of the index ρ in the formula for Z_2.

Using this equality, let us prove that $Z_2 = \{y, \{y\}\}$. Let $u \in Z_2$. Then, as above, $u \in U$. Therefore, the formula for Z_2 presented above implies $u = y$ or $u\{y\}$. Therefore, $u \in \{y, \{y\}\}$. Thus, $Z_2 \subset \{y, \{y\}\}$. Conversely, let $u \in \{y, \{y\}\}$. Then $u = y \in U$ or $u = \{y\} = Z_3 \in U$. Therefore, $u \in U$ in both cases. Therefore, $u \in Z_2$ by the same formula . Therefore, $\{y, \{y\}\} \subset Z_2$, which implies the required equality. This equality leads to the disappearance of the index σ in the formula for Z_1.

Finally, let us verify that if $y \in U$, then $Z_1 = y \cup \{y\}$. Let $u \in Z_1$. Since $Z_1 \in U$ and U is transitive, it follows that $u \in U$. Therefore, the formula for Z_1 implies that there exists $z \in U$ such that $u \in z$ and $z \in \{y, \{y\}\}$. Therefore, $u \in \cup\{y, \{y\}\} \equiv Z$, i.e., $Z_1 \subset Z$. Conversely, let $u \in Z$. Then there exists $z \in \{y, \{y\}\}$ such that $u \in z$. From $z = y \in U$ or $z = \{y\} = Z_3 \in U$, we conclude that $z \in U$. Therefore, the mentioned formula implies $u \in Z_1$. Therefore, $Z \subset Z_1$, which implies the required equality. This equality leads to the disappearance of the index τ in the formula $\mathbf{A8}^\tau$.

All that was said above implies $\mathbf{A8}^\tau \equiv \exists Y \in U(\emptyset \in Y \wedge \forall y \in U(y \in Y \Rightarrow y \cup \{y\} \in Y))$. If $y \in Y$, then $Y \in U$ and the transitivity of U imply $y \in U$. Then $y \cup \{y\} \in Y$ is deduced from this formula. By the deduction theorem, we deduce the formula $y \in Y \Rightarrow y \cup \{y\} \in Y$, and by the generalization rule, we deduce the formula $\forall y \in Y(y \cup \{y\} \in Y)$. Therefore, from $\mathbf{A8}^t$, we deduce the formula $\exists Y \in U(\emptyset \in Y \wedge \forall y \in Y(y \cup \{y\} \in Y))$ almost coinciding with the infinity axiom, which asserts that there exists an inductive set $Y \in U$.

Using the obtained transformations, let us prove that the set U is universal.

Consider the formula $\mathbf{A2}^U$. According to this formula, for any $u, v \in U$, there exists the corresponding set $x \in U$. If $z \in x$, then the transitivity of U implies $z \in U$. Therefore, from this formula, we deduce the formula $z = u \vee z = v$. If $z = u \vee z = v$, then $z \in U$, and, therefore, from $\mathbf{A2}^U$, we deduce the formula $z \in x$. Since $\mathbf{A2}^U$ is deducible in ZF, by the deduction theorem and the generalization rule, we deduce the formula $\forall z(z \in x \Leftrightarrow z = u \vee z = v)$, which means that $x = \{u, v\}$. Therefore, $\{u, v\} \in U$. By the deduction theorem, we deduce the formula $u, v \in U \Rightarrow \{u, v\} \in U$. This implies $\{u\} \in U$ and $\langle u, v\rangle \in U$.

Consider the formula $\mathbf{A4}^U$. According to this formula, for any $X \in U$, there exists the corresponding set $Y \in U$. As above, the transitivity of U implies $Y = \cup X$. Therefore, $\cup X \in U$, and by the deduction theorem, we deduce the formula $X \in U \Rightarrow \cup X \in U$. It follows from this that $X, Y \in U$ implies $X \cup Y \equiv \cup\{X, Y\} \in U$.

Consider the formula $\mathbf{A5}^U$. According to this formula, for any $X \in U$, there exists the corresponding set $Y \in U$. Clearly, $Y \subset \mathcal{P}(X)$. Let $y \in \mathcal{P}(X)$. Then by the quasi-transitivity of U, $y \subset X \in U$ implies $y \in U$. Hence, $Y = \mathcal{P}(X)$. Therefore, $\mathcal{P}(X) \in U$, and by the deduction theorem, we deduce the formula $X \in U \Rightarrow \mathcal{P}(X) \in U$.

If $X, Y \in U$, then by the quasi-transitivity property, $X * Y \subset \mathcal{P}(\mathcal{P}(X \cup Y)) \in U$ implies $X * Y \in U$.

Consider an inductive set $Y \in U$ whose existence was proved above. Since ω is minimal among all inductive sets, it follows that $\omega \subset Y$. By the quasi-transitivity property, this implies $\omega \in U$.

Property (4) from the definition of a universal set holds automatically.

Therefore, we have proved that (1) $\vdash$ (2).

(2) $\vdash$ (1). Let U be a universal set. According to Sec. 3.1, it is supertransitive. Consider the standard interpretation $M \equiv (U, I)$ of the ZF theory. In the above, we have carried out the transformation of certain axioms and axiom schemes of the ZF theory under the interpretation M on the sequence s. Let us prove that they are deducible in ZF.

Consider the formula $\mathbf{A1}^U$. Let $X, Y \in U$, and let $\chi \equiv \forall u \in U(u \in X \Leftrightarrow u \in Y)$. Take an arbitrary set u. If $u \in X$, then the transitivity of U implies $u \in U$, and then $u \in Y$ is deduced. Analogously, from $u \in X$, we deduce $u \in Y$. Therefore, by the deduction theorem, we deduce the formula $u \in X \Leftrightarrow u \in Y$, and by the generalization rule, we deduce the formula $\forall u(u \in X \Leftrightarrow u \in Y)$. According to the volume axiom, the equality $X = Y$ is deduced from this. By the deduction theorem in ZF, we deduce the formula $\chi \Rightarrow X = Y$. Further, by logical tools, we deduce $\mathbf{A1}^t$.

Consider the formula $\mathbf{A2}^U$. Let $u, v \in U$. By the property of a universal set, we have $\{u, v\} \in U$. By the axiom of pair, it follows that $\forall z \in U(z \in \{u, v\} \Leftrightarrow z = u \vee z = v)$. Therefore, by **LAS13**, we deduce the formula $\exists x \in U \forall z \in U(z \in x \Leftrightarrow z = u \vee z = v)$. Further, by logical tools, we deduce the formula $\mathbf{A2}^t$.

The axiom separation scheme **AS3** transforms into the formula scheme $\mathbf{AS3}^t \Leftrightarrow \forall X \in U \exists Y \in U \forall u \in U(u \in Y \Leftrightarrow u \in X \wedge \varphi^\tau(u))$, where Y is not a free variable of the formula $\varphi(u)$, and φ^τ denotes the formula $M \vDash \varphi[s^\tau]$ in which s^τ denotes the corresponding change of the sequence s under the transformation of the mentioned quantor overformulas $\forall X(\dots)$, $\exists Y(\dots)$ and $\forall u(\dots)$. According to **AS3**, for $X \in U$, there exists Y such that $\forall u \in U(u \in Y \Leftrightarrow u \in X \wedge \varphi^\tau(u))$. Since $Y \subset X \in U$, it follows by Lemma 1 (Sec. 3.1) that $Y \in U$. Therefore, $\mathbf{AS3}^t$ is deduced in ZF.

Similar to the deducibility of $\mathbf{A2}^t$, we verify the deducibility of $\mathbf{A4}^t$ and $\mathbf{A5}^t$.

Let us verify the deducibility of $\mathbf{AS6}^t$. Let the formula α hold. Consider any set $X \in U$. According to the axiom separation scheme, there exists the set $F \equiv \{z \in U | \exists x, y \in U(z = \langle x, y\rangle \wedge \varphi^\sigma(x, y))\}$. Clearly, $F \subset U*U$. The transitivity of U implies $X \subset U$. Therefore, there exists a set $Z \equiv F[X] \subset U$. Consider the set $G \equiv \{z \in U | \exists x, y \in U(z = \langle x, y\rangle \wedge \varphi^\sigma(x, y) \wedge x \in X)\} = F|X \subset X * Z$. Let $x \in X \subset U$. If $x \notin dom\, G$, then $G\langle x\rangle = \emptyset \in U$. Let $x \in dom\, G$, i.e., $G\langle x\rangle \neq \emptyset$. If $y, y' \in G\langle x\rangle \subset U$, then $\varphi^\sigma(x, y) \wedge \varphi^\sigma(x, y')$ holds, or, more precisely, $\varphi^\sigma(x, y, X, Y) \wedge \varphi^\sigma(x, y', X, Y)$, since X and Y can be free variables of the formula φ^σ. Since $\varphi^\tau(x, y) = \varphi^\sigma(x, y, X\|X_M[s], Y\|Y_M[s])$ and similarly for y', by **LAS11**, $\varphi^\tau(x, y) \wedge \varphi^\tau(x, y')$ holds. Therefore, the formula α implies $y = y'$. Therefore, $G\langle x\rangle = \{y\} \in U$. Thus, $G\langle x\rangle \in U$ for any $x \in X$. By Lemma 3 (Sec. 3.1), $Y_0 \equiv rng\, G = \cup\big(G\langle x\rangle | x \in X\big) \in U$.

If $x \in X \subset U$, $y \in U$, and $\varphi^\sigma(x, y)$, then $\langle x, y\rangle \in G$ implies $y \in Y_0$. This means that the formula β is deduced from the formula α. By the deduction scheme, the formula $\alpha \Rightarrow \beta$ and, therefore, the scheme $\mathbf{AS6}^t$ are deduced.

According to Lemma 2 (Sec. 3.1), $\emptyset \in U$. From this and **A7**, we deduce $\mathbf{A7}^t$.

Consider the formula $\mathbf{A8}^\tau$ and the set $\omega \in U$. It follows from the previous paragraph that $\emptyset^t = \emptyset \in \omega$. Let $y \in U$ and $y \in \omega$. Then, as above, we verify that $Z_3 = \{y\}$, $Z_2 = \{y, \{y\}\}$, and $Z_1 = y \cup \{y\} \in \omega$. By the deduction theorem, we deduce the formula $(y \in \omega \Rightarrow Z_1 \in \omega)$. Further, by logical tools, we deduce the formula $(\emptyset^t \in \omega \wedge \forall y \in U(y \in \omega \Rightarrow (y \cup \{y\})^\tau \in \omega))$ and, therefore, the formula $\mathbf{A8}^t$.

The regularity axiom **A9** translates into the formula $\mathbf{A9}^t \Leftrightarrow \mathbf{A9}^\tau \equiv \forall X \in U(X \neq \emptyset^t \Rightarrow \exists x \in U(x \in X \wedge (x \cap X)^\tau = \emptyset^t))$, where the set $\emptyset^t$ is defined from the formula $\mathbf{A7}^U$ and, as was proved above, coincides with the empty set $\emptyset$ and the set $Z \equiv (x \cap X)^\tau$ is determined from the formula $\exists Z \in U \forall u \in U(u \in Z \Leftrightarrow u \in x \wedge u \in X)$.

Let us verify that if $X \in U$ and $x \in U$, then $Z = x \cap X$. Let $u \in Z$. Since $Z \in U$ and U is transitive, it follows that $u \in U$. Therefore, the formula for Z implies $u \in x \wedge u \in X$, i.e., $u \in x \cap X$. Therefore, $Z \subset x \cap X$. Conversely, let $u \in x \cap X$, i.e., $u \in x \wedge u \in X$. By the transitivity, $u \in U$. Therefore, the mentioned formula implies $u \in Z$. Thus, $x \cap X \subset Z$, which proves the required equality. This equality leads to the disappearance of the index τ in the formula $\mathbf{A9}^\tau$.

Let $X \in U$, and let $X \neq \emptyset^t = \emptyset$. By the regularity axiom, there exists $x \in X$ such that $x \cap X = \emptyset$. By the transitivity, $x \in U$. By logical tools, we deduce $\mathbf{A9}^t$ from this.

Finally, the axiom of choice **A10** transforms into the formula

$$\mathbf{A10}^t \Leftrightarrow \mathbf{A10}^\tau \equiv \forall X \in U(X \neq \emptyset^t \Rightarrow \exists z \in U((z \leftrightharpoons \mathcal{P}(X) \setminus \{\emptyset\} \to X)^\tau \wedge \forall Y \in U(Y \in (\mathcal{P}(X) \setminus \{\emptyset\})^\sigma$$
$$\Rightarrow \forall x \in U(x \in X \wedge \langle Y, x\rangle^\sigma \in z \Rightarrow x \in Y))))),$$

where the set $Z_1 \equiv Z_1(X) \equiv (\mathcal{P}(X) \setminus \{\emptyset\})^\sigma$ is defined from the formula $\exists Z_1 \in U \forall u \in U(u \in Z_1 \Leftrightarrow u \in \mathcal{P}(X)^\rho \wedge u \notin \{\emptyset\}^\rho)$; the set $Z_2 \equiv \langle Y, x\rangle^\sigma$ is defined from the formula $\exists Z_2 \in U \forall u \in U(u \in Z_2 \Leftrightarrow (u = \{Y\}^\sigma \vee u = \{Y, x\}^\sigma))$; the set $Z_3 \equiv \{Y, x\}^\sigma$ is defined from the formula $\exists Z_3 \in U \forall u \in U(u \in Z_3 \Leftrightarrow (u = Y \vee u = x))$; the set $Z_4 \equiv \{Y\}^\sigma$ is defined from the formula $\exists Z_4 \in U \forall u \in U(u \in Z_4 \Leftrightarrow u = y)$; $\varphi^\tau \equiv (z \leftrightharpoons \mathcal{P}(X) \setminus \{\emptyset\} \to X)^\tau$ denotes the formula $M \vDash \varphi[s^\tau]$, in wich s^τ denotes the corresponding change of the sequence s in transforming the quantor overformulas $\forall X(\dots)\ \exists z(\dots)$ mentioned above.

Fix the conditions $X \in U$ and $X \neq \emptyset^t = \emptyset \in U$. In the above, we have proved that this implies $\mathcal{P}(X)^\rho = \mathcal{P}(X)$ and $\{\emptyset\}^\rho = \{\emptyset\}$. This leads to the disappearance of the index ρ in the formula for Z_1.

Let us verify that $Z_1 = \mathcal{P}(X) \setminus \{\emptyset\} \equiv Z$. Let $u \in Z_1$. Since $Z_1 \in U$ and U is transitive, it follows that $u \in U$. Therefore, the formula for Z_1 implies $u \in Z$. Therefore, $Z_1 \subset Z$. Conversely, let $u \in Z$. Since $\mathcal{P}(X) \in U$, it follows that $\mathcal{P}(X) \subset U$ by transitivity. This implies $u \in U$. Therefore, the mentioned formula implies $u \in Z_1$. Therefore, $Z \subset Z_1$, which proves the required equality. This leads to the replacement of Z_1 by Z in the formula $\mathbf{A10}^\tau$.

Consider the formula $\varphi \equiv (z \leftrightharpoons Z \to X)$. It is the conjunction of the following three formulas: $\varphi_1 \equiv (z \subset Z * X)$, $\varphi_2 \equiv (dom\, z = Z)$, and $\varphi_3 \equiv (\forall x(x \in Z \Rightarrow \forall y(y \in X \Rightarrow \forall y'(y' \in X \Rightarrow (\langle x,y\rangle \in z \wedge \langle x,y'\rangle \in z \Rightarrow y = y')))))$.

Therefore, $\varphi^\tau = \varphi_1^\tau \wedge \varphi_2^\tau \wedge \varphi_3^\tau$. Since $\varphi_1 = (\forall u(u \in z \Rightarrow \exists x \exists y(x \in Z \wedge y \in X \wedge u = \langle x,y\rangle)))$, it follows that $\varphi_1^\tau \Leftrightarrow (\forall u \in U(u \in z \Rightarrow \exists x \in U \exists y \in U(x \in Z \wedge y \in X \wedge u = \langle x,y\rangle^\sigma)))$. Analogously, $\varphi_2 = (\forall x(x \in Z \Rightarrow \exists y(y \in X \wedge \langle x,y\rangle \in z)))$ implies $\varphi_2^\tau \Leftrightarrow (\forall x \in U(x \in Z \Rightarrow \exists y \in U(y \in X \wedge \langle x,y\rangle^\sigma \in z)))$.

Finally, $\varphi_3^\tau \Leftrightarrow (\forall x \in U(x \in Z \Rightarrow \forall y \in U(y \in X \Rightarrow \forall y' \in U(y' \in X \Rightarrow (\langle x,y\rangle^\sigma \in z \wedge \langle x,y'\rangle^\sigma \in z \Rightarrow y = y')))))$.

By the transitivity property, for x, y, and y' in the formulas φ_1^τ, φ_2^τ, and φ_3^τ, we have $x, y, y' \in U$. Therefore, by what was proved above, the following equalities hold in these formulas: $\langle x,y\rangle^\sigma = \langle x,y\rangle$, and $\langle x,y'\rangle^\sigma = \langle x,y'\rangle$. This implies that the formulas φ_1^τ, φ_2^τ, and φ_3^τ differ from the formulas φ_1, φ_2, and φ_3, respectively, by only the bounded quantor prefices $\forall \cdots \in U$ and $\exists \cdots \in U$.

For X, by the axiom of choice **A10**, there exists z such that $\chi \equiv (z \leftrightharpoons Z \to X) \wedge \forall Y(Y \in Z \Rightarrow \forall x(x \in X \wedge \langle Y,x\rangle \in z \Rightarrow x \in Y))$.

Therefore, the formula $\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is deduced, and hence the formulas φ_1, φ_2, and φ_3 are deduced.

Let $u \in U$ and $u \in z$. Then we deduce from the formula φ_1 that there exist $x \in Z$ and $y \in X$ such that $u = \langle x,y\rangle$. Since $x \in Z \in U$ and $y \in X \in U$, by the transitivity property, $x, y \in U$. This means that for given conditions $u \in U$ and $u \in z$, the formula $\exists x \in U \exists y \in U(x \in Z \wedge y \in X \wedge u = \langle x,y\rangle^\sigma)$ is deduced. Applying the deduction theorem two times and the deduction rule, we deduce the formula φ_1^τ.

Let $x \in U$ and $x \in Z$. Then we deduce from the formula φ_2 that for x, there exists $y \in X$ such that $\langle x,y\rangle \in z$. It follows from $y \in X \in U$ that $y \in U$. This means that for given conditions $x \in U$ and $x \in Z$, the formula $\exists y \in U(y \in X \wedge \langle x,y\rangle^\sigma \in z)$ is deduced. As in the previous paragraph, we deduce from this the formula φ_2^τ.

Let $x \in U$, $x \in Z$, $y \in U$, $y \in X$, $y' \in U$, $y' \in X$, $\langle x,y\rangle \in z$, and $\langle x,y'\rangle \in z$. Then $y = y'$ is deduced from the formula φ_3. Applying alternately the deduction theorem several times and the deduction rules, we deduce the formula φ_3^τ.

Thus, the formula φ^τ is deduced.

Let us veryfy that $Z_4 = \{Y\}$ under the conditions $X \in U$, $Y \in U$, and $Y \in Z$. Let $u \in \{Y\}$, i.e., let $u = Y \in U$. Then the formula for Z_4 implies $u \in Z_4$. Conversely, if $u \in Z_4 \in U$, then $u \in U$, and hence, $u = Y \in \{Y\}$. This yields the required equality.

Let us veryfy that $Z_3 = \{Y, x\}$ under the conditions $X \in U$, $x \in X$, $Y \in U$, and $Y \in Z$. Let $u \in \{Y, x\}$. Then $u = Y \in U$ or $u = x \in X \in U$ implies $u \in U$, and, therefore, $u \in Z_3$. Conversely, if $u \in Z_3 \in U$, then $u \in U$ and the formula for Z_3 imply $u = Y \vee u = x$, i.e., $u \in \{Y, x\}$. This yields the required equality.

Finally, let us verify that $Z_2 = \langle Y, x\rangle$ under the above conditions. Let $u \in \langle Y, x\rangle$, i.e., $u = \{Y\}$ or $u = \{Y, x\}$. The above equalities lead to the disappearance of the index σ in the formula for Z_2. Since $Y \in U$ and $x \in X \in U$, it follows that $x \in U$ and the universality of U imply $u = \{Y\} \in U$ or $u = \{Y, x\} \in U$.

Therefore, $u \in U$ implies $u \in Z_2$. Conversely, if $u \in Z_2 \in U$, then $u \in U$ and the formula for Z_2 imply $u = \{Y\}$ or $u = \{Y, x\}$, i.e. $u = \langle Y, x\rangle$. This yields the required equality.

Since $Z =\in U$ and $X \in U$, it follows that $Z * X \in U$. By Lemma 1 (Sec. 3.1), $z \subset Z * X$ implies $z \in U$.

All this means that from Axiom **A10**, we deduce the existence of an object $z \in U$ satisfying the formula χ from which the formula $\xi \equiv (\varphi^\tau \wedge \forall Y \in U(Y \in Z_{\Rightarrow} \forall x \in U(x \in X \wedge \langle Y,x\rangle \in z \Rightarrow x \in Y))$ is deduced. Thus from the fixed conditions, we deduce the formula $\exists z \in U \xi$. Applying alternately the deduction theorem and the generalization rule several times, as a result, we deduce the formula $\mathbf{A10}^t$.

Therefore, M is a model of the ZF theory. □

Corollary. *Any inaccessible cumulative set $V_\varkappa$ is a standard model for the ZF theory.*

Proof. The assertion follows from the proved Proposition 1 and Theorem 2 (Sec. 3.2). □

This proposition implies that for supertransitive standard model sets, analogs of all the assertions presented in Sec. 3 for universal sets hold.

The corollary of Proposition 1 is well known (see, e.g., ([13], 13, Theorem 21)). Using Theorems 1 and 2 (Sec. 3.2) and Proposition 1, we prove the following converse theorem.

Theorem 1. *In the ZF theory, the following assertions are equivalent for a set U:*

(1) *$U = V_\varkappa$ for the inaccessible cardinal number $\varkappa = |U| = \sup(\mathbf{On} \cap U)$;*

(2) *U is a supertransitive standard model for the ZF theory and has the strong substitution property.*

Proof. (1) $\vdash$ (2). By Theorem 2 (Sec. 3.2), the set $U = V_\varkappa$ is universal. By Proposition 1, the set U is a hypermodel.

(2) $\vdash$ (1). By Proposition 1, U is universal. By Theorem 1 (Sec. 3.2), $U = V_\varkappa$ and $\varkappa = \sup(\mathbf{On} \cap U)$. By Corollary 1 of Theorem 1 (Sec. 3.2), $\varkappa = |U|$. □

Unfortunately, this theorem does not yield the description of all natural models and all supertransitive standard models of the ZF theory. This description will be given in Sec. 7.3.

Theorem 1 is equivalent to the Zermelo–Shepherdson theorem (see [28] and [23]) on the canonical forms of supertransitive standard model sets for the NBG theory in the ZF set theory (see Sec. 5.2 below).

5.2. Supertransitive standard models of the NBG theory in the ZF theory. The NBG theory is a first-order theory (without equalities) with a single binary predicate symbol of *belonging* $\in$ (write $A \in B$).

The objects of the NBG theory are called *classes*.

A class A is called a *set* if $\exists X(A \in X)$. We will denote this formula by $S(A)$.

The formula $\forall x(x \in A \Rightarrow x \in B)$ is denoted by $A \subset B$. Two classes A and B are said to be *equal* (denoted by $A = B$) if $A \subset B \wedge B \subset A$.

Let us present a list of proper axioms and axiom schemes of the NBG theory.

A1 (*volume axiom*).

$$\forall y \forall z((y = z) \Rightarrow \forall X(y \in X \Leftrightarrow z \in X)).$$

A formula φ is said to be a *predicate* (see [3], Chap. 4, § 1) if for any variables x, all strings of symbols $\forall x$ and $\exists x$ entering the formula φ are located at positions of the following form: $\forall x(S(x) \Rightarrow \dots)$ and $\exists x(S(x) \wedge \dots)$.

AS2 (*axiom scheme of complete envelopment*). Let $\varphi(x)$ be a predicate formula such that the substitution $\varphi(x\|y)$ is admissible and φ does not freely contain the variable Y. Then

$$\exists Y \forall y(y \in Y \Leftrightarrow (S(y) \wedge \varphi(y))).$$

This axiom scheme postulates the existence of classes, which are denoted by $\{x | \varphi(x)\}$.

The *universal class* is the class of all sets $V \equiv \{x | x = x\}$. The *empty class* is the class $\emptyset \equiv \{x | x \neq x\}$.

A3 (*axiom of set of subclasses* ($\equiv$ *of a complete ensemble*)).

$$\forall X(S(X) \Rightarrow \exists Y(S(Y) \wedge \forall Z(Z \subset X \Leftrightarrow Z \in Y))).$$

For each class A, the class $\mathcal{P}(A) \equiv \{x | x \subset A\}$ is called the *class of subsets of the class A*.

Axiom **A3** is equivalent to the conjunction of the following two axioms:

A3′ (*axiom of subset*).

$$\forall X \forall Y(S(X) \wedge Y \subset X \Rightarrow S(Y)).$$

A3″ (*axiom of set of subsets*).

$$\forall X(S(X) \Rightarrow S(\mathcal{P}(X))).$$

For two classes A and B, the class $A \cup B \equiv \{x | x \in A \vee x \in B\}$ is called the *union of the classes A and B*; the class $A \cap B \equiv \{x | x \in A \wedge x \in B\}$ is called the *intersection of the classes A and B*.

A4 (*axiom of binary union*).

$$\forall X \forall Y (S(X) \wedge S(Y) \Rightarrow S(X \cup Y)).$$

For a class A, consider the *individual class* $\{A\} \equiv \{x | x = A\}$. For two classes A and B consider an *unodered pair* $\{A, B\} \equiv \{A\} \cup \{B\}$, a *coordinate pair* $\langle A, B\rangle \equiv \{\{A\}, \{A, B\}\}$, and the *coordinate product* $A * B \equiv \{x | \exists y \exists z (y \in A \wedge z \in B \wedge x = \langle y, z\rangle)\}$.

As was done in Sec. 1.1 in the ZF theory, in the NBG theory we define a *correspondence* C *with domain* $dom\, C$ and *range* $rng\, C$, a *mapping* ($\equiv$ *function*) F, a *correspondence* $C : A \prec B$ with the *set of values* $C\langle a\rangle$ *at a point* $a \in A$, a *function* $F : A \to B$ with *value* $F(a)$ *at a point* $a \in A$, etc.

A5 (*axiom of general union*).

$$\forall X \forall Y \forall Z (S(X) \wedge (Z \subset X * Y) \wedge \forall x (x \in X \Rightarrow S(Z\langle x\rangle)) \Rightarrow S(rng\, Z)).$$

For each class A, the class $\cup A \equiv \{x | \exists y \in A (x \in y)\}$ is called the *union of the class* A.

Axiom **A5** is the conjunction of the following two axioms:

A5′ (*axiom of union*).

$$\forall X (S(X) \Rightarrow S(\cup X)).$$

A5″ (*axiom of replacement* ($\equiv$ *substitution*)).

$$\forall X \forall Y \forall Z (S(X) \wedge (Z \fallingdotseq X \to Y) \Rightarrow S(rng\, Z)).$$

A6 (*regularity axiom*).

$$\forall X (X \neq \emptyset \Rightarrow \exists x (x \in X \wedge x \cap X = \emptyset)).$$

A7 (*axiom of infinity*).

$$\exists X (S(X) \wedge \emptyset \in X \wedge \forall x (x \in X \Rightarrow x \cup \{x\} \in X)).$$

A8 (*axiom of choice*).

$$\forall X (S(X) \wedge X \neq \emptyset \Rightarrow \exists z ((z \fallingdotseq \mathcal{P}(X) \setminus \{\emptyset\} \to X) \wedge \forall Y (Y \in \mathcal{P}(X) \setminus \{\emptyset\} \Rightarrow z(Y) \in Y))).$$

Theorem 1. *For a set* P *the following assertions are equivalent in the ZF theory*:

(1) P *is a supertransitive standard model for the NBG theory*;

(2) $P = \mathcal{P}(U)$ *for a certain universal set* U.

Proof. 1) $\vdash$ 2). Consider an arbitrary sequence $s \equiv x_0, \dots, x_q, \dots$ of elements of the set P and translations of axioms and axioms schemes of the NBG theory into sequences s under the standard interpretation $M \equiv (P, I)$.

Instead of $\theta_M[s]$ and $M \vDash \varphi[s]$, we will write θ^t and φ^t for terms θ and formulas φ, respectively.

To simplify the further presentation, we first consider translations of certain simple formulas. Let u and v be certain classes.

Exactly in the same way as in proving Proposition 1 from the previous subsection, we verify that the equivalences $(u \in_{NBG} v)^t \Leftrightarrow u^t \in_{ZF} v^t$ and $(u \subset_{NBG} v)^t \Leftrightarrow u^t \subset_{ZF} v^t$ hold. This implies that the equivalence $(u =_{NBG} v)^t \Leftrightarrow (u^t \subset_{ZF} v^t) \wedge (v^t \subset_{ZF} u^t)$ holds.

The formula $X \subset Y \wedge Y \subset X$ of the ZF theory, equivalent to the formula $\forall a (a \in X \Leftrightarrow a \in Y)$, will be temporary denoted by $X \stackrel{*}{=} Y$.

Let $u^t \stackrel{*}{=} v^t$. By the volume axiom **A1** we deduce the formula $u^t =_{ZF} v^t$ in the ZF theory. By the deduction axiom, we deduce the formula $u^t \stackrel{*}{=} v^t \Rightarrow u^t =_{ZF} v^t$ in the ZF theory. Conversely, let $u^t =_{ZF} v^t$. Take $a \in u^t$. By the axiom of replacement of equals, we deduce the formula $a \in v^t$ from the latter equality. Therefore, by the deduction axiom, we deduce the formula $a \in u^t \Rightarrow a \in v^t$ in ZF, and by the generalization rule, we deduce the formula $u^t \subset v^t$. Similarly we deduce the formula $v^t \subset U^t$. Hence the formula $u^t \stackrel{*}{=} v^t$ is deduced. By the deduction axiom we deduce the formula $u^t =_{ZF} v^t \Rightarrow u^t \stackrel{*}{=} v^t$ in ZF. Thus, the equivalence $u^t \stackrel{*}{=} v^t \Leftrightarrow u^t =_{ZF} v^t$ holds. Therefore, the equivalence $(u =_{NBG} v)^t \Leftrightarrow (u^t =_{ZF} v^t)$ holds.

In what follows, we do not write literal translations of axioms and axiom schemes but their equivalent variants, which are obtained by using the mentioned equivalences.

The volume axiom **A1** translations into the formula $\mathbf{A1}^t \Leftrightarrow \mathbf{A1}^P = \forall y \in P \forall z \in P(y = z \Rightarrow \forall X \in P(y \in X \Leftrightarrow z \in X))$.

The axiom scheme of complete envelopment **AS2** translates into formula scheme $\mathbf{AS2}^t \Leftrightarrow \exists Y \in P \forall y \in P(y \in Y \Leftrightarrow \exists X \in P(y \in X) \wedge \varphi^\tau(y))$, where Y is not a free variable of the formula $\varphi(y)$ and by φ^τ we denote the formula $M \vDash \varphi[s^\tau]$ in which s^τ stands for the corresponding change of the sequence s under the transfer of the quantor overformulas $\exists Y(\dots)$, $\forall y(\dots)$ and $\exists X(\dots)$ mentioned above .

The axiom of subset $\mathbf{A3'}$ translates into the formula $\mathbf{A3'}^t \Leftrightarrow \mathbf{A3'}^P = \forall X \in P \forall Y \in P(\exists E \in P(X \in E) \wedge Y \subset X \Rightarrow \exists F \in P(Y \in F))$.

The axiom of sets of subsets $\mathbf{A3''}$ translates into the formula $\mathbf{A3''}^t \Leftrightarrow \mathbf{A3''}^\tau = \forall X \in P(\exists E \in P(X \in E) \Rightarrow \exists F \in P(\mathcal{P}(X)^\tau \in F))$, where the set $Z \equiv \mathcal{P}(X)^\tau$ is determined from the formula $\exists Z \in P \forall z \in P(z \in Z \Leftrightarrow (\exists G \in P(z \in G) \wedge z \subset X))$.

The axiom of binary union **A4** translates into the formula $\mathbf{A4}^t \Leftrightarrow \mathbf{A4}^\tau = \forall X \in P \forall Y \in P(\exists E \in P(X \in E) \wedge \exists F \in P(Y \in F) \Rightarrow \exists G \in P((X \cup Y)^\tau \in G))$, where the set $Z \equiv (X \cup Y)^\tau$ is determined from the formula $\exists Z \in P \forall z \in P(z \in Z \Leftrightarrow (\exists H \in P(z \in H) \wedge (z \in X \vee z \in Y)))$.

The axiom of general union **A5** translates into the formula $\mathbf{A5}^t \Leftrightarrow \mathbf{A5}^\tau = \forall X \in P \forall Y \in P \forall Z \in P(\exists E \in P(X \in E) \wedge (Z \subset (X * Y)^\tau) \wedge \forall x \in P(x \in X \Rightarrow \exists F \in P(Z\langle x\rangle^\sigma \in F)) \Rightarrow \exists G \in P((rng\, Z)^\tau \in G))$, where:

— the class $Z_1 \equiv (X * Y)^\tau$ is determined from the formula $\exists Z_1 \in P \forall z \in P(z \in Z_1 \Leftrightarrow (\exists H \in P(z \in H) \wedge \exists x \in P \exists y \in P(x \in X \wedge y \in Y \wedge z = \langle x, y\rangle^*)))$;

— the class $Z_2 \equiv Z_2(x) \equiv Z\langle x\rangle^\sigma$ is determined from the formula $\exists Z_2 \in P \forall y \in P(y \in Z_2 \Leftrightarrow (\exists K \in P(y \in K) \wedge y \in Y \wedge \langle x, y\rangle^* \in Z))$;

— the class $Z_3 \equiv (rng\, Z)^\tau$ is determined from the formula $\exists Z_3 \in P \forall y \in P(y \in Z_3 \Leftrightarrow (\exists L \in P(y \in L) \wedge y \in Y \wedge \exists x \in P(x \in X \wedge \langle x, y\rangle^* \in Z))$;

— the class $Z_4 \equiv \langle x, y\rangle^*$ is determined from the formula $\exists Z_4 \in P \forall z \in P(z \in Z_4 \Leftrightarrow \exists M \in P(z \in M) \wedge (z = \{x\}^* \vee z = \{x, y\}^*))$;

— the class $Z_5 \equiv \{x, y\}^*$ is determined from the formula $\exists Z_5 \in P \forall z \in P(z \in Z_5 \Leftrightarrow \exists N \in P(z \in N) \wedge (z = x \vee z = y))$;

— the class $Z_6 \equiv \{x\}^*$ is determined from the formula $\exists Z_6 \in P \forall z \in P(z \in Z_6 \Leftrightarrow \exists Q \in P(z \in Q) \wedge z = x)$.

The regularity axiom **A6** translates into the formula $\mathbf{A6}^t \Leftrightarrow \mathbf{A6}^\tau \equiv \forall X \in P(X \neq \emptyset^t \Rightarrow \exists x \in P(x \in X \wedge (x \cap X)^\tau = \emptyset^t))$, where:

— the class $Z_1 \equiv \emptyset^t$ is determined from the formula $\exists Z_1 \in P \forall z \in P(z \in Z_1 \Leftrightarrow (\exists E \in P(z \in E) \wedge z \neq z))$;

— the class $Z_2 \equiv (x \cap X)^\tau$ is determined from the formula $\exists Z_2 \in P \forall z \in P(z \in Z_2 \Leftrightarrow (\exists F \in P(z \in F) \wedge z \in x \wedge z \in X))$.

The infinity axiom **A7** translates into the formula $\mathbf{A7}^t \Leftrightarrow \mathbf{A7}^\tau \equiv \exists X \in P(\exists E \in P(X \in E) \wedge \emptyset^t \in X \wedge \forall x \in P(x \in X \Rightarrow (x \cup \{x\})^\tau \in X))$, where:

— the class $Z_1 \equiv \emptyset^t$ is determined from the formula presented above;

— the class $Z_2 \equiv Z_2(x) \equiv (x \cup \{x\})^\tau$ is determined from the formula $\exists Z_2 \in P \forall z \in P(z \in Z_2 \Leftrightarrow (\exists F \in P(z \in F) \wedge (z \in x \vee z \in \{x\}^\sigma)))$;

— the class $Z_3 \equiv Z_3(x) \equiv \{x\}^\sigma$ is determined from the formula $\exists Z_3 \in P \forall z \in P(z \in Z_3 \Leftrightarrow (\exists G \in P(z \in G) \wedge z = x))$.

Since M is a model of the NBG theory, all the translations written above are deducible formulas in the ZF theory ZF.

Using the obtained transfers, we prove that $P = \mathcal{P}(U)$ for a certain set U.

In the NBG theory, let us consider the formula $\varphi(x) \equiv (x = x)$. Then **AS2** defines an implicit axiom of the NBG theory of the form $\exists Y \forall y(y \in Y \Leftrightarrow \exists X(y \in X) \wedge y = y)$. According to the translations made above, this implicit axiom translates into the formula equivalent to the formula $\Phi \equiv \exists Y \in P \forall y \in P(y \in Y \Leftrightarrow \exists X \in P(y \in X) \wedge y = y)$. Since this formula is deducible in ZF, it defines a certain element $U \in P$ in ZF.

Consider an arbitrary element $X \in P$. Let $y \in X$. Then the transitivity of P implies $y \in P$. Hence for y, the formula $\exists X \in P(y \in X \wedge y = y)$ is deduced. By the formula Φ, we have $y \in U$. Therefore, $X \subset U$, i.e., $X \in \mathcal{P}(U)$. Thus, we have deduced the embedding $P \subset \mathcal{P}(U)$.

Conversely, if and $X \in \mathcal{P}(U)$, then the quasi-transitivity of P implies $X \in P$. Therefore, $P = \mathcal{P}(U)$.

Let us prove that the set U is universal.

Let $y \in x \in U \in P$. The transitivity of P implies $x \in P = \mathcal{P}(U)$. Hence $y \in x \subset U$ implies $y \in U$. Therefore, the set U is transitive.

Let $y \subset x \in U \in P$. Then $x \in P = \mathcal{P}(U)$, and $y \subset x \subset U$ implies $y \in P$. By $\mathbf{A3'}^P$, we conclude that $y \in F$ for a certain $F \in P$. Therefore, $y \in F \subset U$ implies $y \in U$. Therefore, the set U is quasitransitive.

Let us verify that in $\mathbf{A3''}^\tau$, the following equality holds for $X \in E \in P$: $\mathcal{P}(X)^\tau = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is defined in Sec. 1.1. Let $z \in \mathcal{P}(X)$. Then by the quasi-transitivity of P, $z \subset X \in P$ implies $z \in P$. Further $z \subset X \in E \in P = \mathcal{P}(U)$ implies $z \subset X \in E \subset U$. By the quasi-transitivity of U, $z \subset X \in U$ implies $z \in U \in P$. Therefore, the formula defined by the set $Z \equiv \mathcal{P}(X)^\tau$, which was presented above, implies $z \in \mathcal{P}(X)^\tau$. Therefore, $\mathcal{P}(X) \subset \mathcal{P}(X)^\tau$. The mentioned formula also implies the inverse embedding.

Let $X, Y \in U \in P$. Then by $\mathbf{A3''}^\tau$, $X, Y \in P$ implies $\mathcal{P}(X) = \mathcal{P}(X)^\tau \in F$ for a certain $F \in P$. Therefore, $\mathcal{P}(X) \in F \subset U$ implies $\mathcal{P}(X) \in U$.

Let us verify that in $\mathbf{A4}^\tau$, the equality $(X \cup Y)^\tau = X \cup Y$ holds for $X \in E \in P$ and $Y \in F \in P$. Let $z \in X \cup Y$. Then $z \in X$ or $z \in Y$. The transitivity of P implies $X \in P$ and $Y \in P$, which, in turn, implies $z \in P$. Moreover, by the transitivity of U, $z \in X \in E \subset U$ or $z \in Y \in F \subset U$ implies $z \in U \in P$. Therefore, the formula presented above, which defines the set $Z \equiv (X \cup Y)^\tau$, implies $z \in (X \cup Y)^\tau$. Therefore, $X \cup Y \subset (X \cup Y)^\tau$. The mentioned formula implies the inverse embedding.

Let $X, Y \in U$. Then by $\mathbf{A4}^\tau$, $X, Y \in P$ implies $X \cup Y = (X \cup Y)^\tau \in G$ for a certain $G \in P$. Therefore, $X \cup Y \in G \subset U$ implies $X \cup Y \in U$.

Let $X \in U \in P$. By what was proved above, $\mathcal{P}(X) \in U$. Then by the quasi-transitivity of the set U, $\{X\} \subset \mathcal{P}(X) \in U$ implies $\{X\} \in U$.

Let $X, Y \in U$. Then by what was proved above, $\{X, Y\} = \{X\} \cup \{Y\} \in U$. This implies $\langle X, Y \rangle \in U$.

If $X, Y \in U$, then by the quasi-transitivity property of U, $X * Y \subset \mathcal{P}(\mathcal{P}(X \cup Y)) \in U$ implies $X * Y \in U$.

Before proving other universality properties, let us simplify the formula $\mathbf{A5}^\tau$ obtained under the transfer of the axiom of general union $\mathbf{A5}$.

Let $z \in \{x\}$. Then $z = x \in X \in P$ implies $z \in P$, and, therefore, $z \in Z_6$. Conversely, if $z \in Z_6 \in P$, then $z \in P$ and the formula for Z_6 imply $z = x \in \{x\}$. Hence $Z_6 = \{x\}$.

Let $z \in \{x, y\}$. Then $z = x \in X \in P$ or $z = y \in Y \in P$ implies $z \in Z_5$. Conversely, if $z \in Z_5$, then $z = x$ or $z = y$ implies $z \in \{x, y\}$. Hence $Z_5 = \{x, y\}$.

These equalities lead to the disappearance of the star in the formula for Z_4. Let $z \in \langle x, y \rangle$. Then $z = \{x\}$ or $z = \{x, y\}$. Since $x \in X \in P = \mathcal{P}(U)$, it follows that $x \in U$. Analogously, $y \in U$. By what was proved above, this implies $\{x\} \in U$ and $\{x, y\} \in U$. Therefore, $z \in U \in P$ implies $z \in Z_4$. Conversely, if $z \in Z_4 \in P$, then $z \in P$, and the formula for Z_4 implies $z = \{x\}$ or $z = \{x, y\}$, i.e., $z \in \langle x, y \rangle$. Hence $Z_4 = \langle x, y \rangle$.

This equality leads to the disappearance of the star in the formulas for Z_3, Z_2, and Z_1.

Using this conclusion, we verify that $Z_1 = X * Y$. Let $z \in Z_1 \in P$. The transitivity of P implies $z \in P$. Therefore, the formula for Z_1 implies $z = \langle x, y \rangle$ for certain $x \in X$ and $y \in Y$. Therefore, $z \in X * Y$. Conversely, let $z \in X * Y$. Then $z = \langle x, y \rangle$ for certain $x \in X \in P$ and $y \in Y \in P$. The transitivity of P implies $x, y \in P$. Since $x \in X \subset U$ and $y \in Y \subset U$, by what was proved above, we have $z = \langle x, y \rangle \in U \in P$ and $z \in P$. Therefore, the formula for Z_1 implies $z \in Z_1$. This proves the necessary equality.

Hence $Z \subset X * Y$.

Using this conclusion, let us verify that $Z_3 = rng\, Z$. Let $y \in Z_3 \in P$. The transitivity of P implies $y \in P$. Therefore, the formula for definition of Z_3 implies $y \in rng\, Z$. Conversely, let $y \in rng\, Z \subset Y \in P$. Then there exists $x \in X \in P$ such that $\langle x, y\rangle \in Z$. The transitivity of P implies $x, y \in P$. Therefore, the formula for Z_3 implies $y \in Z_3$. This proves the necessary equality.

Finally, let us verify that $Z_2 = Z\langle x\rangle$.

Let $y \in Z_2 \in P$. The transitivity of P implies $y \in P$. Therefore, the formula for Z_2 implies $y \in Y$ and $\langle x, y\rangle \in Z$.Therefore, $y \in Z\langle x\rangle$. Conversely, let $y \in Z\langle x\rangle \subset Y \in P$. Then $\langle x, y\rangle \in Z$. The transitivity of P implies $y \in P$. Therefore, the formula for Z_2 implies $y \in Z_2$. This proves the necessary equality.

All that was proved above imply that in the formula $\mathbf{A5}^\tau$, the indices τ and σ disappear.

Using this conclusion, we prove that $X \in U$ implies $\cup X \in U$. In ZF, let us consider the sets $Y \equiv \cup X$ and $Z \equiv \{z \in X * Y | \exists x \in X \exists y \in y(z = \langle x, y\rangle \wedge y \in x)\}$. If $y \in x \in X \in U$, then the transitivity of U implies $y \in U$. Therefore, $Y \subset U$ implies $Y \in P$. Let $z \in Z$, i.e., $z = \langle x, y\rangle$ for certain $x \in X$ and $y \in Y$ such that $y \in x$. Then $y \in x \in U$ implies $y \in U$. By what was proved above, $z = \langle x, y\rangle \in U$. Hence $Z \subset U$, i.e., $Z \in P$.

Let us verify that for any $x \in P$ such that $x \in X$, $Z\langle x\rangle = x$ holds. If $y \in Z\langle x\rangle$, then $\langle x, y\rangle \in Z$ implies $\langle x, y\rangle = \langle x', y'\rangle$ for certain $x' \in X$ and $y' \in Y$ such that $y' \in x'$. This implies $y = y' \in x' = x$. Conversely, if $y \in x \in X$, then $y \in Y$, and $\langle x, y\rangle \in Z$ implies $y \in Z\langle x\rangle$.

This implies $Z\langle x\rangle = x \in U \in P$ for any $x \in P$ such that $x \in X \in U \in P$. Since the formula $\mathbf{A5}^\tau$ is also deducible in ZF, it implies $Y = rng\, Z \in G$ for a certain $G \in P$. Therefore, $Y \in G \subset U$.

Let us verify that $X \in U$ and $f \in U^X$ imply $rng\, f \in U$. If $x \in X \in U$ and $y \in U$, by what was proved above, $x \in U$ implies $\langle x, y\rangle \in U$. Hence $f \subset X * U \subset U$ implies $f \in P$. Moreover, by what was proved above, $f(x) \in U$ implies $f\langle x\rangle = \{f(x)\} \in U \in P$ for any $x \in X$. Applying the formula $\mathbf{A}5^\tau$, we conclude that $rng\, f \in G$ for a certain $G \in P$. Therefore, $rng\, f \in U$.

Now let us simplify the formula $\mathbf{A7}^\tau$. We verify that $Z_1 = \emptyset_{ZF}$. Let $z \in P$. Assume that $z \in Z_1$. Then according to the formula for Z_1, we obtain $z \neq z$. But, according to the equality axiom, $z = z$. The obtained contradiction implies $z \notin Z_1$. Now let $z \notin P$. Since $Z_1 \subset P$, it follows that $z_1 \notin Z_1$. Thus, for any z, $z \notin Z_1$ holds. According to the axiom of empty set $\mathbf{A7}$ in the ZF theory, we conclude that $Z_1 = \emptyset_{ZF}$.

In simplifying the formula $\mathbf{A5}^\tau$, we have proved that the formula for $Z_3 \equiv \{x\}^\sigma$ implies the equality $Z_3 = \{x\}$.

Let $x \in X \in P$. By what was proved above, $x \in U$ implies $\{x\} \in U \in P$. The transitivity of P implies $\{x\} \in P$. In simplifying the formula $\mathbf{A4}^\tau$, we have proved that the equality $Z_2 = x \cup \{x\}$ is deduced from these properties.

Thus, the formula $\mathbf{A7}^\tau$ becomes $\exists X \in P(\exists E \in P(X \in E) \wedge \emptyset_{ZF} \in X \wedge \forall x \in P(x \in X \Rightarrow x \cup \{x\} \in X))$. Let $x \in X$, where $X \in E \in P$. The transitivity of P implies $x \in P$. Then the formula $x \cup \{x\} \in X$ is deduced from the formula $\mathbf{A7}^\tau$. By the deduction axiom, we deduce the formula $(x \in X \Rightarrow x \cup \{x\} \in X)$, and by the generalization rule, we deduce the formula $\forall x \in X(x \cup \{x\} \in X)$. Thus, from $\mathbf{A7}^\tau$, we deduce the formula $\exists X \in P(\exists E \in P(X \in E) \wedge \emptyset_{ZF} \in X \wedge \forall x \in X(x \cup \{x\} \in X))$, which almost coincides with the infinity axiom $\mathbf{A8}$ in the ZF theory and which says that there exists an inductive set $X \in E \in P$. Since ω is the minimal set among all inductive sets, it follows that $\omega \subset X \in U$. By the quasi-transitivity of the set U, this implies $\omega \in U$.

Therefore, we have proved that $1) \vdash 2)$.

$2) \vdash 1)$. Let $P = \mathcal{P}(U)$ for a certain universal set U. Consider the standard interpretation $M \equiv (P, I)$ of the NBG theory. In the above, we carried out the translation of axioms and axiom schemes of the NBG theory into sequences s under the interpretation M. Let us prove that they are deduced in NBG.

We verify that P is supertransitive. Let $x \in y \in P$. Then $x \in y \subset U$ implies $x \in U$. Since U is transitive, it follows that $x \subset U$ and hence $x \in P$. Therefore, P is transitive. Let $x \subset y \in P$. Then $x \subset y \subset U$ implies $x \in P$. Therefore, P is quasi-transitive.

Let $y, z \in P$, $y = z$, and let $X \in P$. Consider the formula $\varphi(y) \equiv (y \in X)$. By the scheme of replacing equals in ZF, from $y = z$, we deduce the formula $\varphi(z) = (z \in X)$. By the deduction theorem, the formula

$y \in X \Rightarrow z \in X$ is deduced. In a similar way, the formula $z \in X \Rightarrow y \in X$ is deduced. Thus, the formula $y \in X \Leftrightarrow z \in X$ and, therefore, the formula $X \in P \Rightarrow (y \in X \Leftrightarrow z \in X)$ are deduced. By the generalization rule, the formula $\psi \equiv \forall X \in P(y \in X \Leftrightarrow z \in X)$ is deduced. By the deduction theorem, the formula $y = z \Rightarrow \psi$ is deduced. By logical means, from this, we deduce the formula $\mathbf{A1}^t$.

According to **AS3** in ZF, for the formula $\varphi^\tau(y)$ and the set U, there exists a set Y such that $\forall y(y \in Y \Leftrightarrow y \in U \wedge \varphi^\tau(y))$. Let $y \in Y$. Then $y \in U \wedge \varphi^\tau(y)$. Since $U \in P$, it follows that $\exists X \in P(y \in X) \wedge \varphi^\tau(y)$. By the deduction theorem, we deduce the formula $y \in Y \Rightarrow \exists X \in P(y \in X) \wedge \varphi^\tau(y)$. Conversely, let $\exists X \in P(y \in X) \wedge \varphi^\tau(y)$. Then $y \in X \subset U$ implies $y \in U$. Therefore, $y \in U \wedge \varphi^\tau(y)$ implies $y \in Y$. By the deduction theorem, we deduce the formula $\exists X \in P(y \in X) \wedge \varphi^\tau(y) \Rightarrow y \in Y$. Thus, the formula $y \in Y \Leftrightarrow \exists X \in P(y \in X) \wedge \varphi^\tau(y)$ is deduced. From this, we deduce the formula $\forall y \in P(y \in Y \Leftrightarrow \exists X \in P(y \in X) \wedge \varphi^\tau(y))$. Since $Y \subset U \in P$, the quasi-transitivity of P implies $Y \in P$. Therefore, $\mathbf{AS2}^t$ is deduced in ZF.

Let $X, Y \in P$, $X \in E \in P$, and let $Y \subset X$. Then by the quasi-transitivity of U proved in Lemma 1 (Sec. 3.1), $Y \subset X \in E \subset U$ implies $Y \in U \in P$. This means that $\mathbf{A3'}^t$ is deduced in ZF.

We have deduced early that for $X \in E \in P$, the equality $\mathcal{P}(X)^\tau = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is defined in Sec. 1.1, holds. By Axiom **A5**, in ZF, there exists $\mathcal{P}(X)$. Since U is universal, $X \in E \subset U$ implies $\mathcal{P}(X) \in U \in P$. This means that $\mathbf{A3''}^t$ is deduced in ZF.

Let $X, Y \in P$, $X \in E \in P$, and let $Y \in F \in P$. We have deduced early that under these conditions, the equality $(X \cup Y)^\tau = X \cup Y$ holds. But, by the universality of U, $X \in U$ and $Y \in U$ imply $X \cup Y \in U \in P$. This means that $\mathbf{A4}^t$ is deduced in ZF.

Let $X, Y, Z \in P$, and let $X \in E \in P$. We have deduced early that under these conditions, the equality $(X * Y)^\tau = X * Y$ holds, and if $Z \subset X * Y$, then the equalities $Z\langle x\rangle^\sigma = Z\langle x\rangle$ $(rng\, Z)^\tau = rng\, Z$ hold. If $x \in X$ and $Z\langle x\rangle \in F \in P$, then $X \in U$, $x \in U$, and $Z\langle x\rangle \in U$. Since U is universal, it follows that $U * U \subset U$. Consider the set $f \equiv \{s \in U * U | \forall x \in X(s = \langle x, Z\langle x\rangle\rangle) \wedge \forall x(x \notin X \Rightarrow s = \langle x, \emptyset\rangle)\}$. Clearly, f is a function from X into U such that $f(x) = Z\langle x\rangle$. By the universality of U, we conclude that $S \equiv rng\, f \in U$, and hence $T \equiv \cup S \in U$. If $t \in T$, then $t \in s \in S$ implies $t \in Z\langle x\rangle$ for a certain $x \in X$. Therefore, $t \in rng\, Z$. Conversely, if $t \in rng\, Z$, then $\langle x, t\rangle \in Z$ for a certain $x \in dom\, Z \subset X$. Hence $t \in Z\langle x\rangle = f(x) \in S$. Therefore, $t \in T$. Thus, $rng\, Z = T \in U \in P$. This means that $\mathbf{A5}^t$ is deduced in ZF.

We have deduced earlier that $\emptyset^t = \emptyset_{ZF}$. Let $X \in P$, and let $X \neq \emptyset_{ZF}$. Let us verify that $Z_2 \equiv (x \cap X)^\tau$ for $x \in X$ in the formula $\mathbf{A6}^\tau$ coincides with $x \cap X$. Let $z \in x \cap X$. Then $z \in X \in P$ implies $z \in P$ by the transitivity of P. By the formula for Z_2, we obtain $z \in Z_2$. Conversely, let $z \in Z_2 \in P$. Then by the formula for Z_2, $z \in P$ implies $z \in x \cap X$. Therefore, $Z_2 = x \cap X$.

By the regularity axiom **A9** in ZF, there exists $x \in X$ such that $x \cap X = \emptyset_{ZF}$. It follows from $x \in X \in P$ that $x \in P$. This means that $\mathbf{A6}^t$ is deduced in ZF.

We have deduced early that $\emptyset^t = \emptyset_{ZF}$ in the formula $\mathbf{A7}^\tau$, and, if $x \in X \in E \in P$, then $(x \cup \{x\})^\tau = x \cup \{x\}$. Since U universal, it follows that $\omega \in U$. Hence $\omega \in P$. Since ω is an inductive set, $\emptyset_{ZF} \in X$ and $x \in X \Rightarrow x \cup \{x\} \in X$. From this, by logical means, we deduce formula $\mathbf{A7}^t$.

The axiom of choice **A8** in NBG transfers into the formula $\mathbf{A8}^t \Leftrightarrow \mathbf{A8}^\tau \equiv \forall X \in P(\exists E \in P(X \in E) \wedge X \neq \emptyset^t \Rightarrow \exists z \in P((z \leftrightharpoons \mathcal{P}(X) \setminus \{\emptyset\} \to X)^\tau \wedge \forall Y \in P(Y \in (\mathcal{P}(X) \setminus \{\emptyset\})^\sigma \Rightarrow \forall x \in P(x \in X \wedge \langle Y, x\rangle^\sigma \in z \Rightarrow x \in Y)$, where:

— the set $Z_1 \equiv Z_1(X) \equiv (\mathcal{P}(X) \setminus \{\emptyset\})^\sigma$ is determined from the formula $\exists Z_1 \in P \forall u \in P(u \in Z_1 \Leftrightarrow u \in \mathcal{P}(X)^\rho \wedge u \notin \{\emptyset\}^\rho)$;

— the set $Z_2 \equiv \langle Y, x\rangle^\sigma$ is determined from the formula $\exists Z_2 \in P \forall u \in P(u \in Z_2 \Leftrightarrow \exists F \in P(u \in F) \wedge (u = \{Y\}^\sigma \vee u = \{Y, x\}^\sigma))$;

— the set $Z_3 \equiv \{Y, x\}^\sigma$ is determined from the formula $\exists Z_3 \in P \forall u \in P(u \in Z_3 \Leftrightarrow \exists G \in P(u \in G) \wedge (u = Y \vee u = x))$;

— the set $Z_4 \equiv \{Y\}^\sigma$ is determined from the formula $\exists Z_4 \in P \forall u \in P(u \in Z_4 \Leftrightarrow \exists H \in P(u \in H) \wedge u = Y)$;

— the set $Z_5 \equiv \{\emptyset\}^\rho$ is determined from the formula $\exists Z_5 \in P\, \forall z \in P\, (z \in Z_5 \Leftrightarrow (\exists K \in P(z \in K) \wedge z = \emptyset^t))$;

— $\varphi^\tau \equiv (z \leftrightharpoons \mathcal{P}(X) \setminus \{\emptyset\} \to X)^\tau$ stands for the formula $M \vDash \varphi[s^\tau]$, where s^τ stands for the corresponding change of the sequence s under the transfer of the quantifier overformulas $\forall X(\dots)$ and $\exists z(\dots)$ mentioned above.

In the above, we have established that $\emptyset^t = \emptyset_{ZF}$. Since $\emptyset_{ZF} \in \omega \in U \in P$, these conditions imply $Z_5 = \{\emptyset_{ZF}\}$ as was proved above.

Fix the conditions $X \in P$, $X \in E \in P$, and $X \neq \emptyset^t = \emptyset_{ZF}$. We have proved above that this implies $\mathcal{P}(X)^\rho = \mathcal{P}(X)$.

Let us verify that $Z_1 = \mathcal{P}(X) \setminus \{\emptyset_{ZF}\} \equiv Z$. Let $u \in Z_1 \in P$. Since $X \in E \subset U$ and U is universal, $\mathcal{P}(X) \in U$. Now the quasi-transitivity of U implies $Z \in U$. Since $u \in P$, the formula for Z_1 implies $u \in Z$. Therefore, $Z_1 \subset Z$. Conversely, let $u \in Z \in U \in P$. The transitivity of P implies $u \in P$. Now the formula for Z_1 implies $u \in Z$. Therefore, $Z \subset Z_1$, which proves the required equality. This leads to the replacement of Z_1 by Z in the formula $\mathbf{A8}^\tau$.

Consider the formula $\varphi \equiv (z \leftrightharpoons Z \to X)$. It is the conjunction of the following three formulas: $\varphi_1 \equiv (z \subset Z * X)$, $\varphi_2 \equiv (dom\, z = Z)$, and $\varphi_3 \equiv (\forall x(x \in Z \Rightarrow \forall y(y \in X \Rightarrow \forall y'(y' \in X \Rightarrow (\langle x, y\rangle \in z \wedge \langle x, y'\rangle \in z \Rightarrow y = y')))))$. Therefore, $\varphi^\tau = \varphi_1^\tau \wedge \varphi_2^\tau \wedge \varphi_3^\tau$. Since $\varphi_1 = (\forall u(u \in z \Rightarrow \exists x \exists y(x \in Z \wedge y \in X \wedge u = \langle x, y\rangle)))$, it follows that $\varphi_1^\tau \Leftrightarrow (\forall u \in P(u \in z \Rightarrow \exists x \in P \exists y \in P(x \in Z \wedge y \in X \wedge u = \langle x, y\rangle^\sigma)))$. Analogously, $\varphi_2 = (\forall x(x \in Z \Rightarrow \exists y(y \in X \wedge \langle x, y\rangle \in z)))$ implies $\varphi_2^\tau \Leftrightarrow (\forall x \in P(x \in Z \Rightarrow \exists y \in P(y \in X \wedge \langle x, y\rangle^\sigma \in z)))$.

Finally, $\varphi_3^\tau \Leftrightarrow (\forall x \in P(x \in Z \Rightarrow \forall y \in P(y \in X \Rightarrow \forall y' \in P(y' \in X \Rightarrow (\langle x, y\rangle^\sigma \in z \wedge \langle x, y'\rangle^\sigma \in z \Rightarrow y = y')))))$. This implies that the formulas $\varphi_1^\tau, \varphi_2^\tau$ and φ_3^τ differ from the formulas φ_1, φ_2 and φ_3 only by the quantifier prefixes $\forall \dots \in P$ and $\exists \dots \in P$, respectively.

By the axiom of choice, **A10** in ZF, for X, there exists z such that $\chi \equiv (z \leftrightharpoons Z \to X) \wedge \forall Y(Y \in Z \Rightarrow \forall x(x \in X \wedge \langle Y, x\rangle \in z \Rightarrow x \in Y))$.

Therefore, the formula $\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ and, therefore, the formulas φ_1, φ_2 and φ_3 are deduced.

Let $u \in P$, and let $u \in z$. Then from the formula φ_1, we deduce that there exist $x \in Z$ and $y \in X$ such that $u = \langle x, y\rangle$. Since $x \in Z \in U \in P$ and $y \in X \in P$, by the transitivity property, we have $x, y \in P$. This means that under these conditions $u \in P$ and $u \in z$, the formula $\exists x \in P \exists y \in P(x \in Z \wedge y \in X \wedge u = \langle x, y\rangle)$ is deduced. Applying the deduction theorem twice and the deduction rule, we deduce the formula φ_1^τ.

Let $x \in P$ and $x \in Z$. Then from the formula φ_2 we deduce that for x, there exists $y \in X$ such that $\langle x, y\rangle \in z$. From $y \in X \in P$, it follows that $y \in P$. This means that under these conditions $x \in P$ and $x \in Z$, the formula $\exists y \in P(y \in X \wedge \langle x, y\rangle \in z)$ is deduced. As in the previous paragraph, we deduce from this the formula φ_2^τ.

Let $x \in P$, $x \in Z$, $y \in P$, $y \in X$, $y' \in P$, $y' \in X$, $\langle x, y\rangle \in z$, and let $\langle x, y'\rangle \in z$. Then from the formula φ_3, we deduce that $y = y'$. Alternatively applying the deduction theorem and the deduction rules several times, we deduce the formula φ_3^τ.

Thus, the formula φ^τ is deduced.

Let us verify that $Z_4 = \{Y\}$ under the conditions $X \in E \in P$, $Y \in P$, and $Y \in Z$. Let $u \in \{Y\}$, i.e., $u = Y \in P$. Since $u = Y \in Z \in U \in P$, $u \in U \in P$ by the transitivity. Then from the formula for Z_4, it follows that $u \in Z_4$. Conversely, if $u \in Z_4 \in P$, then $u \in P$ and the formula for Z_4 imply $u = Y \in \{Y\}$. This yields the necessary equality.

Let us verify that $Z_3 = \{Y, x\}$ under the conditions $X \in E \in P$, $x \in X$, $Y \in P$, and $Y \in Z$. Let $u \in \{Y, x\}$. Then $u = Y \in Z \in U \in P$ or $u = x \in X \in E \in P$ implies $u \in P$, and, therefore, $u \in Z_3$. Conversely, if $u \in Z_3 \in P$, then $u \in P$ and the formula for Z_3 imply $u = Y \vee u = x$, i.e., $u \in \{Y, x\}$. This yields the necessary equality.

Finally, let us verify that $Z_2 = \langle Y, x\rangle$ under the previous conditions. Let $u \in \langle Y, x\rangle$, i.e., $u = \{Y\}$ or $u = \{Y, x\}$. The previous equalities lead to the disappearance of the index σ in the formula for Z_2. Since $Y \in Z \in U$, it follows that $Y \in U$. Moreover, $x \in X \in E \in P$ implies $x \in X \in P$, i.e., $x \in X \subset U$. Now the universality of U implies $u = \{Y\} \in U$ or $u = \{Y, x\} \in U$. Therefore, $u \in U \in P$ and $u \in P$ imply

$u \in Z_2$. Conversely, if $u \in Z_2 \in P$, then $u \in P$ and the formula for Z_2 imply $u = \{Y\}$ or $u = \{Y, x\}$, i.e., $u \in \langle Y, x\rangle$. This yields the necessary equality.

Since $Z \in U \in P$ and $X \in E \in P$, i,e., $X \in E \subset U$, from the universality of U, we obtain $z \subset Z * X \in U$. By Lemma 1 (Sec. 3.1), $z \in U \in P$ and, therefore, $z \in P$.

Therefore, from the axiom of choice in ZF is deduced for fixed conditions of existence an object $z \in P$ satisfying the formula χ from which the formula $\xi \equiv (\varphi^\tau \wedge \forall Y \in P(Y \in Z \Rightarrow \forall x \in P(x \in X \wedge \langle Y, x\rangle \in z \Rightarrow x \in Y)))$ is deduced. Thus, from the fixed conditions, we deduce the formula $\exists z \in P\xi$. Applying alternatively the deduction theorem and the generalization rule several times, as a result, we deduce the formula $\mathbf{A8}^t$.

Therefore, M is a model of the NBG theory. □

Now we can prove the Zermelo–Shepherdson theorem (see [28] and [23]).

Theorem 2. *In the ZF theory, the following assertions are equivalent for a set P:*

(1) *P is a supertransitive standard model for the NBG theory*;
(2) *$P = V_{\varkappa+1} = \mathcal{P}(V_\varkappa)$ for a certain inaccessible cardinal number $\varkappa$.*

Proof. 1) $\vdash$ 2). By Theorem 1, $P = \mathcal{P}(U)$ for a certain universal set U. By Theorem 1 (Sec. 3.2), $U = V_\varkappa$ for a certain inaccessible cardinal number $\varkappa$. By Corollary 2 of Lemma 4 (Sec. 2.2) , $P = \mathcal{P}(V_\varkappa) = V_{\varkappa+1}$.

2) $\vdash$ 1). By Corollary 2 of Lemma 4 (Sec. 2.2) $V_{\varkappa+1} = \mathcal{P}(V_\varkappa)$. By Theorem 2 (Sec. 3.2), the set $V_\varkappa$ is universal. Now the assertion follows from Theorem 1. □

The standard models of the ZF and NBG set theories of the form $(V_\alpha, =, \in)$ are said to be *natural* (see [27] and [21]). The Zermelo–Shepherdson theorem yields a complete description of all natural models of the NBG set theory. The complete description of all natural models of the ZF set theory was given by the authors (see [1]) and is further presented in Sec. 7.

6. Tarski Sets and Galactic Sets. Theorem on the Characterization of Natural Models of the NBG Theory

6.1. Tarski sets and their properties. A set U is called a *Tarski set* in ZF if it has the following properties: (see [26] and [17], IX, § 5):

(1) $x \in U \Rightarrow x \subset U$;
(2) $x \in U \Rightarrow \mathcal{P}(x) \in U$;
(3) $((x \subset U) \wedge \forall f(f \in U^x \Rightarrow rng\, f \neq U)) \Rightarrow x \in U$.

To the ZF theory, Tarski added the *Tarski axiom* AT according to which *each set is an element of a certain Tarski set.* In [26], it was proved that Axiom AT is equivalent to the inaccessibility axiom AI (see also [17], IX, § 1, Theorem 6, and § 5, Theorem 1).

Lemma 1. *The following properties are equivalent for any sets U and X:*

(3) $(x \subset U) \wedge \forall f(f \in U^x \Rightarrow rng\, f \neq U) \Rightarrow x \in U$;
(3') $(x \subset U) \wedge (|x| < |U|) \Rightarrow x \in U$.

Proof. $(3') \vdash (3)$. Let $x \subset U$ and $\forall f(f \in U^x \Rightarrow rng\, f \neq U)$. Clearly, $|x| \le |U|$. Assume that $|x| = |U|$. Then there exists a bijection $f : x \leftrightarrows U$, which contradicts the condition. Therefore, $|x| < |U|$. By Property $(3')$, $x \in U$.

$(3) \vdash (3')$. Let $f \in U^x$. Then $|rng\, f| \le |x| < |U|$ implies $rng\, f \neq U$. By Property (3), $x \in U$. □

Let us deduce other properties of Tarski sets from these properties.

Lemma 2. *If U is a Tarski set and $x \in U$, then $|x| \in |U|$.*

Proof. By Properties (1) and (2), $x \in U$ implies $\mathcal{P}(x) \in U$ and $\mathcal{P}(x) \subset U$. By the Cantor theorem, $|x| < |\mathcal{P}(x)| \le |U|$. □

Lemma 3. *If U is a Tarski set, then $x \in U \wedge y \subset x \Rightarrow y \in U$.*

Proof. If $x \in U$, then by Property (2), $\mathcal{P}(x) \in U$, and, by Property (1), $\mathcal{P}(x) \subset U$. Since $y \in \mathcal{P}(x)$, it follows that $y \in U$, which is what was required. □

Lemma 4. *If U is a Tarski set, then $x \in U \wedge (f \in U^x) \Rightarrow rng\, f \in U$.*

Proof. If $x \in U$, then $|x| < |U|$ by Lemma 2. Since $f \in U^x$, it follows that $rng\, f \subset U$ and $|rng\, f| \leq |x| < |U|$. By Property (3′), Lemma 1 implies $rng\, f \in U$. □

Lemma 5. *If U is a Tarski set, then $|U| \subset U$.*

Proof. Consider the class $\mathbf{C} \equiv \{x | x \in \mathbf{On} \wedge x \notin U\}$. This class is nonempty, since otherwise the class **On** is a set. Therefore, it contains a minimal element $\varkappa$. Since $\forall \alpha \in \varkappa(\alpha \in U)$, it follows that $\varkappa \subset U$. Hence $|\varkappa| \leq |U|$. Assume that $|\varkappa| < |U|$. Then by Lemma 1, $\varkappa \in U$, which is not so. Therefore, $|U| = |\varkappa| \leq \varkappa$, i.e., $|U| \subset \varkappa \subset U$. □

Lemma 6. *If U is a Tarski set, then $|U| \notin U$.*

Proof. Assume that $\varkappa \equiv |U| \in U$. Then by Property (2), $\mathcal{P}(\varkappa) \in U$. By Lemma 2, $\alpha \equiv |\mathcal{P}(\varkappa)| \in |U|$. Taking Lemma 5 into account, we conclude that $\alpha \in U$, and, by Property (1), it follows that $\alpha \subset U$. By the Cantor theorem, $\alpha > |U|$. But since $\alpha \subset U$, it follows that $\alpha \leq |U|$. We deduce from the obtained contradiction that $|U| \notin U$. □

Lemma 7. *If U is a Tarski set, then $|\mathcal{P}(\alpha)| \in |U|$ holds for any ordinal number $\alpha \in |U|$.*

Proof. Since $\alpha \in |U|$ and $|U| \subset U$ by Lemma 5, it follows that $\alpha \in U$. Then by Property (2) of a Tarski set, $\mathcal{P}(\alpha) \in U$. By Lemma 2, $|\mathcal{P}(\alpha)| \in |U|$. □

Lemma 8. *If U is a nonempty Tarski set, then $\emptyset \in U$ and $|U| \geq 5$.*

Proof. Since $\forall x(\emptyset \subset x)$, it follows that $x_0 \equiv \emptyset \subset U$. Since $|\emptyset| = 0 < |U|$, $x_0 \in U$ by Lemma 1. By Property (2), $x_1 \equiv \{x_0\} = \mathcal{P}(\emptyset) \in U$. It follows from $x_0 \neq x_1$ that $|U| \geq 2$. It follows from the proved properties that $x_2 \equiv \{x_1\} \subset U$ and $|x_2| = 1 < |U|$. Therefore, $x_2 \in U$ by property (3′) from Lemma 1. Hence $x_3 \equiv \{x_2\} \subset U$, and $|x_3| = 1 < |U|$ implies $x_3 \in U$ once again. Analogously, $x_4 \equiv \{x_3\} \in U$. Since all x_i are different for $i \in 5$, it follows that $|U| \geq 5$. □

Lemma 9. *If U is a Tarski set, then $x, y \in U \Rightarrow \{x\},\ \{x, y\},\ \langle x, y\rangle \in U$.*

Proof. By Lemma 8, $|U| \geq 5$. Therefore, $\{x\}, \{x, y\} \subset U$, and, by Property (3′), $|\{x\}| = 1 \leq |\{x, y\}| \leq 2 < |U|$ implies $\{x\}, \{x, y\} \in U$. Whence $\langle x, y\rangle \equiv \{\{x\}, \{x, y\}\} \in U$. □

Lemma 10. *If U is a Tarski set, then $|x| \geq |U| \Rightarrow x \notin U$.*

Proof. Assume the contrary, i.e., for a certain x, $|x| \geq |U| \wedge x \in U$ holds. By Property (2), $y \equiv \mathcal{P}(x) \in U$. Since $|\mathcal{P}(x)| > |x|$, it follows that $|y| > |U|$. By Property (1), $y \subset U$. But $|y| \leq |U|$ in this case. The obtained contradiction implies $x \notin U$. □

Lemma 11. *If U is a Tarski set, then $x, y \in U \Rightarrow x \cup y \in U$.*

Proof. Since $\forall z(z \in x \vee z \in y \Rightarrow z \in U)$, it follows that $x \cup y \subset U$. Since $x, y \in U$, by Lemma 10, we have $\alpha \equiv |x| < |U| \equiv \varkappa$ and $\beta \equiv |y| < \varkappa$. We need to prove that $|x \cup y| < \varkappa$. First, let us consider the case where $\alpha \leq 2$ and $\beta \leq 2$. Then, obviously, $|x \cup y| \leq 4 < |U|$ by Lemma 8. Therefore, $x \cup y \in U$ by Property (3′) from Lemma 1.

In what follows, we will assume that $\alpha \geq \beta > 2$. Consider the sets $P \equiv \{0\} \times x$, $Q \equiv \{1\} \times y$, $S \equiv x \cup y$, and $T \equiv P \cup Q$. Define the mapping $u : T \to S$, $u(0, a) \equiv a$ for any $(0, a) \in P$ and $u(1, b) \equiv b$ for any $(1, b) \in Q$. Since the mapping u is surjective, it follows that $|S| \leq |T|$.

Clearly, there exist bijective functions $g : P \leftrightarrows \alpha$ and $h : Q \leftrightarrows \beta \subset \alpha$. Define the function $f : T \to \mathcal{P}(\alpha)$ setting $f(p) \equiv \{g(p)\}$ for any $p \in P$ and $f(q) \equiv \alpha \setminus \{h(q)\}$ for any $q \in Q$. Since $P \cap Q = \emptyset$, this definition is correct. The function f is injective. Indeed, the function f is injective on P and Q. Let $p \in P$, $q \in Q$, and let $f(p) = f(q)$. Then $\{g(p)\} = \alpha \setminus \{h(q)\}$ implies $\alpha = \{g(p)\} \cup \{h(q)\} = \{g(p), h(q)\} \leq 2$, which contradicts our assumption. Therefore, $f(p) \neq f(q)$.

By the injectivity of the function f, Lemma 7 implies $|S| \leq |T| \leq |\mathcal{P}(\alpha)| < \varkappa$. Therefore, $S \in U$ by Property $(3')$ of Lemma 1. □

Corollary 1. *If U is a Tarski set, then $\omega \subset U$.*

Corollary 2. *If U is a Tarski set, then $|U| \geq \omega$.*

Corollary 3. *If U is a tarski set, then $x, y \in U \Rightarrow x * y \in U$.*

Proof. By Lemma 11 and Property (2), $B \equiv \mathcal{P}(\mathcal{P}(x \cup y)) \in U$. By Lemma 3, $A \equiv x * y \subset B$ implies $A \in U$. □

Lemma 12. *If U is a Tarski set, then $\alpha < |U| \Rightarrow |\alpha * \alpha| < |U|$ holds for any ordinal number α.*

Proof. We first consider the case $|\alpha| \leq 2$. Then by Lemma 8, $|\alpha * \alpha| \leq |2 * 2| = 4 < |U|$. In what follows, we will assume that $|\alpha| > 2$.

Since $\alpha < |U| \equiv \varkappa$, $|\mathcal{P}(\alpha)| < \varkappa$ by Lemma 7. The set $X \equiv \alpha * \alpha$ consists of ordered pairs $\langle \beta, \gamma \rangle$ such that $\beta, \gamma \in \alpha$. Divide the set X into three disjoint subsets $X_1 \equiv \{\langle \beta, \gamma \rangle | \beta < \gamma < \alpha\}$, $X_2 \equiv \{\langle \beta, \beta \rangle | \beta < \alpha\}$, and $X_3 \equiv \{\langle \beta, \gamma \rangle | \gamma < \beta < \alpha\}$. Obviously, $X_1 \cup X_2 \cup X_3 = X$. Construct the function $f : X \to \mathcal{P}(\alpha)$ as follows: if $x_1 = \langle \beta, \gamma \rangle \in X_1$, then $f(x_1) \equiv \{\beta, \gamma\} \in \mathcal{P}(\alpha)$; if $x_2 \equiv \langle \beta, \beta \rangle \in X_2$, then $f(x_2) \equiv \{\beta\} \in \mathcal{P}(\alpha)$; if $x_3 = \langle \beta, \gamma \rangle \in X_3$, then $f(x_3) \equiv \alpha \setminus \{\beta, \gamma\} \in \mathcal{P}(\alpha)$. The function f is injective on X_1, X_2, and X_3. If $f(x_1) = f(x_2)$, then $\{\beta, \gamma\} = \{\beta\}$ implies $\gamma = \beta < \gamma$, which is impossible. If $f(x_1) = f(x_3)$, then $\{\beta, \gamma\} = \alpha \setminus \{\beta, \gamma\}$, which is impossible, since $\alpha \neq \emptyset$. Finally, if $f(x_2) = f(x_3)$, then $\{\beta\} = \alpha \setminus \{\beta, \gamma\}$ implies $\alpha = \{\beta\} \cup \{\beta, \gamma\} = \{\beta, \gamma\}$, and, therefore, $|\alpha| \leq 2$, which contradicts our assumption. It follows from the obtained contradiction that the function f is injective. Therefore, $|X| \leq |\mathcal{P}(\alpha)| < \varkappa$. □

The following theorem and its Corollary 1 were proved by Tarski in [23] (see also [13], IX, § 5, Theorem 1). Here, we give another proof.

Theorem 1. *If U is a Tarski set, then $\varkappa \equiv |U|$ is a regular cardinal number.*

Proof. Assume that the cardinal number $\varkappa$ is not regular. Then $\alpha' \equiv cf(\varkappa) < \varkappa$ and, by Lemma 5, $\alpha' \in U$. By definition, there exists a function $\varphi : \alpha' \to \varkappa$ such that $\cup rng\, \varphi = \varkappa$. Denote $rng\, \varphi$ by A and consider the cardinal number $\alpha \equiv |A| \leq \alpha' < \varkappa$. By Lemma 5, $A \subset U$ and $\alpha \in U$. Construct the function $g : A \to \mathcal{P}(\varkappa)$ as follows. Consider an arbitrary ordinal number $\beta \in A$ and the set $A_\beta \equiv \{\gamma \in A | \gamma < \beta\}$. Define $\beta' \equiv \sup A_\beta = \cup A_\beta$ for $A_\beta \neq \emptyset$ (see Lemma 2 (Sec. 1.2)) and $\beta' \equiv 0$ for $A_\beta = \emptyset$. Consider the set $C_\beta \equiv \{\gamma \in \varkappa | \beta' \leq \gamma < \beta\}$ and put $g(\beta) \equiv C_\beta$. Let us show that for $\beta_1 \neq \beta_2$, $g(\beta_1) \cap g(\beta_2) = \emptyset$ holds. Indeed, let $\beta_1 < \beta_2$. Then $\beta_1 \in A_{\beta_2}$, and hence $\beta_1 \leq {\beta_2}'$. If $x \in g(\beta_1) \wedge x \in g(\beta_2)$, then $x \in C_{\beta_1} \wedge x \in C_{\beta_2}$, whence $x \in \varkappa \wedge x < \beta_1 \wedge {\beta_2}' \leq x$, which is impossible. We conclude from the obtained contradiction that $g(\beta_1) \cap g(\beta_2) = \emptyset$. Now let us show that $B \equiv \cup\left(g(\beta) | \beta \in A \right) = \varkappa$. It is obvious from the definition of the sets $g(\beta)$ that $B \subset \varkappa$. Now assume that $x \in \varkappa$. Since $\cup A = \varkappa$, there exists $\beta \in A$ such that $x \in \beta$. Therefore, the set $D \equiv \{\gamma \in A | x \in \gamma\}$ is nonempty, and hence it has a minimal element λ. By the definition of the set D, we have $\forall \gamma \in A(\gamma < \lambda \Rightarrow \gamma \leq x)$, which implies $x \geq \lambda'$. , $\lambda' \leq x < \lambda$, i.e., $x \in g(\lambda)$. Therefore, $B = \varkappa$.

Since U is a Tarski set and $\varkappa$ is its cardinality, there exists a one-to-one function $f : \varkappa \leftrightarrows U$. Since $\varkappa = \cup\left(g(\beta) | \beta \in A \right)$ and the sets $g(\beta)$ are pairwise disjoint, it follows that $U = \cup\left(f[g(\beta)] | \beta \in A \right)$. Denote the set $f[g(\beta)]$ by $4U_\beta$. Fix $\beta \in A$. It follows from $C_\beta \subset \beta$ that $|U_\beta| = |C_\beta| \leq |\beta|$.

Consider the (possibly, empty) set $F_\beta \equiv \{q \in U_\beta | \, |q| = \alpha\}$. By what was proved above, $|F_\beta| \leq |U_\beta| \leq |\beta| \leq \beta$. Therefore, $| \cup F_\beta| = | \cup \left(q | q \in F_\beta \right) | \leq | \cup \left(q * \{q\} | q \in F_\beta \right) | \equiv \sum(|q| \, | q \in F_\beta) = \alpha |F_\beta| \leq \alpha |\beta| = \sum(\alpha_q | q \in \beta) \equiv | \cup \left(\alpha * \{q\} | q \in \beta \right) |$, where $\alpha_q \equiv \alpha$ for any $q \in \beta$. Since $\cup\left(\alpha * \{q\} | q \in \beta \right) \subset \alpha * \beta \subset$

$\max(\alpha,\beta) * \max(\alpha,\beta)$ and $\max(\alpha,\beta) < \varkappa$, by Lemma 12, it follows that $|\cup F_\beta| < \varkappa$. Therefore, $\cup F_\beta \in U$ and $\mathcal{P}(\cup F_\beta) \in U$.

It follows from $F_\beta \subset U_\beta \subset U$ and the transitivity of U that $\cup F_\beta \subset U$. Therefore, by the inequality proved above, we conclude that $V_\beta \equiv U \setminus \cup F_\beta \neq \emptyset$ for any $\beta \in A$. Assume that $\mathcal{P}(\cup F_\beta) \in \cup F_\beta$. Since $\cup F_\beta \in \mathcal{P}(\cup F_\beta)$, we obtain an infinitely decreasing sequence $\mathcal{P}(\cup F_\beta) \ni \cup F_\beta \ni \mathcal{P}(\cup F_\beta) \ni \cup F_\beta \ni \ldots$. By the regularity axiom, this is impossible. Hence $\mathcal{P}(\cup F_\beta) \in V_\beta$. Define the function $h : A \to U$ setting $h(\beta) \equiv \mathcal{P}(\cup F_\beta)$. Consider the function $h' = h \circ \varphi : \alpha' \to U$. By Lemma 4, $M \equiv rng\, h = rng\, h' \in U$.

Obviously, $|M| \leq \alpha$. The transitivity of U implies $\alpha \subset U$. If α is an infinite number, then the following inequalities hold for the set $M' \equiv M \cup \alpha \subset U$: $\alpha \leq |M'| \leq |(M * \{0\}) \cup (\alpha * \{1\})| \equiv |M| + \alpha = \alpha$, which implies $|M'| = \alpha$. By Lemma 11, $\alpha \in U$ and $M \in U$ imply $M' \in U$. If α is a finite (i.e., natural) number, then by Corollary 2 of Lemma 11, the set $U \setminus M$ is infinite. Therefore, there exists an injective mapping $v : \omega \to U \setminus M$. Consider the natural number $n \equiv \alpha - |M|$ and the finite set $N \equiv v[n] \subset U \setminus M$. In this case, $|M'| = \alpha$ for the set $M' \equiv M \cup N$. By Corollary 1 of Lemma 11, $n \in \omega \subset U$. Therefore, by Lemma 5, $N = rng\,(u|n) \in U$. By Lemma 11, $M \in U$ and $N \in U$ imply $M' \in U$.

Since we have proved in the above that $U = \cup\big(U_\beta | \beta \in A\big)$, it follows that $M' \in U_\beta$ for a certain $\beta \in A$. Moreover, $|M'| = \alpha$. Therefore, $M' \in F_\beta$. If $x \in M' \in F_\beta$, then $x \in \cup F_\beta$, i.e., $M' \subset \cup F_\beta$. It follows from $h(\beta) \in V_\beta = U \setminus \cup F_\beta$ that $h(\beta) \notin M'$. However, $h(\beta) \in M \subset M'$ by definition. The obtained contradiction implies that the cardinal $\varkappa$ is regular. □

Corollary. *If U is a Tarski set and $\varkappa \equiv |U| > \omega$, then $\varkappa$ is an inaccessible cardinal number.*

Proof. By Theorem 1, $\varkappa$ is a regular cardinal number. By Lemma 7, $|\mathcal{P}(\alpha)| \in \varkappa$ holds for any $\alpha < \varkappa$. By the condition, $\varkappa > \omega$. Therefore, $\varkappa$ is an inaccessible cardinal number. □

Theorem 2. *If U is a Tarski set, then $x \in U \Rightarrow \cup x \in U$.*

Proof. Consider the numbers $\alpha \equiv |x|$ and $\varkappa \equiv |U|$ and a certain bijection $u : \alpha \leftrightarrows x$. By Lemma 2, $\alpha \in \varkappa$. The transitivity of U implies $\cup x \subset U$. Therefore, $|\cup x| \leq \varkappa$.

Assume that $|\cup x| = \varkappa$. Then there exists a bijection $f : \cup x \leftrightarrows \varkappa$. Fix an element $a \in \alpha$. Then $u(a) \in x$ implies $u(a) \subset \cup x$. Therefore, we can consider the injective mapping $g_a \equiv f|u(a) : u(a) \rightarrowtail \varkappa$. Consider the number $\beta_a \equiv |u(a)|$, a certain bijection $v_a : \beta_a \leftrightarrows u(a)$, and the injective mapping $h_a \equiv g_a \circ v_a : \beta \rightarrowtail \varkappa$. Assume that $\cup rng\, h_a = \varkappa$. Then by Theorem 1, $\beta_a \geq \varkappa$. However, by Lemma 2, $u(a) \in x \subset U$ implies $\beta_a \equiv |u(a)| < \varkappa$. This contradiction implies $\sup rng\, h_a = \cup rng\, h_a < \varkappa$.

Therefore, we can define the function $\eta : \alpha \to \varkappa$ setting $4\eta(a) \equiv \sup rng\, h_a$. Consider the set $Z \equiv rng\, \eta \subset \varkappa$. By what was proved, $z \leq \varkappa$ for any $z \in Z$. Let π be an order number such that $z \leq \pi$ for any $z \in Z$. Take any element $q \in \varkappa$ and consider $p \equiv f^{-1}(q) \in \cup x$. Then $p \in y \in x$ for a certain $y \in x$. Consider the element $a \equiv u^{-1}(y) \in \alpha$. Since $p \in y = u(a)$, it follows that $q = f(p) \in f[u(a)] = rng\, q_a = rng\, h_a$. Hence $q \leq \sup rng\, h_a \equiv \eta(a) \leq \pi$. This means that $\sup \varkappa \leq \pi$. Since $\varkappa$ is a limit number, $\varkappa = \sup \varkappa \leq \pi$ by Lemma 3. This implies $\varkappa = \sup Z = \cup Z = \cup rng\, \eta$. Now Theorem 1 implies $\alpha \geq \varkappa$, which contradicts the inequality $\alpha < \varkappa$.

Therefore, $|\cup x| < \varkappa$. By Property $(3')$ of Lemma 1, we now conclude that $\cup x \in U$. □

6.2. Galactic sets and their connection with Tarski sets. Consider a certain set x. Any finite sequence $(x_i | i \in n+1)$ such that $x_0 = x$ and $x_{i+1} \in x_i$ for any $i \in n$ is called a *sequence of subelements of the set x (of length n)*.

A set U is said to be *dominant* if the following conditions are equivalent for any set x:

(1) $x \in U$;

(2) all elements of all chains of subelements of the set x are of cardinality less than $|U|$.

Lemma 1. *Any dominant set is transitive.*

Proof. Let a set U be dominant, $x \in U$, and let $y \in x$. Any chain of subelements of the set y is a subchain of a certain chain of subelements of the set x, and, therefore, all its members are of cardinality less than $|U|$. Therefore, $y \in U$. □

Lemma 2. *Any dominant set U has Property $(3')$ of a Tarski set, i.e., $x \subset U \wedge |x| < |U| \Rightarrow x \in U$.*

Proof. Let $x \subset U$, and let $|x| < |U|$. Consider an arbitrary chain of subelements $(x_0, x_1, \ldots, x_n)$ of the set x. Since $x_1 \in x_0 = x \subset U$, by induction, we deduce from $x_1 \in U$ and the transitivity of the set U proved in Lemma 1 that $x_i \in U$ for any $i = 1, \ldots, n$. Therefore, $|x_i| < |U|$ for any $i = 1, \ldots, n$. Moreover, $|x_0| = |x| < |U|$ by condition. Therefore, $x \in U$. □

Proposition 1. *Any Tarski set is dominant.*

Proof. Let $(x_i | i \in n+1)$ be a chain of subelements of a set x, and let $x \in U$. By induction, we deduce from the transitivity that $x_i \in U$ for any $i \in n+1$. By Lemma 2 (Sec. 6.1), $|x_i| < |U|$.

Denote by $\mathbf{C}$ the class consisting of sets satisfying the Condition (2) from the definition of dominance. Let us show that $\mathbf{C} \subset U$. Consider the class $\mathbf{D} \equiv \{x | (x \in U \wedge x \in \mathbf{C}) \vee x \notin \mathbf{C}\}$ and show that it satisfies the $\in$-induction principle. Consider a certain $y \subset \mathbf{D}$. Then $(z \in U \wedge z \in \mathbf{C}) \vee z \notin \mathbf{C}$ holds for all $z \in y$. If $z \notin \mathbf{C}$ for a certain $z \in y$, then $y \notin \mathbf{C}$, and, therefore, $y \in \mathbf{D}$. Now let $\forall z \in y (z \in U \wedge z \in \mathbf{C})$. Consider $\alpha \equiv |y|$ and $\varkappa \equiv |U|$. If $\alpha \geq \varkappa$, then $y \notin \mathbf{C}$, which implies $y \in \mathbf{D}$. Let $\alpha < \varkappa$. In this case, $y \in U$ by Lemma 1 (Sec. 6.1). Let us show that $y \in \mathbf{C}$. Indeed, consider any chain of subelements $(y_i | i \in n+1)$ of the set y. Then the sequence $(y_i | i \in (n+1) \setminus 1)$ is a chain of subelements of the set $y_1 \in y = y_0$. Since $y_1 \in \mathbf{C}$ by assumption, it follows that all elements of the sequence $(y_i | i \in (n+1) \setminus 1)$ are of cardinality less than $\varkappa$. Therefore, $y \in \mathbf{C}$. Hence $y \in U \wedge y \in \mathbf{C}$, and, therefore, $y \in \mathbf{D}$. We conclude from this that the class $\mathbf{D}$ satisfies the $\in$-induction principle. Therefore, $\mathbf{D} = \mathbf{V}$. Thus, $\forall x ((x \in U \wedge x \in \mathbf{C}) \vee x \notin \mathbf{C}$, i.e., $\mathbf{C} \subset U$. □

Lemma 3. *For each cardinal number α, there can exist no more than one Tarski set of cardinality α.*

Proof. Assume that there exist two Tarski sets U_1 and U_2 of the same cardinality α. Let $x \in U_1$. Then by Proposition 1, $|x| < \alpha = |U_1|$, and any chain of subelements of the set x consists of sets of cardinality less than $\alpha = |U_1|$, which implies that $|x| < |U_2|$ and any chain of subelements of the set x consists of sets of cardinality less than $|U_2|$. Therefore, $x \in U_2$ by the same proposition. Thus, $U_1 \subset U_2$. Analogously, $U_2 \subset U_1$, which implies $U_1 = U_2$. □

A set U is said to be *exponential* if $\forall x \in U (\mathcal{P}(x) \in U)$. A dominant and exponential set is said to be *galactic.*

Theorem 1. *The following assertions are equivalent for a set U:*

(1) *U is a Tarski set;*
(2) *U is galactic.*

Proof. (1) $\vdash$ (2). This deducibility follows from the exponentiality property of a Tarski set and Proposition 1.

(2) $\vdash$ (1). This deducibility follows from Lemmas 1 and 2 and Lemma 1 (Sec. 6.1). □

Let us show that under the assumption of the continuum hypothesis, there exists a dominant set that is not exponential.

Lemma 4. *If $|2^\omega| = \omega_1$, then there exists a dominant set of cardinality ω_1.*

Proof. If $|2^\omega| = \omega_1$, then to prove the existence of a dominant set of cardinality ω_1, it suffices to show that the set consisting of all sets whose chains of subelements consist of only countable sets is of cardinality ω_1.

Denote this set by X.

Since $\omega_1 \subset X$, it follows that $|X| \geq \omega_1$.

Now let us show that $|X| \leq \omega_1$, i.e., there exists an injective function from the set X into the set of infinite sequences consisting of zeros and units.

Any set $x \in X$ can be represented as a tree whose root is the set x itself, branches are chains of subelements, and leaves are last elements of these chains, i.e., sets containing no sets (empty sets). All branches of such a tree are of finite length, and, moreover, the number of these branches is countable. The number of stores of the tree is also countable, and on each store, there are countably many sets (nodes or leaves of the tree). Clearly, certain trees correspond to the same set in X (these are the trees obtained from each other by a renumbering of vertices), but only one set in X corresponds to each three X. We will consider not trees themselves, but their "isomorphism classes" .

Let us enumerate leaves of the tree in a certain way (this can be done, since their number is countable). To each such "numbered" tree, we put in correspondence the function $f \in \omega^{\omega\times\omega}$ as follows: $f(n,m)$ is the maximum natural k such that the nth and the mth leaves end branches of a certain set from the kth store (in the case $n = m$, we set $f(n,m) \equiv n$). Such a number k can always be defined, since first, any two leaves end the branches of the initial set x, i.e., $k \geq 1$, and, second, $k \leq \min(m,n)$. According to such a function $f \in \omega^{\omega\times\omega}$, the isomorphism class of a tree is uniquely reconstructed, and, therefore, $|X| \leq |\omega^{\omega\times\omega}|$.

Let us show that $|\omega^{\omega\times\omega}| = 2^\omega = \omega_1$. Since $|\omega \times \omega| = \omega$, it follows that $|\omega^{\omega\times\omega} = |\omega^\omega|$. The set ω^ω is the set of infinite sequences of natural numbers. Since $|\omega^\omega| \geq |2^\omega|$, it suffices to prove that $|\omega^\omega| \leq |2^\omega|$, i.e., to construct an injective mapping from the set of infinite sequences of natural numbers into the set of infinite sequences consisting of zeros and units. We do this as follows. Let $N \equiv (n_i \in \omega|i \in \omega)$ be an infinite sequence of natural numbers. To this sequence, we put in correspondence the following sequence $M \equiv \{m_j \in 2|j \in \omega\}$ of zeros and units: for each $i \in \omega$, for $j = \sum(n_k|k \in i) + i$, we set $m_j \equiv 0$, and for all other j, we set $m_j \equiv 1$. Foe example, if we have the sequence $1, 2, 3, 4, 5, \ldots$, then it is mapped into the sequence $1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0 \ldots$. Such a mapping is injective, and, therefore, $\omega^\omega| = |2^\omega|$; thus, $|X| \leq |2^\omega|$. Since $|X| \geq |2^\omega|$, by the Cantor theorem, $|X| = |2^\omega|$. Since $|2^\omega| = \omega_1$ by the assumption, it follows that $|X| = \omega_1$, and, therefore, X is dominant. □

The fact that such a set X is not exponential is obvious, since ω_1 is not an inaccessible cardinal number.

6.3. Characterization of Tarski sets. Characterization of natural models of the NBG theory.

Proposition 1. *Let U be a Tarski set, and let $|U| = \omega$. Then $U = V_\omega$.*

Proof. By Lemma 1 (Sec. 2.2), $\omega \subset V_\omega$ implies $\omega \leq |V_\omega|$. Since $V_\omega = \cup\big(V_n|n \in \omega\big)$ and $|V_n| < \omega$, it follows that $|V_\omega| \leq \omega$. Hence $|V_\omega| = \omega$. Let us show that V_ω is a Tarski set. By Lemma 4 (Sec. 2.2), the set V_ω is transitive, and, by Lemma 7 (Sec. 2.2), it is exponential. Let us show that V_ω satisfies Property (3′). Consider a certain set $x \subset V_\omega$ such that $|x| < |V_\omega| = \omega$. If $y \in x \subset V_\omega = \cup\big(V_n|n \in \omega\big)$, then $N(y) \equiv \{n \in \omega|y \in V_n\} \neq \emptyset$. Therefore, the set $N(y)$ has a minimal element $n(y) \in \omega$. Since the set x is finite, the set $M \equiv \{m \in \omega|\exists y \in x(m = n(y))\}$ has a maximum element n. Hence $x \subset V_n$ implies $x \in V_{n+1} \subset V_\omega$. Therefore, V_ω is a Tarski set. Since a Tarski set of cardinality ω is unique by Lemma 1 (Sec. 6.2), it follows that $U = V_\omega$. □

Theorem 1. *Let U be a Tarski set, and let $\varkappa \equiv |U| > \omega$. Then*:

(1) *U is a universal set*;
(2) *$U = V_\varkappa$ for an inaccessible cardinal number $\varkappa = \sup(\mathbf{On} \cap U)$.*

Proof. (1) Let us show that the set U has all the properties of a universal set.

Property (1) follows from Property (1) of a Tarski set.

The property $x \in U \Rightarrow \mathcal{P}(x) \in U$ follows from Property (2) of a Tarski set. The property $x \in U \Rightarrow \cup x \in U$ follows from Theorem 2 (Sec. 6.1);

The property $x, y \in U \Rightarrow x \cup y \in U$ follows from Lemma 11 (Sec. 6.1). The properties $x, y \in U \Rightarrow \{x, y\}, \langle x, y\rangle \in U$ follow from Lemma 9 (Sec. 6.1). The property $x, y \in U \Rightarrow x * y \in U$ follows from Corollary 3 of Lemma 1 (Sec. 6.1).

Property (4) follows from Lemma 4 (Sec 6.1).

Since $|U| > \omega$ by condition, it follows by Lemma 1 (Sec. 6.1) that $\omega \in U$.

Therefore, the set U is universal.

(2) By Theorem 1 (Sec. 3.2), $U = V_\varkappa$ for the inaccessible cardinal number $\varkappa = \sup(\mathbf{On} \cap U)$. □

Now we can prove the main theorem on the characterization of natural models of the NBG theory.

Theorem 2. *For a set U, the following assertions are equivalent in the ZF theory*:

(1) *U is an uncountable Tarski set*;
(2) *U is a universal set*;
(3) *U is an inaccessible cumulative set, i.e., $U = V_\varkappa$ for a certain inaccessible cardinal number $\varkappa$*;
(4) *U is a supertransetive standard model set for the ZF theory and U has the strong subtitution property*;
(5) *$\mathcal{P}(U)$ is a supertransitive standard model set for the NBG theory*;
(6) *U is an uncountable galactic set.*

Proof. The deducibility (1) $\vdash$ (3) follows from Theorem 1.

The deducibility (3) $\vdash$ (1) follows from Lemma 4 (Sec. 2.2), Lemma 7 (Sec. 2.2), and Lemma 5 (Sec. 2.3).

The equivalence of (2) and (3) follows from Theorem 2 (Sec. 3.2).

The equivalence of (2) and (4) follows from Proposition 1 (Sec. 5.1).

The equivalence of (1) and (6) follows from Theorem 1 (Sec. 6.2).

The equivalence of (2) and (5) follows from Theorem 1 (Sec. 5.2).

The deducibility (3) $\vdash$ (1) was proved by Tarski [26].

The equivalence of (3) and (4) was in fact proved by Zermelo [28] and Shepherdson [23] (see also [2], Theorem 1.3). All other assertions of the Theorem 2 belong to the authors (see also the announcement in [2]). □

Corollary. *In the ZF theory, the following axioms are equivalent*:

(1) *the Tarski axiom AT*;
(2) *the universality axiom AU*;
(3) *the inaccessibility axiom AI*;
(4) *the galacticity axiom AH according to which each set is an element of a certain galactic set.*

The equivalence of (1) and (3) in this corollary was proved by Tarski in [26] (see also [17], IX, § 5, Theorem 1). Here, we give another proof using the theorem characterization.

7. Characterization of Natural Models of the ZF Theory

7.1. Scheme-inaccessible cardinal numbers and scheme-inaccessible cumulative sets. If all free variables of a formula φ are among the variables $x_0, \dots, x_{m-1}, p_0, \dots, p_{n-1}$, then we will denote this by $\varphi(\vec{x}; \vec{p})$. When using the notation $\varphi(\vec{x}; \vec{p})$, the variables $p_0, \dots, p_{n-1}$ will be called *parameters*. Instead of $x_0 \in A \wedge \dots \wedge x_{m-1} \in A$, $\forall x_0 \in A \dots \forall x_{m-1} \in A$ and $\exists x_0 \in A \dots x_{m-1} \in A$, we will write $\vec{x} \in A$, $\forall \vec{x} \in A$ and $\exists \vec{x} \in A$, respectively.

For each transitive set A, any formula $\varphi(x, y; \vec{p})$ of the ZF theory defines the *scheme correspondence* $[\varphi(x, y; \vec{p})|A] \equiv \{z \in A * A | \exists x, y \in A(z = \langle x, y\rangle \wedge \varphi^A(x, y; \vec{p}))\} \subset A * A$ depending on the parameter $\vec{p}$.

An ordinal number $\varkappa$ is said to be *scheme-regular* if $\forall \vec{p} \in V_\varkappa \forall \alpha(\alpha \in \varkappa \wedge ([\varphi(x, y; \vec{p})|V_\varkappa] \leftrightharpoons \alpha \to \varkappa \Rightarrow \cup rng\, [\varphi(x, y; \vec{p})|V_\varkappa] \in \varkappa)$, where φ is a metavariable used for designation of any formula of the ZF theory.

An ordinal number $\varkappa > \omega$ is said to be (*strongly*) *scheme-inaccessible* if:

(1) $\forall \alpha(\alpha \in \varkappa \Rightarrow |\mathcal{P}(\alpha)| \in \varkappa)$;
(2) $\varkappa$ is scheme-regular.

Lemma 1. *Let an ordinal number $\varkappa$ satisfy the quasi-exponentiality condition $\forall \alpha(\alpha \in \varkappa \Rightarrow |\mathcal{P}(\alpha)| \subset \varkappa)$. Then $\varkappa$ is a cardinal number.*

Proof. Let α be an ordinal number such that $\alpha \le \varkappa$ and $\alpha \sim \varkappa$. Then $|\alpha| = |\varkappa|$. Assume that $\alpha < \varkappa$. By the condition, $|\mathcal{P}(\alpha)| \subset \varkappa$. Applying the Cantor theorem, we obtain $|\alpha| < |\mathcal{P}(\alpha)| \le |\varkappa|$, which contradicted the previous relation. Therefore, $\alpha = \varkappa$. □

Corollary. *A scheme-inaccessible ordinal number $\varkappa$ is a cardinal number.*

Proof. If $\alpha \in \varkappa$, then $|\mathcal{P}(\alpha)| \in \varkappa$ by Property (1). By the transitivity, $|\mathcal{P}(\alpha)| \subset \varkappa$. Hence $\varkappa$ satisfies the condition of Lemma 1. Therefore, $\varkappa$ is a cardinal number. □

The sets $V_\varkappa$ of scheme-inaccessible cardinal numbers $\varkappa$ will be called *scheme-inaccessible cumulative sets.*

Lemma 2. *For any scheme-inaccessible cardinal number $\varkappa$ and any ordinal number $\alpha \in \varkappa$, we have $|V_\alpha| < \varkappa$.*

Proof. Consider the set $C' \equiv \{x \in \varkappa | \, |V_x| < \varkappa\}$ and the classes $\mathbf{C}'' \equiv \mathbf{On} \setminus \varkappa$ and $\mathbf{C} \equiv C' \cup \mathbf{C}''$. Since $V_0 = \emptyset$, it follows that $|V_0| = 0 < \varkappa$. Therefore, $0 \in \mathbf{C}$.

Let $\alpha \in \mathbf{C}$. If $\alpha \ge \varkappa$, then $\alpha+1 \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha < \varkappa$. Then $\alpha \in C'$. If $\alpha+1 = \varkappa$, then $\alpha+1 \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha + 1 < \varkappa$. Since $V_\alpha \sim |V_\alpha|$, it follows that $\mathcal{P}(V_\alpha) \sim \mathcal{P}(|V_\alpha|)$. Hence $|\mathcal{P}(V_\alpha)| = |\mathcal{P}(|V_\alpha|)|$. By Corollary 2 of Lemma 4 (Sec. 2.2), $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = |\mathcal{P}(|V_\alpha|)|$. Since $|V_\alpha| < \varkappa$ and the ordinal number $\varkappa$ is scheme-inaccessible, $|\mathcal{P}(|V_\alpha|)| < \varkappa$. Hence $|V_{\alpha+1}| < \varkappa$. Therefore, $\alpha + 1 \in C' \subset \mathbf{C}$.

Let α be a limit ordinal number, and let $\alpha \subset \mathbf{C}$. If $\alpha \cap \mathbf{C}'' \neq \emptyset$, then there exists $\beta \in \alpha$ such that $\beta \ge \varkappa$. Hence $\alpha > \beta \ge \varkappa$ implies $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha \cap \mathbf{C}'' = \emptyset$, i.e., $\alpha \subset C' \subset \varkappa$. If $\alpha = \varkappa$, then $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha < \varkappa$. By $\alpha \subset C'$, $|V_\beta| < \varkappa$ holds for any $\beta \in \alpha$. Therefore, $\sup\{|V_\beta| \, |\beta \in \alpha\} \le \varkappa$.

Consider the formula $\varphi(x, y) \equiv (x \in \alpha \Rightarrow y = |V_x|) \wedge (x \notin \alpha \Rightarrow y = \emptyset)$.

Let us show that under our conditions (i.e., for $x \in \alpha \in V_\varkappa$ and $y \in V_\varkappa$), the equivalence $(y = |V_x|)^{V_\varkappa} \Leftrightarrow y = |V_x|$ holds.

The formula $(y = |V_x|)^{V_\varkappa}$ is rewritten as $(Cn(y))^{V_\varkappa} \wedge \exists f \in V_\varkappa (f \leftrightharpoons y \leftrightarrows V_x)^{V_\varkappa}$. The formula $Cn(y)^{V_\varkappa}$ can be rewritten as $On(y)^{V_\varkappa} \wedge \forall \alpha \in V_\varkappa (On(\alpha)^{V_\varkappa} \wedge (\alpha \subset y)^{V_\varkappa} \wedge \exists h \in V_\varkappa (h \leftrightharpoons \alpha \leftrightarrows y)^{V_\varkappa} \Rightarrow \alpha = y)$. Consider the formula $On(y)^{V_\varkappa}$ under the condition $y \in V_\varkappa$. This formula has the form

$On(y)^{V_\varkappa} \equiv \forall x \in V_\varkappa (x \in y \Rightarrow (x \subset y)^{V_\varkappa}) \wedge \forall x, x', x'' \in V_\varkappa (x \in y \wedge x' \in y \wedge x'' \in y \wedge x \in x' \wedge x' \in x'' \Rightarrow x \in x'') \wedge \forall x, x' \in V_\varkappa (x \in y \wedge x' \in y \Rightarrow x \in x' \vee x = x' \vee x' \in x) \wedge \forall T \in V_\varkappa ((\emptyset \neq T \subset y)^{V_\varkappa} \Rightarrow \exists x \in V_\varkappa (x \in T \wedge \forall x' \in V_\varkappa (x' \in T \Rightarrow x \in x')))$.

Note that under the condition $y \in V_\varkappa$, the formula $(x \subseteq y)^{V_\varkappa} \equiv \forall z \in V_\varkappa (z \in x \Rightarrow z \in y)$ is equivalent to the formula $x \subseteq y$ by the supertransitivity of the set $V_\varkappa$. Analogously, $(\emptyset \neq T \subseteq y)^{V_\varkappa} \Leftrightarrow \emptyset \neq T \subseteq y$. The formula $\forall x \in V_\varkappa (x \in y \Rightarrow x \subseteq y)$ is equivalent to the formula $\forall x (x \in y \Rightarrow x \subseteq y)$, since $x \in y$ implies $x \in V_\varkappa$. The formula $\forall x, x', x'' \in V_\varkappa (x \in y \wedge x' \in y \wedge x'' \in y \wedge x \in x' \wedge x' \in x'' \Rightarrow x \in x'')$ is equivalent to the formula $\forall x, x', x'' (x \in y \wedge x' \in y \wedge x'' \in y \wedge x \in x' \wedge x' \in x'' \Rightarrow x \in x'')$, since $x, x', x'' \in y$ implies $x, x', x'' \in V_\varkappa$. The formula $\forall x, x' \in V_\varkappa (x \in y \wedge x' \in y \Rightarrow x \in x' \vee x = x' \vee x' \in x)$ is equivalent to the formula $\forall x, x' (x \in y \wedge x' \in y \Rightarrow x \in x' \vee x = x' \vee x' \in x)$, since $x, x' \in y$ implies $x, x' \in V_\varkappa$. Finally, the formula $\forall T \in V_\varkappa (\emptyset \neq T \subseteq y \Rightarrow \exists x \in V_\varkappa (x \in T \wedge \forall x' \in V_\varkappa (x' \in T \Rightarrow x \in x')))$ is equivalent to the formula $\forall T (\emptyset \neq T \subset y \Rightarrow \exists x (x \in T \wedge \forall x' (x' \in T \Rightarrow x \in x')))$, since $T \subset y$ implies $T \in V_\varkappa$, and $x, x' \in T$ implies $x, x' \in V_\varkappa$. Therefore, $On(y)^{V_\varkappa} \Leftrightarrow On(y)$.

It is seen from this that the formula $Cn(y)^{V_\varkappa}$ can be rewritten as $On(y) \wedge \forall \alpha \in V_\varkappa (On(\alpha) \wedge (\alpha \subset y)^{V_\varkappa} \wedge \exists h \in V_\varkappa (h \leftrightharpoons \alpha \leftrightarrows y)^{V_\varkappa} \Rightarrow \alpha = y)$. Since $y \in V_\varkappa$, it follows that $(\alpha \subset y)^{V_\varkappa} \Leftrightarrow \alpha \subset y$ and, automatically, $\alpha \in V_\varkappa$.

The formula $\exists h \in V_\varkappa (h \leftrightharpoons \alpha \leftrightarrows y)^{V_\varkappa}$ is rewritten as

$\exists h \in V_\varkappa (\forall x \in V_\varkappa (x \in h \Leftrightarrow \exists z \in V_\varkappa \exists z' \in V_\varkappa (z \in \alpha \wedge \exists z' \in y \wedge x = \langle z, z' \rangle)) \wedge \forall z \in V_\varkappa (z \in \alpha \Rightarrow \exists z' \in V_\varkappa (z' \in y \wedge \langle z, z' \rangle \in h)) \wedge \forall z' \in V_\varkappa (z' \in y \Rightarrow \exists z \in V_\varkappa (z \in \alpha \wedge \langle z, z' \rangle \in h)) \wedge \forall z, z', z'' \in V_\varkappa (z \in \alpha \wedge z', z'' \in y \wedge \langle z, z' \rangle \in h \wedge \langle z, z'' \rangle \in h \Rightarrow z' = z'') \wedge \forall z, z', z'' \in V_\varkappa (z, z' \in \alpha \wedge z'' \in y \wedge \langle z, z'' \rangle \in h \wedge \langle z', z'' \rangle \in h \Rightarrow z = z'))$.

The formula $\forall x \in V_\varkappa (x \in h \Leftrightarrow \exists z, z' \in V_\varkappa (z \in \alpha \wedge z' \in y \wedge x = \langle z, z' \rangle))$ is equivalent to the formula $\forall x (x \in h \Leftrightarrow \exists z \in \alpha \exists z' \in y (x = \langle z, z' \rangle))$, since $z \in \alpha$ implies $z \in V_\varkappa$, $z' \in y$ implies $z' \in V_\varkappa$, and $x = \langle z, z' \rangle$ implies $x \in V_\varkappa$.

The formula $\forall z \in V_\varkappa(z \in \alpha \Rightarrow \exists z' \in V_\varkappa(z' \in y \wedge \langle z, z'\rangle \in h))$ is equivalent to the formula $\forall z \in \alpha \exists z' \in y(\langle z, z'\rangle \in h)$, since $z \in \alpha$ implies $z \in V_\varkappa$ and $z' \in y$ implies $z' \in V_\varkappa$. Analogously, the formula $\forall z' \in V_\varkappa(z' \in y \Rightarrow \exists z \in V_\varkappa(z \in \alpha \wedge \langle z, z'\rangle \in h))$ is equivalent to the formula $\forall z' \in y \exists z \in \alpha(\langle z, z'\rangle \in h)$.

The formula $\forall z, z', z'' \in V_\varkappa(z \in \alpha \wedge z', z'' \in y \wedge \langle z, z'\rangle \in h \wedge \langle z, z''\rangle \in h \Rightarrow z' = z'')$ is equivalent to the formula $\forall z \in \alpha \forall z', z'' \in y(\langle z, z''\rangle \in h \wedge \langle z, z''\rangle \in h \Rightarrow z' = z'')$, since $z \in \alpha$ and $z', z'' \in y$ imply $z, z', z'' \in V_\varkappa$. Analogously, $\forall z, z', z'' \in V_\varkappa(z, z' \in \alpha \wedge z'' \in y \wedge \langle z, z''\rangle \in h \wedge \langle z', z''\rangle \in h \Rightarrow z = z')$.

Therefore, the formula $Cn(y)^{V_\varkappa}$ is equivalent to the formula $On(y) \wedge \forall \alpha(On(\alpha) \wedge \alpha \subset y \wedge \exists h \in V_\varkappa(h \leftrightharpoons \alpha \leftrightarrows y) \Rightarrow \alpha = y)$. Since $h \leftrightharpoons \alpha \leftrightarrows y$, it follows that $h \subset \alpha * y$, and by Corollary 2 of Lemma 7 (Sec. 2.2), $\alpha, y \in V_\varkappa$ implies $\alpha * y \in V_\varkappa$ and, therefore, $h \in V_\varkappa$. We obtain from this that $Cn(y)^{V_\varkappa} \Leftrightarrow Cn(y)$.

We know that for $x < \varkappa$, $V_x \in V_\varkappa$ holds. Therefore, exactly in the same way as was done in the above, we prove that the formula $\exists f \in V_\varkappa(f \leftrightharpoons y \leftrightarrows V_x)^{V_\varkappa}$ is equivalent to the formula $\exists f(f \leftrightharpoons y \leftrightarrows V_x)$.

We conclude from all that was said that $(y = |V_x|)^{V_\varkappa} \Leftrightarrow (y = |V_x|)$.

Then $[\varphi|V_\varkappa] = \{z|\exists x \in V_\varkappa \exists y \in V_\varkappa(z = \langle x, y\rangle \wedge (x \in \alpha \Rightarrow y = |V_x|) \wedge (x \notin \alpha \Rightarrow y = \emptyset) \wedge \alpha \in V_\varkappa)\}$. If $y \in rng\,[\varphi|V_\varkappa]$, then $\exists x(\langle x, y\rangle \in [\varphi|V_\varkappa])$, i.e., $y \in V_\varkappa \wedge \exists x(x \in V_\varkappa \wedge ((x \in \alpha \wedge y = |V_x|) \vee (x \notin \alpha \wedge y = \emptyset)))$. Therefore, either $y = \emptyset$ or $y = V_x$ for a certain $x \in \alpha$. Conversely, if $y = V_x$ for a certain $x \in \alpha$, then $y \in rng\,[\varphi|V_\varkappa]$. Therefore, $rng\,[\varphi|V_\varkappa] = \{|V_\beta|\,|\beta \in \alpha\}$. By the corollary of Theorem 1, $\cup rng\, f = \cup\{|V_\beta|\,|\beta \in \alpha\} = \sup\{|V_\beta|\,|\beta \in \alpha\} = |V_\alpha|$. By the inequality proved above, we obtain $|V_\alpha| \leq \varkappa$. Assume that $|V_\alpha| = \varkappa$. Then by the scheme regularity of the number $\varkappa$, $\varkappa = \cup rng\,[\varphi|V_\varkappa]$ implies $\varkappa \leq \alpha$, which contradicts the initial inequality $\alpha < \varkappa$. Thus, $|V_\alpha| < \varkappa$. Therefore, $\alpha \in C' \subset \mathbf{C}$.

By the transfinite induction principle, $\mathbf{C} = \mathbf{On}$. Therefore, $C' = \varkappa$. □

Lemma 3. *If $\varkappa$ is a scheme-inaccessible cardinal number, then $\varkappa = |V_\varkappa|$.*

Proof. By Lemma 2, $\varkappa \subset V_\varkappa$. Hence $\varkappa = |\varkappa| \leq |V_\varkappa|$. By the corollary to Theorem 1 (Sec. 2.2), $|V_\varkappa| = \sup(|V_\beta|\,|\beta \in \varkappa)$. Since $|V_\beta| < \varkappa$ by Lemma 2, it follows that $|V_\varkappa| \leq \varkappa$. As a result, we obtain $\varkappa = |V_\varkappa|$. □

Lemma 4. *If $\varkappa$ is a scheme-inaccessible cardinal number, α is an ordinal number such that $\alpha < \varkappa$, and $\varphi(x, y; \vec{p})$ is a formula, then $\forall \vec{p} \in V_\varkappa([\varphi(x, y; \vec{p})|V_\varkappa \leftrightharpoons V_\alpha \to V_\varkappa \Rightarrow rng\,[\varphi(x, y; \vec{p})|V_\varkappa] \in V_\varkappa)$.*

Proof. Since $\varkappa$ is a limit ordinal number, it follows that $V_\varkappa = \cup\left(V_\delta|\delta \in \varkappa\right)$. For $x \in V_\alpha$, there exists $\delta \in \varkappa$ such that $[\varphi|V_\varkappa](x) \in V_\delta$. Therefore, the nonempty set $\{y \leq \delta|[\varphi|V_\varkappa](x) \in V_y\}$ has a minimal element z.

Consider a certain bijective mapping $h : |V_\alpha| \to V_\alpha$.

We have the formula $\forall x \in V_\alpha \exists v[\varphi|V_\varkappa](x) = v \wedge \varphi^{V_\varkappa}(x, v, \vec{p}) \wedge \vec{p} \in V_\varkappa$, since $v \in V_\varkappa$ by the condition. Consider the functional formula $\psi(u, z) \equiv (u \in |V_\alpha| \Rightarrow z = sm\{y \leq \delta|[\varphi|V_\varkappa](h(u)) \in V_y\}) \wedge (u \notin |V_\alpha| \Rightarrow z = \emptyset)$. In this case, $[\psi|V_\varkappa] = \{v|\exists u \in V_\varkappa \exists z \in V_\varkappa(v = \langle u, z\rangle \wedge \psi^{V_\varkappa}(u, z))$. Consider the formula $\psi^{V_\varkappa}(u, z)$ in more detail. It is equivalent to the formula $((u \in |V_\alpha|)^{V_\varkappa} \Rightarrow (z = sm\{y \leq \delta|[\varphi|V_\varkappa](h(u)) \in V_y\})^{V_\varkappa}) \wedge ((u \notin |V_\alpha|)^{V_\varkappa} \Rightarrow (z = \emptyset)^{V_\varkappa})$, which, in turn, is equivalent to the formula $(u \in |V_\alpha|^{V_\varkappa} \Rightarrow ((\forall y \leq \delta([\varphi|V_\varkappa](h(u)) \in V_y \Rightarrow z \subset y) \wedge ([\varphi|V_\varkappa](h(u)) \in V_z)^{V_\varkappa}) \wedge (u \notin |V_\alpha|^{V_\varkappa} \Rightarrow z = \emptyset)$, which is equivalent to $(u \in |V_\alpha| \Rightarrow ([\varphi|V_\varkappa](h(u)) \in V_z)^{V_\varkappa} \wedge \forall y \in V_\varkappa(y \leq \delta \wedge ([\varphi|V_\varkappa](h(u)) \in V_y)^{V_\varkappa} \Rightarrow z \subset y) \wedge (u \notin |V_\alpha| \Rightarrow z = \emptyset)$.

The formula $([\varphi|V_\varkappa](h(u)) \in V_z)^{V_\varkappa} \wedge \forall y \in V_\varkappa(y \leq \delta \wedge ([\varphi|V_\varkappa](h(u)) \in V_y)^{V_\varkappa} \Rightarrow z \subset y)$ is equivalent to the formula $([\varphi|V_\varkappa](h(u)) \in V_z)^{V_\varkappa} \wedge \forall y \leq \delta(([\varphi|V_\varkappa](h(u)) \in V_y)^{V_\varkappa} \Rightarrow z \subset y)$, since $y \leq \delta$ implies $y \in V_\varkappa$.

Consider the formula $([\varphi|V_\varkappa](h(u)) \in V_z)^{V_\varkappa}$ in more detail. It is equivalent to the formula $\exists w(w \in V_z \wedge \langle h(u), w\rangle \in [\varphi|V_\varkappa]))^{V_\varkappa}$, which is equivalent to $\exists w \in V_\varkappa(w \in V_z \wedge (\langle h(u), w\rangle \in \{a|\exists b \in V_\varkappa \exists c \in V_\varkappa(a = \langle b, c\rangle \wedge \varphi^{V_\varkappa}(b, c, \vec{p}) \wedge \vec{p} \in V_\varkappa)\})^{V_\varkappa})$, which means that $\exists w \in V_\varkappa(w \in V_z \wedge (u \in V_\varkappa \wedge w \in V_\varkappa \wedge \varphi^{V_\varkappa}(h(u), w, \vec{p}) \wedge \vec{p} \in V_\varkappa)^{V_\varkappa})$, which, in turn, means that $\exists w(w \in V_z \wedge \varphi^{V_\varkappa}(h(u), w, \vec{p}) \wedge \vec{p} \in V_\varkappa)$.

Therefore, the formula $[\psi|V_\varkappa]$ is equivalent to the formula $\{v|\exists u \in V_\varkappa \exists z \in V_\varkappa(v = \langle u, z\rangle \wedge (u \in |V_\alpha| \Rightarrow \exists w(w \in V_z \wedge \varphi^{V_\varkappa}(h(u), w, \vec{p}) \wedge \vec{p} \in V_\varkappa) \wedge \forall y \in \varkappa(\exists w(w \in V_y \wedge \varphi^{V_\varkappa}(h(u), w, \vec{p}) \wedge \vec{p} \in V_\varkappa) \Rightarrow z \subset y)) \wedge (u \notin |V_\alpha| \Rightarrow z = \emptyset)$. This formula is equivalent to the formula $\{v|\exists u \in V_\varkappa \exists z \in V_\varkappa(v = \langle u, z\rangle \wedge (u \in |V_\alpha| \Rightarrow z = sm\{y|\exists w(w \in V_y \wedge \varphi^{V_\varkappa}(h(u), w, \vec{p}) \wedge \vec{p} \in V_\varkappa)\}) \wedge (u \notin |V_\alpha| \Rightarrow z = \emptyset)\}$.

We easily deduce from this that $[\psi|V_\varkappa] \leftrightharpoons |V_\alpha| \to \varkappa$.

Consider the ordinal number $\gamma \equiv \cup rng\,[\psi|V_\varkappa] = \sup rng\,[\psi|V_\varkappa] \leq \varkappa$.

Assume that $\gamma = \varkappa$. Since the cardinal $\varkappa$ is quasi-regular, this is impossible. Therefore, $\gamma < \varkappa$.

Let $rng\,[\varphi|V_\varkappa] \notin V_\varkappa$. Then there exists $t \in rng\,[\varphi|V_\varkappa]$ such that $t \notin V_\gamma$. A certain $s \in dom\,[\varphi|V_\varkappa]$ such that $\langle s,t\rangle \in [\varphi|V_\varkappa$ and $s \in V_\alpha$ corresponds to the set t. Moreover, $h^{-1}(s) \in |V_\alpha|$.

Consider $\beta \equiv [\psi|V_\varkappa](h^{-1}(s))$. Since $h^{-1}(s) \in |V_\alpha|$, it follows that $\beta = sm\{y|\exists w(w \in V_y \wedge \varphi^{V_\varkappa}(s,w,\vec{p}) \wedge \vec{p} \in V_\varkappa)\}$, and since the formula φ is functional, it follows that $w = t$, i.e., $\beta = sm\{y|t \in V_y\}$. Since $t \notin V_\gamma$ by the condition, it follows that $\beta > \gamma$, which contradicts the condition $rng\,[\psi|V_\varkappa] \subset \gamma$.

Therefore, $rng\,[\varphi|V_\varkappa] \subset V_\gamma \in V_\varkappa$. □

Corollary. *If $\varkappa$ is a scheme-inaccessible cardinal number, $A \in V_\varkappa$, $\varphi(x,y;\vec{p})$ is a formula, and*

$$[\varphi(x,y;\vec{p})|V_\varkappa] \leftrightharpoons A \to V_\varkappa,$$

then $rng\,[\varphi(x,y;\vec{p})|V_\varkappa] \in V_\varkappa$.

Proof. Since $\varkappa$ is a limit number, it follows that $V_\varkappa = \cup\big(V_\alpha|\alpha \in \varkappa\big)$. Therefore, $A \in V_\alpha$ for a certain $\alpha \in \varkappa$. By Lemma 4 (Sec. 2.2), $A \subset V_\alpha$. Consider the formula $\psi(x,y;\vec{p},A,\alpha) \equiv (x \in A \wedge \varphi(x,y;\vec{p}) \vee (x \in V_\alpha \setminus A \wedge y = \emptyset)$. It follows from $x \in V_\alpha \setminus A \subset V_\alpha \in V_\varkappa$ and the supertransitivity of V_α that $x, V_\alpha \setminus A \in V_\varkappa$. Therefore, $(x \in V_\alpha \setminus A)^{V_\varkappa} \Leftrightarrow x \in V_\alpha \setminus A$. Hence $\psi^{V_\varkappa} \Leftrightarrow (x \in A \wedge \varphi^{V_\varkappa}) \vee (x \in V_\alpha \setminus A) \wedge y = \emptyset)$. Since $g \equiv [\psi|V_\varkappa] \leftrightharpoons V_\alpha \to V_\varkappa$, by Lemma 4, $rng\,g \in V_\varkappa$. It follows from $B \equiv rng\,[\varphi|V_\varkappa] \subset rng\,g \in V_\varkappa$ that $B \in V_\varkappa$. □

Lemma 5. *If $\varkappa$ is a scheme-inaccessible cordinal number and $A \in V_\varkappa$, then $\cup A \in V_\varkappa$.*

Proof. Since $\varkappa$ is a limit ordinal number, it follows that $V_\varkappa = \cup\big(V_\delta|\delta \in \varkappa\big)$. For $A \in V_\varkappa$, there exists $\delta \in \varkappa$ such that $A \in V_\delta$. Then for each $a \in A$, $a \in V_\delta$. This implies that for each $a \in A$, the nonempty set $\{y \leq \delta|a \in V_y\}$ has a minimal element z_a.

Consider a certain bijective mapping $h : |V_\delta| \to V_\delta$.

Consider the formula $\psi(u,z) \equiv (u \in |V_\delta| \wedge h(u) \in A \Rightarrow z = sm\{y \leq \delta|h(u) \in V_y\}) \wedge (u \notin |v_\delta| \vee h(u) \notin A \Rightarrow z = \emptyset)$. In this case, $[\psi|V_\varkappa] = \{v|\exists u \in V_\varkappa \exists z \in V_\varkappa(v = \langle u,z\rangle \wedge \psi^{V_\varkappa}(u,z))$. Consider the formula $\psi^{V_\varkappa}(u,z)$ in more detail. It is equivalent to the formula $((u \in |V_\delta|)^{V_\varkappa} \wedge (h(u) \in A)^{V_\varkappa} \Rightarrow (z = sm\{y \leq \delta|h(u) \in V_y\})^{V_\varkappa}) \wedge ((u \notin |V_\delta|)^{V_\varkappa} \vee (h(u) \notin A)^{V_\varkappa} \Rightarrow (z = \emptyset)^{V_\varkappa})$. Analogously to the previous lemma, this formula is equivalent to the formula $(u \in |V_\delta| \wedge h(u) \in A \Rightarrow ((\forall y \leq \delta(h(u) \in V_y) \Rightarrow z \subset y) \wedge (h(u) \in V_z))^{V_\varkappa}) \wedge (u \notin |V_\delta| \vee h(u) \notin A \Rightarrow z = \emptyset)$.

The formula $((\forall y \leq \delta(h(u) \in V_y) \Rightarrow z \subset y) \wedge (h(u) \in V_z))^{V_\varkappa}$ is equivalent to the formula $\forall y \in V_\varkappa(y \leq \delta \wedge h(u) \in V_y \Rightarrow z \subset y) \wedge (h(u) \in V_z)$, since $h, h(u), \delta, V_y, V_z \in V_\varkappa$. Since $y \leq \delta$ implies $y \in V_\varkappa$ in this case, this formula is equivalent to the formula $(\forall y \leq \delta(h(u) \in V_y) \Rightarrow x \subset y) \wedge (h(u) \in V_z)$, i.e., to the formula $z = sm\{y \leq \delta|h(u) \in V_y\}$.

Therefore, $[\psi|V_\varkappa] = \{v|\exists u \in V_\varkappa \exists z \in V_\varkappa(v = \langle u,z\rangle \wedge ((u \in |V_\delta| \wedge h(u) \in A \Rightarrow z = sm\{v \leq \delta|h(u) \in V_y\}) \wedge (u \notin |V_\delta| \vee h(u) \notin A \Rightarrow z = \emptyset))$.

We easily obtain from this that $[\psi|V_\varkappa] \leftrightharpoons |V_\delta| \to \varkappa$.

Consider the ordinal number $\gamma \equiv \cup rng\,[\psi|V_\varkappa] \in \varkappa$.

Let $\cup A \notin V_\varkappa$. Then there exists $t \in \cup A$ such that $t \notin V_\gamma$. Since $t \in \cup A$, there exists $a \in A$ such that $t \in a$, which implies $a \notin V_\gamma$. But if we consider $s \equiv [\psi|V_\varkappa](h^{-1}(a))$, then we obtain $s \leq \gamma \wedge a \in V_s$, which implies $a \in V_\gamma$. Therefore, $\cup A \in V_\varkappa$. □

Any formula $\sigma(x;\vec{u})$ of the ZF theory and any transitive set A define the *scheme set* $\langle\sigma(x;\vec{u})|A\rangle \equiv \{x \in A|\sigma^A(x;\vec{u})\}$, which depends on the parameter $\vec{u}$.

Lemma 6. *If $\varkappa$ is a scheme-inaccessible cardinal number and $\varphi(x,y;\vec{p})$ and $\sigma(x;\vec{u})$ are formulas, then $\forall\vec{p}\forall\vec{u} \in V_\varkappa\forall\varepsilon \in |V_\varkappa|(([\varphi(x,y,;\vec{p})|V_\varkappa] \leftrightharpoons \langle\sigma(x;\vec{u})|V_\varkappa\rangle \leftrightarrows \varepsilon \Rightarrow \langle\sigma(x;\vec{u})|V_\varkappa\rangle \in V_\varkappa)$.*

Proof. Denote $[\varphi|V_\varkappa]$, $rng[\varphi|V_\varkappa]$, and $\langle\sigma|V_\varkappa\rangle$ for given $\vec{p},\vec{u} \in V_\varkappa$ from the condition of the lemma by f, R, and S, respectively. Consider the formula $\rho(y;\vec{p},\vec{u}) \equiv \exists x(\sigma(x;\vec{u}) \wedge \varphi(x,y;\vec{p}))$. Then $\rho^{V_\varkappa} = \exists x \in V_\varkappa(\sigma^{V_\varkappa}(x;\vec{u}) \wedge \varphi^{V_\varkappa}(x,y;\vec{p}))$ implies $\langle\rho(y;\vec{p},\vec{u})|V_\varkappa\rangle \equiv \{y \in V_\varkappa|\exists x(x \in V_\varkappa \wedge \sigma^{V_\varkappa}(x;\vec{u}) \wedge \varphi^{V_\varkappa}(x,y;\vec{p}))\} = R$ for given $\vec{p},\vec{u} \in V_\varkappa$. Since $R \subset \varepsilon \in |V_\varkappa| = \varkappa \subset V_\varkappa$ by Lemma 3, it follows that $R \in V_\varkappa$.

Consider the formula $\psi(y,x;\vec{p},\vec{u}) \equiv \sigma(x;\vec{u}) \wedge \varphi(x,y;\vec{p})$. Then $\psi^{V_\varkappa} = \sigma^{V_\varkappa}(x;\vec{u}) \wedge \varphi^{V_\varkappa}(x,y;\vec{p})$ implies $g \equiv [\psi|V_\varkappa] \equiv \{t \in V_\varkappa * V_\varkappa | \exists y,x \in V_\varkappa(t = \langle y,x\rangle \wedge \sigma^{V_\varkappa}(x;\vec{u}) \wedge \varphi^{V_\varkappa}(x,y;\vec{p}))\} = f^{-1}$. Therefore, g is a bijective mapping from R onto S. Since $R \in V_\varkappa$, $S = rng\, g \in V_\varkappa$ by the corollary to Lemma 4. □

7.2. Scheme-universal sets and their connection with scheme-inaccessible cumulative sets. A set U is said to be *scheme-universal* if it has the following properties:

(1) $x \in U \Rightarrow x \subset U$ (*transitivity property*);
(2) $x \in U \Rightarrow \mathcal{P}(x), \cup x \in U$;
(3) $x, yU \Rightarrow x \cup y, \{x,y\}, \langle x,y\rangle, x * y \in U$;
(4) $\forall \vec{p} \in U \forall x(x \in U \wedge [\varphi(x,y;\vec{p})|U] \leftrightharpoons x \to U \Rightarrow rng\,[\varphi(x,y;\vec{p})|U] \in U$;
(5) $\omega \in U$.

The following two lemmas are proved analogously to the corresponding lemmas of Sec. 3.1.

Lemma 1. *If a set U is scheme-universal, then $x \in U \wedge y \subset x \Rightarrow y \in U$.*

This lemma shows that a scheme-universal set is quasi-transitive. This and the transitivity property imply that a quasi-universal set is supertransitive.

Lemma 2. *If a set U is scheme-universal, then $\emptyset \in U$ and $1 \in U$.*

The following theorem has a completely different proof than Lemma 4 (Sec. 3.1) analogous to it.

Theorem 1. *If U is a scheme-universal set, then $X \in U \Rightarrow |X| \in U$.*

Proof. If $X = \emptyset$, then $|X| = 0 \in U$. In what follows, we will assume that $X \neq \emptyset$. By the Zermelo principle, we can assume that X is completely ordered. Take a minimal element m of the set X. Consider the nonempty set $A \equiv \mathbf{On} \cap U$.

For all $x \in X$, by X_x we denote the interval $\{t \in X | t < x\}$. By Lemma 1, $X_x \subset X \in U$ implies $X_x \in U$.

If for X_x, there exists a mapping f such that $dom\, f = X_x$ and $rng\, f \in A$, then by Lemma 1, $f \subset X_x * rng\, f \in U$ implies $f \in U$.

Assume that for $x \in X$, there exist isotone bijective mappings f and g such that $dom\, f = dom\, g = X_x$ and $rng\, f, rng\, g \in \mathbf{On}$. If $x = m$, then $f = g = \emptyset$. If $x > m$, then consider the set $X' \equiv \{y \in X_x | f(y) \neq g(y)\}$. Assume that $X' \neq \emptyset$. Then X' contains a minimal element n. Consider the set $X_n \subset X_x$. Assume that $f(m) > 0$ and consider $z \in X_x$ such that $f(z) = 0$. Since f is isotone, it follows that $f(m) > f(z)$ implies $m > z$, which is impossible. Therefore, $f(m) = 0 = g(m)$ implies $m < n$, i.e., $m \in X_n$. Clearly, $f|X_n = g|X_n$. Since a bijective isotone mapping preserves all order boundaries, it follows that $f(n) = f(\sup X_n) = \sup f[X_n] = \sup g[X_n] = g(\sup X_n) = g(n)$, which contradicts the property $n \in X'$. The obtained contradiction implies $X' = \emptyset$, i.e., $f = g$.

Denote by $bij(f)$ and $isot(f)$ the formulas expressing the property of the mapping f to be bijective and isotone, respectively. Consider the formula $\varphi(x,a;X) \equiv (X \neq \emptyset \wedge x \in X \wedge \exists f(func(f) \wedge dom\,(f) = X_x \wedge rng\,(f) = a \wedge bij(f) \wedge isot(f) \wedge On(a))$. It follows from what was proved in the previous paragraph that for $x \in X$, there can exist a unique f and, therefore, a unique a, i.e., the formula $\varphi(x,a;X)$ is functional. Consider the function $H \equiv [\varphi|U] \equiv \{z \in U * U | \exists x,a \in U(z = \langle x,a\rangle \wedge \varphi^U(x,a;X))\} \subset U * U$ depending on the parameter $X \in U$.

Consider the formula $\varphi^U(x,a;X) = (X \neq \emptyset \wedge x \in X \wedge \exists f \in U(func^U(f) \wedge (dom\,(f) = X_x)^U \wedge (rng\,(f) = a)^U \wedge bij^U(f) \wedge isot^U(f)) \wedge On^U(a))$ for $x,a,X \in U$. It follows from the absoluteness of these formula and operations that $func^U(f) \Leftrightarrow func(f)$, $(dom\,(f) = X_x)^U \Leftrightarrow (dom\,(f) = X_x)$, $(rng\,(f) = a)^U \Leftrightarrow (rng\,(f) = a)$, $On^U(a) \Leftrightarrow On(a)$, $bij^U(f) \Leftrightarrow bij(f)$ and $isot^U(f) \Leftrightarrow isot(f)$ (see [13], 10). Hence $\varphi^U(x,a;X) \Leftrightarrow (X \neq 0 \wedge x \in X \wedge \exists f \in U(func(f) \wedge dom\, f = X_x \wedge rng(f) = a \wedge bij(f) \wedge isot(f)) \wedge On(a))$ for $a,x,X \in U$. Consider the set $Z \equiv dom\, H$. Since $Z \subset X \in U$ and U is scheme-transitive, it follows that $Z \in U$. Hence, by Property (4) from the definition of a quasitransitive set, it follows that $c \equiv rng\, H \in U$. Therefore, H is a function from Z onto c. Since the set $e \equiv \emptyset \in U$ is an isotone bijective mapping such that $dom\, e = X_m = \emptyset \in U$ and $rng\, e = \emptyset = 0 \in A$, it follows that $H \neq \emptyset$.

Let $\alpha \in \beta \in c$. Then $\beta = H(y)$ for certain $\beta \in \mathbf{On}$ and $y \in Z$ such that $\varphi^U(y, \beta; X)$. This means that $y \in X$ and there exists an isotone bijection $g \in U$ such that $dom\,(g) = X_y$ and $rng\,(g) = \beta$. Since β is an ordinal number, it follows that α is also an ordinal number and $\alpha \subset \beta$. Consider $x \equiv g^{-1}(\alpha) \in X_y$. If $t \in X_x \subset X_y$, then $g(t) < g(x) = \alpha$, i.e., $g(t) \in \alpha$. If $\gamma \in \alpha$, then $\gamma \in \beta$, and we can take an element $z \equiv g^{-1}(\gamma) \in X_y$. It follows from $g(z) = \gamma < \alpha = g(x)$ that $z < x$, i.e., $z \in X_x$. Moreover, $g(z) = \gamma$. This implies that the function $f \equiv g|X_x$ maps X_x onto α. Clearly, it is bijective and isotone. Since $f \subset g \in U$, the quasi-transitivity of U implies $f \in U$. Hence $\alpha = H(x) \in c$. This means that the set c is transitive.

Let $\alpha, \beta \in c$. Then $\alpha = H(x)$ and $\beta = H(y)$ for certain $\alpha, \beta \in \mathbf{On}$ and $x, y \in Z$ such that $\varphi^U(x, \alpha; X)$ and $\varphi^U(y, \beta; X)$. This means that $x, y \in X$ and there exist isotone bijections $f, g \in U$ such that $dom\,(f) = X_x$, $dom\,(g) = X_y$, $rng\,(f) = \alpha$, and $rng\,(g) = \beta$. Since α and β are ordinal numbers, it follows that either $\alpha \in \beta$, or $\beta \in \alpha$, or $\alpha = \beta$. Therefore, the set c is linearly ordered with respect to the binary relation $\in \cup =$.

Let $\emptyset \neq \alpha \subset c$. By the regularity axiom, there exists $r \in \alpha$ such that $r \cap \alpha = \emptyset$. Take any $s \in \alpha$ such that $s \in r$ or $s = r$. It follows from $r \cap \alpha = \emptyset$ that $s \notin r$. Therefore, $s = r$. This means that r is a minimal element in α. Therefore, c is completely ordered.

Therefore, we have proved that c is an ordinal number.

Let us verify that the function H is bijective and isotone. Let $x, y \in Z$, and let $x < y$. Then for ordinal numbers $a \equiv H(x)$ and $b \equiv H(y)$, there exist isotone bijections $f, g \in U$ such that $dom(f) = X_x$, $dom(g) = X_y$, $rng(f) = a$, and $rng(g) = b$. Consider the ordinal number $a' \equiv g(x) \in b$. If $t \in X_x$, then $t < x < y$ implies $g(t) < g(x) \equiv a'$, i.e., $g(t) \in a'$. If $\alpha \in a' \subset b$, then for the element $s \equiv g^{-1}(\alpha)$, $\alpha < a'$ implies $s < x$, i.e., $s \in X_x$ and $g(s) = \alpha$. Therefore, the function $f' \equiv g|X_x$ is an isotone bijection from X_x onto a'. It follows from the uniqueness proved early that $f' = f$. Therefore, $f \subset g$ implies $a \subset b$. Assume that $a = b$. Then $X_x = f^{-1}[a] = f'^{-1}[a] = g^{-1}[a] = g^{-1}[b] = X_y$, which contradicts the inequality $x < y$. Therefore, $a \in b$, i.e., $a < b$. Conversely, let $x, y \in Z$, and let $a < b$. Since $a \in b$, we can take the element $x'' \equiv g^{-1}(a) \in X_y$. If $t \in X_{x''}$, then $t < x'' < y$ implies $g(t) < g(x'') = a$, i.e., $g(t) \in a$. If $\alpha \in a \subset b$, then for the element $s \equiv g^{-1}(\alpha)$, $\alpha < a$ implies $s = g^{-1}(\alpha) < g^{-1}(a) = x''$, i.e., $s \in X_{x''}$ and $g(s) = \alpha$. Then the function $f'' \equiv g|X_{x''}$ is an isotone bijection from $X_{x''}$ onto a. Consider the isotone bijections $p \equiv f^{-1} : a \leftrightarrows X_x$ and $p'' \equiv f''^{-1} : a \leftrightarrows X_{x''}$. In the same way as above, we prove that $p = p''$. Therefore, $X_x = X_{x''}$ implies $x = x'' < y$.

Therefore, the surjective function H is isotone. Therefore, it is bijective.

Thus, H is an isotone bijection from $Z \subset X$ onto $c \in A$. Assume that $Z \neq X$. Then the set $X \setminus Z$ contains a minimal element y. Consider the initial interval X_y. If $x \in X_y$, then $x \in Z$, i.e., $X_y \subset Z$. Conversely, let $x \in Z$. Assume that $y \leq x$. Consider the ordinal number $a \equiv H(x)$. For it, there exists an isotone bijection $f \in U$ such that $dom(f) = X_x$ and $rng(f) = a$. If $y = x$, then $y \in Z$, which is impossible. Let $y < x$. Consider the ordinal number $b \equiv f(y) \in a$ and the isotone bijection $g \equiv f|X_y$ from X_y onto b. Since $g \subset f \in U$, by the transitivity of U, we obtain $g \in U$. Therefore, $b = H(y)$ and $y \in Z$, which is impossible. The obtained contradiction implies $x < y$, i.e., $x \in X_y$. As a result, we obtain $X_y = Z$.

Consider a new set $Y \equiv Z \cup \{y\}$ and define the function f from Y onto $a \equiv c + 1$ setting $f|Z \equiv H$ and $f(y) \equiv c$. Let $x, x' \in Y$, and let $x < x'$. If $x, x' \in Z$, then $f(x) = H(x) < H(x') = f(x')$. If $x \in Z$ and $x' = y$, then $f(x) = H(x) \in c$ implies $f(x) < c = f(x')$. Therefore, f is strictly monotone. Conversely, let $f(x) < f(x')$ for $x, x' \in Y$. If $x, x' \in Z$, then $H(x) < H(x')$ implies $x < x'$. If $x \in Z$ and $x' = y$, then $x < y = x'$. If $x' \in Z$ and $x = y$, then $f(x') = H(x') \in c$. Hence $f(x') < c = f(x)$, which contradicts the condition. As a result $x < x'$. Therefore, f is isotone. Since f is surjective, f is bijective.

Assume that $X \setminus Z \neq \{y\}$. Then the nonempty set $X \setminus Y$ contains a minimal element x. If $x = y$, then $x \in Y$, which is impossible. If $x < y$, then $x \notin X \setminus Z$, i.e., $x \in Z \subset Y$, which is also impossible. Therefore, $y < x$. Let $t \in Y$. If $t \in Z = X_y$, then $t < y < x$, i.e., $t \in X_x$. If $t = y < x$, then $t \in X_x$ once again. Therefore, $Y \subset X_x$. Conversely, if $t \in X_x$, then $t < x$ implies $t \notin X \setminus Y$, i.e., $t \in Y$. As a result, $Y = X_x$. Hence f is a bijective isotone function from X_x onto a. It follows from $y \in X \in U$ that $y \in U$. Therefore,

$\langle y, c\rangle \in U$ and $\{\langle y, c\rangle\} \in U$. Further, by the quasi-transitivity of U, $H \subset Z * c \in U$ implies $H \in U$. This implies $f = H \cup \{\langle y, c\rangle\} \in U$. This means that $a = H(x) \in c \in a$, which is impossible. The obtained contradiction implies $X = Y$. Therefore, f is an isotone bijection from X onto a. Since $a = c \cup \{c\} \in U$, it follows that $a \in A$.

If $Z = X$, then we set $a \equiv c$ and $f \equiv H$.

Thus, in both cases, we have constructed the isotone bijective mapping $f \in U$ from X onto $a \in A$. Since $|X| \subset a \in U$, the scheme-transitivity of U implies $|X| \in U$. □

Let us prove that in a scheme-universal set, as in a universal set, there exists the $\in$-induction principle, which is analogous to the $\in$-induction principle in ZF (see Lemma 4 (Sec. 1.2) and Lemma 5 (Sec. 3.1)).

Lemma 3. *Let U be a scheme-universal set, $C \subset U$, and $\forall x \in U <$ let $(x \subset C \Rightarrow x \in C)$. Then $C = U$.*

Proof. The proof is analogous to that of Lemma 5 (Sec. 5.1), except of the central part, which changes as follows.

Denote R_x^x by R_x. Consider the following formula of the ZF theory: $\varphi(x, y) \equiv (x \in \omega \wedge y = R_x)$. Clearly, this formula is functional. Consider the formula $\varphi^U(x, y) \equiv ((x \in \omega)^U \wedge (y = R_x)^U)$ for $x, y \in U$. Since $x, \omega, y, R \in U$, using the transitivity property of the set U, we can prove that $(x \in \omega)^U \Leftrightarrow x \in \omega$ and $(y = R_x)^U \Leftrightarrow y = R_x$. Therefore, $\varphi^U(x, y) \Leftrightarrow \varphi(x, y)$ for $x, y \in U$. Consider the function $[\varphi|U] \equiv \{z \in U * U | \exists x, y \in U(z = \langle x, y\rangle \wedge \varphi^U(x, y))\} = \{z \in U * U | \exists x, y \in U(z = \langle x, y\rangle \wedge \varphi(x, y))\} = \{z \in U * U | \exists x, y \in U(z = \langle x, y\rangle \wedge x \in \omega \wedge y = R_x)\} \subset U * U$. Clearly, $dom\,[\varphi|U] = \omega$ and $A \equiv rng\,[\varphi|U] = \{q \in U | \exists p \in \omega(q = R_p)\}$. Since $[\varphi|U] \leftrightarrows \omega \to U$, then by Properties (5), (4), and (2), from the definition of scheme-universal set, it follows that $A \in U$ and $Q \equiv \cup A \in U$. Clearly $Q = \{y | \exists n \in \omega(y \in R_n)\}$. Therefore, $R_n \subset Q$ for any $n \in \omega$, and hence $P = R_0 \subset Q$. It follows from the uniqueness property mentioned above that $u(m) = u(n)|(m + 1)$ for all $m \leq n$, i.e., $R_k^m = R_k^n$ for any $k \in m + 1$. Therefore, $\cup R_k = \cup R_m^m = \cup R_m^{m+1} = R_{m+1}^{m+1} \equiv R_{m+1}$. □

For a scheme-universal set, as well as for a universal set, the following analog of the von Neumann equality from Lemma 4 (Sec. 2.3) holds.

Lemma 4. *Let U be a scheme-universal set. Then*:

(1) $V_\alpha \in U$ *for any* $\alpha \in \mathbf{On} \cap U$;

(2) $U = \cup\left(V_\alpha \subset U | \alpha \in \mathbf{On} \cap U\right)$.

Proof. (1) Consider the sets $A \equiv \mathbf{On} \cap U$ and $C' \equiv \{\alpha \in A | V_\alpha \in U\}$ and the classes $\mathbf{C}'' \equiv \mathbf{On} \setminus U$ and $\mathbf{C} \equiv C' \cup \mathbf{C}''$. By Lemma 2, $0 = V_0 = \emptyset \in U$. Hence $0 \in \mathbf{C}$. Let $\alpha \in \mathbf{C}$. Assume that $\alpha + 1 \in A$. Since $\alpha \in \alpha + 1 \in U$, by Property (1), $\alpha \in U$, and, therefore, $\alpha \in A \cap \mathbf{C} = C'$. Then by Properties (2) and (3), the condition $V_\alpha \in U$ implies $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha) \in U$. Therefore, $\alpha + 1 \in C' \subset \mathbf{C}$. In the case $\alpha + 1 \notin A$, we immediately obtain $\alpha + 1 \in \mathbf{C}'' \subset \mathbf{C}$.

Let α be a limit ordinal number, and let $\alpha \subset \mathbf{C}$. Assume that $\alpha \in A$. If $\beta \in \alpha$, then $\beta \in \alpha \in U$ implies $\beta \in A \cap \mathbf{C} = C'$.

Consider the functional formula $\varphi(x, y) \equiv (x \in \alpha \Rightarrow y = V_x) \wedge (x \notin \alpha \Rightarrow y = \emptyset)$. Then $[\varphi|U] = \{z | \exists x \in U \exists y \in U(z = \langle x, y\rangle \wedge (x \in \alpha \Rightarrow y = V_x)^U \wedge (x \notin \alpha \Rightarrow y = \emptyset)^U \wedge \alpha \in U\}$. Since $\alpha \in U$ and $x \in \alpha \Rightarrow v_x \in U$ by condition, this formula is equivalent to the formula $\{z | \exists x \in U \exists y \in U(z = \langle x, y\rangle \wedge (x \in \alpha \Rightarrow y = V_x) \wedge (x \notin \alpha \Rightarrow y = \emptyset))\}$. Obviously, in this case, $[\varphi|U] \leftrightarrows \alpha \to U$ and $rng[\varphi|U] = \left(V_y | y \in \alpha\right)$. By Property (4), $\left(V_y | y \in \alpha\right) \in U$, and, therefore, $V_\alpha = \cup\left(V_y | y \in \alpha\right) \in U$. Therefore, $\alpha \in C' \subset \mathbf{C}$. In the case $\alpha \notin A$, we immediately obtain $\alpha \in \mathbf{C}'' \subset \mathbf{C}$.

By the transfinite induction principle, $\mathbf{C} = \mathbf{On}$, and hence $C' = A$.

(2) It follows from what was proved above that $V_\alpha \subset U$ for any $\alpha \in A$. Therefore, $P \equiv \cup\left(V_\alpha | \alpha \in A\right) \subset U$. Let us show that P satisfies the $\in$-induction principle from Lemma 3. Consider the formula $\varphi(u, z) \equiv (u \in P \Rightarrow z = sm\,\{\alpha \in A | p \in V_\alpha\}) \wedge (u \notin P \Rightarrow z = \emptyset)$.

Let $x \in U$, and let $x \subset P$. If $x = \emptyset$, then $x \in P$. In what follows, we will assume that $x \neq \emptyset$. If $y \in x \subset P$, then $y \in V_\alpha$ for a certain $\alpha \in A$. Hence, by Lemma 1, $\varphi(y) \leq \alpha \in U$ implies $\varphi(y) \in A$.

Therefore, we can consider the functional formula $\psi \equiv \varphi|x$. It is easy to show that $[\psi|U] \leftrightharpoons x \to A$. By Property (4), $R \equiv rng\,[\psi|A] \in U$, and by Property (2), $\rho \equiv \cup R \in U$. Since $\emptyset \neq R \subset \mathbf{On}$, by Lemma 2 (Sec. 1.2), ρ is an ordinal number. Hence $\rho \in A$.

If $y \in x$, then by Lemma 1 (Sec. 1.2), $[\psi|U](y) \subset \rho$ implies $y \in V_{[\psi|U](y)} \subset V_\rho$. Therefore, by Lemma 3 (Sec. 1.2), $x \subset V_\rho \in V_{\rho+1}$ implies $x \in V_{\rho+1}$. By Property (3), $\rho + 1 = \rho \cup \{\rho\} \in U$ implies $\rho + 1 \in A$. Therefore, $x \in P$.

Now Lemma 3 implies $P = U$. □

Theorem 2. *Let U be an arbitrary scheme-universal set. Then*:

(1) *$U = V_\varkappa$ for $\varkappa \equiv \sup(\mathbf{On} \cap U) = \cup(\mathbf{On} \cap U) \subset U$*;

(2) *$\varkappa$ is a scheme-inaccessible cardinal number*;

(3) *the correspondence $q : U \mapsto \varkappa$ such that $U = V_\varkappa$ is an injective isotone mapping from the class $\mathbf{U}'$ of all scheme-universal sets into the class $\mathbf{In}'$ of all scheme-inaccessible cardinal numbers*

Proof. (1) Since $A \equiv \mathbf{On} \cap U$ is a nonempty set, because by Property (5), it contains the element ω, by Lemma 2 (Sec. 1.2), $\varkappa$ is an ordinal number.

Let $\varkappa \in U$. Then by the properties of a scheme-universal set, $\varkappa + 1 = \varkappa \cup \{\varkappa\} \in U$. Since $\varkappa + 1 \in \mathbf{On}$, $\varkappa + 1 \in (\mathbf{On} \cap U)$ in this case, i.e., $\varkappa + 1 \leq \varkappa$, which is not true. Therefore, $\varkappa \notin U$.

Assume that $\varkappa = \alpha + 1$ for a certain ordinal number α. In this case, $\alpha \in U$, since $\varkappa \subset U$ and $\alpha \in \varkappa$. Since $\varkappa = \alpha \cup \{\alpha\}$, by the properties of a quasi-universal set, it follows that $\varkappa \in U$, which is impossible.

Therefore, $\varkappa$ is a limit ordinal number.

Hence $V_\varkappa = \cup\langle V_\beta | \beta \in \varkappa\rangle$. By Lemma 4, $U = \cup\langle V_\alpha | \alpha \in A\rangle$. If $\alpha \in A$, then $\alpha \leq \varkappa$ implies $V_\alpha \subset V_\varkappa$. Therefore, $U \subset V_\varkappa$. If $\beta \in \varkappa = \cup A$, then $\beta \in \alpha \in A$ for a certain α. By Property (1), $\beta \in A$. Therefore, $V_\varkappa \subset U$.

Therefore, $U = V_\varkappa$.

(2) Obviously, $\varkappa \neq 0$.

Assume that the ordinal number $\varkappa$ is not scheme-regular. Then $\exists\alpha(\alpha \in \varkappa \wedge [\varphi|U] \leftrightharpoons \alpha \to \varkappa \wedge \cup rng\,[\varphi|U] = \varkappa)$ for a certain functional formula $\varphi(x, y, \vec{p})$. But by Property (4) of a scheme-universal set, $\alpha \in U$ and $\varkappa \subset U$ imply $rng\,[\varphi|U] \in U$, and, therefore, $\cup rng\,[\varphi|U] \in U$. Therefore $\cup rng\,[\varphi|U] \neq \varkappa$, and the obtained contradiction implies that the ordinal number $\varkappa$ is scheme-regular.

Let λ be an ordinal number such that $\lambda < \varkappa$. Since $\lambda \in \varkappa \subset U$, by Property (2), $\mathcal{P}(\lambda) \in U$. By Theorem 1, $|\mathcal{P}(\lambda)| \in U$. Hence $|\mathcal{P}(\lambda)| \leq \varkappa$. Assuming that $\varkappa = |\mathcal{P}(\lambda)| \in U$, as above, we arrive at a contradiction. Therefore, $|\mathcal{P}(\lambda)| < \varkappa$.

Assertion (2) is proved.

(3) Lemma 1 (Sec. 1.2) implies that $\varkappa$ is unique. Therefore, we can define the mapping $q : \mathbf{U}' \to \mathbf{In}'$ such that $q(U) = \varkappa$, where $U = V_\varkappa$. Lemma 1 (Sec. 1.2) also implies that q is isotone. □

Corollary 1. *If U is a scheme-universal set, then $|U|$ is a scheme-inaccessible cardinal number, $|U| = \sup(\mathbf{On} \cap U)$ and $U = V_{|U|}$.*

Proof. By Theorem 1 $U = V_\varkappa$, for a scheme-inaccessible cardinal number, $\varkappa \equiv \sup(\mathbf{On} \cap U)$. By Lemma 3 Sec. 7.1, $\varkappa = |V_\varkappa| = |U|$. □

Theorem 3. *For any set U, the following assertions are equivalent*:

(1) *U is a scheme-inaccessible cumulative set*;

(2) *U is scheme-universal set.*

Proof. (1) $\vdash$ (2). Let $U = V_\varkappa$ for a certain scheme-inaccessible cardinal number $\varkappa > \omega$. Let us show that the set U is quasi-universal.

The property $x \in U \Rightarrow x \subset U$ follows from Lemma 4 (Sec. 2.2).

The property $x \in U \Rightarrow \mathcal{P}(x) \in U$ follows from Lemma 7 (Sec. 2.2).

The property $x \in U \wedge y \in U \Rightarrow x \cup y \in U$ follows from Lemma 6 (Sec. 2.2).

The properties $x \in U \wedge y \in U \Rightarrow \{x, y\}, \langle x, y\rangle, x \times y \in U$ follow from Colloraries 1 and 2 of Lemma 7 (Sec. 2.2).

The property $\omega \in U$ follows from lemma 8 (Sec. 2.2).

The property $x \in U \Rightarrow \cup x \in U$ follows from Lemma 5 (Sec. 7.1).

The property $x \in U \wedge [\varphi|U] \leftrightharpoons x \to U \Rightarrow rng\,[\varphi|U] \in U$ follows from Lemma 4 (Sec. 7.1).

Therefore, the set U is quasi-universal.

(2) $\vdash$ (1). This deducibility obviously follows from the previous theorem □

7.3. Supertransitive standard models of the ZF theory in the ZF theory. In this subsection, we consider supertransitive standard models of the ZF theory in the ZF theory.

Proposition 1. *In the ZF theory, the following assertions are equivalent for a set U:*

(1) *U is supertransitive standard model for the ZF theory in ZF;*
(2) *U is scheme-universal.*

Proof. (1) $\vdash$ (2). Consider an arbitrary sequence $s \equiv x_0, \dots, x_q, \dots$ of elements of the set U and translations of certain axioms and axiom scheme of the ZF theory under the standard interpretation $M \equiv (U, I)$ on a sequence s.

Instead of $\theta_M[s]$ and $M \vDash \varphi[s]$, we write θ^t and φ^t for the terms θ and formulas φ, respectively.

To simplify the further presentation, we first consider translations of certain simple formulas. Let u and v be certain sets.

The formula $u \in v$ translates into the formula $(u \in v)^t = (\langle u^t, v^t\rangle \in B)$. Denote the latter formula by γ. By definition, this formula is equivalent to the formula $(\exists x \exists y(x \in U \wedge y \in U \wedge \langle u^t, v^t\rangle = \langle x, y\rangle \wedge x \in y))$. Using the property of an ordered pair, we conclude that $u^t = x$ and $v^t = y$. Hence the formula $\delta \equiv (u^t \in v^t)$ is deduced from γ. By the deduction theorem, $\gamma \Rightarrow \delta$. Conversely, consider the formula δ. In the ZF theory, it is proved that for sets u^t and v^t, there exists a set z such that $z = \langle u^t, v^t\rangle$. By the logical axiom scheme **LAS3** from ([16], III, § 1), the formula $(z = \langle u^t, v^t\rangle \Rightarrow u^t \in U \wedge v^t \in U \wedge z = \langle u^t, v^t\rangle \wedge u^t \in v^t)$ is deduced from the formula δ. Since the formula $z = \langle u^t, v^t\rangle$ is deduced from axioms, the formula $(u^t \in U \wedge v^t \in U \wedge z = \langle u^t, v^t\rangle \wedge u^t \in v^t)$ is deduced. By **LAS13**, the formula $\exists x \exists y(x \in U \wedge y \in U \wedge z = \langle x, y\rangle \wedge x \in y)$ is deduced; it is equivalent to the formula $z \in B$ and, therefore, to the formula γ. By the deduction theorem, $\delta \Rightarrow \gamma$. Therefore, the first equivalence $(u \in v)^t \Leftrightarrow u^t \in v^t$ holds.

The formula $v \subset w$ translates into the formula $(v \subset w)^t$. Denote the latter formula by ε. By the first equivalence proved above, it is equivalent to the formula $\varepsilon' \equiv \forall u \in U(u \in v^t \Rightarrow u \in w^t)$. According to **LAS11**, the formula $\varepsilon'' \equiv (x \in U \Rightarrow (x \in v^t \Rightarrow x \in w^t))$ is deduced from the formula ε'. If $x \in v^t$, then $v^t \in U$ and the transitivity of the set U imply $x \in U$. Then the formula ε'' implies $x \in v^t \Rightarrow x \in w^t$. Hence, by the deduction theorem, the formula $(\varepsilon \Rightarrow (x \in v^t \Rightarrow x \in w^t))$ is deduced. By the inversion rule, the formula $\forall x(\varepsilon \Rightarrow (x \in v^t \Rightarrow x \in w^t))$ is deduced. By **LAS12**, the formula $(\varepsilon \Rightarrow \forall x(x \in v^t \Rightarrow x \in w^t))$, i.e., the formula $(\varepsilon \Rightarrow v^t \subset w^t)$, is deduced.

Conversely, assume that we have the formula $v^t \subset w^t$. By logical axioms, from it, we sequentially deduce the formulas $(u \in v^t \Rightarrow u \in w^t)$ and $(u \in U \Rightarrow (u \in v^t \Rightarrow u \in w^t))$. By the generalization rule, we deduce the formula ε'. Hence, by the deduction theorem, the formula $(v^t \subset w^t \Rightarrow \varepsilon)$ is deduced. Therefore, the second equivalence $(v \subset w)^t \Leftrightarrow v^t \subset w^t$ holds.

Exactly in the same way as the first equivalence was deduced, the third equivalence $(u = v)^t \Leftrightarrow u^t = v^t$ is deduced.

In what follows, we will write not literal translations of axioms and axiom schemes, but their equivalent variants, which are obtained by using the mentioned equivalence.

The volume axiom **A1** translates into the formula $\mathbf{A1}^t \Leftrightarrow \mathbf{A1}^U = \forall X \in U \forall Y \in U(\forall u \in U(u \in X \Leftrightarrow u \in Y) \Rightarrow X = Y)$.

The pair axiom **A2** translates into the formula $\mathbf{A2}^t \Leftrightarrow \mathbf{A2}^U = \forall u \in U \forall v \in U \exists x \in U \forall z \in U(z \in x \Leftrightarrow z = u \vee z = v)$.

The union axiom **A4** translates into the formula $\mathbf{A4}^t \Leftrightarrow \mathbf{A4}^U = \forall X \in U \exists Y \in U \forall u \in U(u \in X \Leftrightarrow \exists z \in U(u \in z \wedge z \in X))$.

The axiom of set of subsets **A5** translates into the formula $\mathbf{A5}^t \Leftrightarrow \mathbf{A5}^U = \forall X \in U \exists Y \in U \forall u \in U(u \subset X \Leftrightarrow u \in Y)$.

The axiom scheme of substitution **AS6** translates into the formula scheme $\mathbf{AS6}^t \Leftrightarrow \forall x \in U \forall y \in U \forall y' \in U(\varphi^\tau(x,y) \wedge \varphi^\tau(x,y') \Rightarrow y = y') \Rightarrow \forall X \in U \exists Y \in U \forall x \in U(x \in X \Rightarrow \forall y \in U(\varphi^\sigma(x,y) \Rightarrow y \in Y))$, where φ^τ and φ^σ stand for the formulas $M \vDash \varphi[s^\tau]$ and $M \vDash \varphi[s^\sigma]$ in which s^τ and s^σ stand for the corresponding changes of the sequence s under the translation of the quantor overformulas mentioned above.

The axiom of empty set **A7** translates into the formula $\mathbf{A7}^t \Leftrightarrow \mathbf{A7}^U = \exists x \in U \forall z \in U(z \notin x)$.

The infinity axiom **A8** translates into the formula $\mathbf{A8}^t \Leftrightarrow \mathbf{A8}^\tau \equiv \exists Y \in U(\emptyset^t \in V \wedge \forall y \in U(y \in Y \Rightarrow (y \cup \{y\})^\tau \in Y))$, where:

— the set $\emptyset^t$ is defined from the formula $A7^U$;

— the set $Z_1 \equiv Z_1(y) \equiv (y \cup \{y\})^\tau$ is defined from the formula $\exists Z_1 \in U \forall u \in U(u \in Z_1 \Leftrightarrow \exists z \in U(u \in z \wedge z \in \{y, \{y\}\}^\sigma))$,

— the set $Z_2 \equiv Z_2(y) \equiv \{y, \{y\}\}^\sigma$ is defined from the formula $\exists Z_2 \in U \forall u \in U(u \in Z_2 \Leftrightarrow u = y \vee u = \{y\}^\rho)$;

— the set $Z_3 \equiv Z_3(y) \equiv \{y\}^\rho$ is defined from the formula $\exists Z_3 \in U \forall u \in U(u \in Z_3 \Leftrightarrow u = y)$.

Since M is a model of the ZF theory, all the translations written above are deducible formulas in the ZF theory.

Therefore, the formula $\mathbf{A7}^U$ states the existence of a certain $x \in U$, which is denoted by $\emptyset^t$. If $z \in U$, then $\mathbf{A7}^U$ implies $z \notin x$. Now let $z \notin U$; assume that $z \in x$. Then by the transitivity of the set U, we obtain $z \in U$, which contradicts the condition. Therefore, $z \notin x$. Thus, $z \notin x$ is deduced. By the generalization rule, we deduce the formula $\forall z(z \notin x)$, which means that $x = \emptyset$. Therefore, $\emptyset^t = \emptyset$ and $\emptyset \in U$.

Now let us verify that if $y \in U$, then $Z_3 = \{y\}$. Let $u \in Z_3$. Sience $Z_3 \in U$ and U is transitive, then $u \in U$. If $u \in U$, then the formula for Z_3 presented above implies $u = y$. Therefore, $u \in \{y\}$. Thus, $Z_3 \subset \{y\}$. Conversely, let $u \in \{y\}$. Then $u = y$. Since $y \in U$, it follows that $u \in U$. Therefore, by the same formula, $u \in Z_3$. Therefore, $\{y\} \subset Z_3$, which implies the required equality. This equality leads to the disappearance of the index ρ in the formula for Z_2.

Using this equality, let us prove that $Z_2 = \{y, \{y\}\}$. Let $u \in Z_2$. Then as above, $u \in U$. Therefore, the formula for Z_2 presented above implies $u = y$ or $u\{y\}$. Therefore, $u \in \{y, \{y\}\}$. Therefore, $Z_2 \subset \{y, \{y\}\}$. Conversely, let $u \in \{y, \{y\}\}$. Then $u = y \in U$ or $u = \{y\} = Z_3 \in U$. Therefore, $u \in U$ in both cases. Hence, by the same formula, $u \in Z_2$. Therefore, $\{y, \{y\}\} \subset Z_2$, which implies the required equality. This equality leads to the disappearance of the index σ in the formula for Z_1.

Finally, let us verify that if $y \in U$, then $Z_1 = y \cup \{y\}$. Let $u \in Z_1$. Since $Z_1 \in U$ and U is transitive, it follows that $u \in U$. Therefore, the formula for Z_1 implies that there exists $z \in U$ such that $u \in z$ and $z \in \{y, \{y\}\}$. Therefore, $u \in \cup\{y, \{y\}\} \equiv Z$, i.e., $Z_1 \subset Z$. Conversely, let $u \in Z$. Then there exists $z \in \{y, \{y\}\}$ such that $u \in z$. We conclude from $z = y \in U$ or $z = \{y\} = Z_3 \in U$ that $z \in U$. Therefore, the mentioned formula implies $u \in Z_1$. Therefore, $Z \subset Z_1$, which implies the required equality. This equality leads to the disappearance of the index τ in the formula for $\mathbf{A8}^\tau$.

All that was said above implies $\mathbf{A8}^\tau \equiv \exists Y \in U(\emptyset \in Y \wedge \forall y \in U(y \in Y \Rightarrow y \cup \{y\} \in Y))$. If $y \in Y$, then $Y \in U$ and the transitivity of U imply $y \in U$. Then $y \cup \{y\} \in Y$ is deduced from this formula. By the deduction theorem, the formula $y \in Y \Rightarrow y \cup \{y\} \in Y$ is deduced, and by the generalization rule, the formula $\forall y \in Y(y \cup \{y\} \in Y)$ is deduced. Therefore, from $\mathbf{A8}^t$, we deduce the formula $\exists Y \in U(\emptyset \in Y \wedge \forall y \in Y(y \cup \{y\} \in Y))$, which almost coincides with the infinity axiom and asserts that there exists an inductive set $Y \in U$.

Using the obtained translations, let us prove that the set U is scheme-universal.

Consider the formula $\mathbf{A2}^U$. According to this formula, for any $u, v \in U$, there exists the corresponding set $x \in U$. If $z \in x$, then the transitivity of U implies $z \in U$. Therefore, the formula $z = u \vee z = v$ is deduced from this formula. If $z = u \vee z = v$, then $z \in U$, and, therefore, the formula $z \in x$ is deduced from

$\mathbf{A2}^U$. Since $\mathbf{A2}^U$ is deducible in ZF, by the deduction theorem and the generalization rule, we deduce the formula $\forall z(z \in x \Leftrightarrow z = u \vee z = v)$, which means that $x = \{u, v\}$. Therefore, $\{u, v\} \in U$. By the deduction theorem, we deduce the formula $u, v \in U \Rightarrow \{u, v\} \in U$. This implies $\{u\} \in U$ and $\langle u, v\rangle \in U$.

Consider the formula $\mathbf{A4}^U$. According to this formula, for any $X \in U$, there exists the corresponding set $Y \in U$. As above, the transitivity of U implies $Y = \cup X$. Therefore, $\cup X \in U$, and by the deduction theorem, we deduce the formula $X \in U \Rightarrow \cup X \in U$. This implies that $X, Y \in U$ implies $X \cup Y \equiv \cup\{X, Y\} \in U$.

Consider the formula $\mathbf{A5}^U$. According to this formula, for any $X \in U$, there exists the corresponding set $Y \in U$. Clearly, $Y \subset \mathcal{P}(X)$. Let $y \in \mathcal{P}(X)$. Then by the cumulativity of U, $y \subset X \in U$ implies $y \in U$. Hence $Y = \mathcal{P}(X)$. Therefore, $\mathbf{P}(X) \in U$, and by the deduction theorem, we deduce the formula $X \in U \Rightarrow \mathcal{P}(X) \in U$.

If $X, Y \in U$, then by the cumulativity property, $X * Y \subset \mathcal{P}(\mathcal{P}(X \cup Y)) \in U$ implies $X * Y \in U$.

Consider an inductive set $Y \in U$ whose existence was proved above. Since ω is a minimal set among all inductive sets, it follows that $\omega \subset Y$. By the cumulativity property, this implies $\omega \subset U$.

Property (4) from the definition of a scheme-universal set holds automatically.

Therefore, we have proved that (1) $\vdash$ (2).

(2) $\vdash$ (1). Let U be a scheme-universal set. According to Sec. 3.1, it is supertransitive. Consider the standard interpretation $M \equiv (U, I)$ of the ZF theory. In above, we have translated certain axioms and axiom schemes of the ZF theory on the sequence s under the interpretation M. Let us prove that they are deducible in ZF.

Consider the formula $A1^U$. Let $X, Y \in U$, and let $\chi \equiv \forall u \in U(u \in X \Leftrightarrow u \in Y)$. Take an arbitrary set u. If $u \in X$, then the transitivity of U implies $u \in U$, and then $u \in Y$ is deduced. Analogously, $u \in Y$ is deduced from $u \in X$. Therefore, by the deduction theorem, the formula $u \in X \Leftrightarrow u \in Y$ is deduced, and by the generalization rule, the formula $\forall u(u \in X \Leftrightarrow u \in Y)$ is deduced. According to the volume axiom, the equality $X = Y$ is deduced from this. By the deduction theorem, the formula $\chi \Rightarrow X = Y$ is deduced in ZF. Further, by logical means $\mathbf{A1}^t$ is deduced.

Consider the formula $\mathbf{A2}^U$. Let $u, v \in U$. By the property of a quasi-universal set, $\{u, v\} \in U$. The pair axiom implies $\forall z \in U(z \in \{u, v\} \Leftrightarrow z = u \vee z = v)$. Therefore, by **LAS13**, the formula $\exists x \in U \forall z \in U(z \in x \Leftrightarrow z = u \vee z = v)$ is deduced. Further, by logical means, the formula $A2^t$ is deduced.

The axiom separation scheme **AS3** transforms into the formula scheme $\mathbf{AS3}^t \Leftrightarrow \forall X \in U \exists Y \in U \forall u \in U(u \in Y \Leftrightarrow u \in X \wedge \varphi^\tau(u))$, where Y is not a free variable of the formula $\varphi(u)$ and φ^τ stands for the formula $M \vDash \varphi[s^\tau]$, in which s^τ stands for the corresponding change of the sequence s under the transformation of the quantor overformulas $\forall x(\dots)$, $\exists Y(\dots)$ and $\forall u(\dots)$ mentioned above. According to **AS3**, for $X \in U$, there exists Y such that $\forall u \in U(u \in Y \Leftrightarrow u \in X \wedge \varphi^\tau(u))$. Since $Y \subset X \in U$, by Lemma 1 (Sec. 3.1), $Y \in U$. Therefore, $\mathbf{AS3}^t$ is deduced in ZF.

The deducibilities $\mathbf{A4}^t$ and $\mathbf{A5}^t$ are verified similarly to the deducibility $\mathbf{A2}^t$.

Let us verify the deducibility of $\mathbf{AS6}^t$. Denote the formula $\varphi^U(x, y, \vec{p}_M[s])$ by $\psi(x, y)$. Let the formula α hold. Consider any set $X \in U$. According to the axiom separation scheme, there exists the set $F \equiv \{z \in U | \exists x, y \in U(z = \langle x, y\rangle \wedge \varphi^\sigma(x, y))\}$. Clearly, $F \subset U * U$. The transitivity of U implies $X \subset U$. Therefore, there exists the set $Z \equiv F[X] \subset U$. Consider the set $G \equiv \{z \in U | \exists x, y \in U(z = \langle x, y\rangle \wedge \varphi^\sigma(x, y) \wedge x \in X)\} = F|X \subset X * Z$. Let $x \in X \subset U$. If $x \notin dom\, G$, then $G\langle x\rangle = \emptyset \in U$. Let $x \in dom\, G$, i.e., $G\langle x\rangle \neq \emptyset$. If $y, y' \in G\langle x\rangle \subset U$, then $\varphi^\sigma(x, y) \wedge \varphi^\sigma(x, y')$, or more precisely, $\varphi^\sigma(x, y, X, Y) \wedge \varphi^\sigma(x, y', X, Y)$ holds, since X and Y cannot be free variables of the formula φ^σ. Since $\varphi^\tau(x, y) = \varphi^\sigma(x, y, X \| X_M[s], Y \| Y_M[s])$, and similar for y', by **LAS11**, $\varphi^\tau(x, y) \wedge \varphi^\tau(x, y')$ holds. Therefore, the formula α implies $y = y'$. Therefore, $G\langle x\rangle = \{y\} \in U$. Therefore, $G\langle x\rangle \in U$ for any $x \in X$. By Lemma 3 (Sec. 3.1), $Y_0 \equiv rng\, G = \cup\left(G\langle x\rangle | x \in X\right) \in U$.

If $x \in X \subset U$, $y \in U$ and $\varphi^\sigma(x, y)$, then $\langle x, y\rangle \in G$ implies $y \in Y_0$. This means that the formula β is deduced from the formula α. By the deduction theorem, the formula $\alpha \Rightarrow \beta$ and, therefore, the scheme $\mathbf{AS6}^t$ is deduced.

According to Lemma 2, $\emptyset \in U$. From this and **A7**, $\mathbf{A7}^t$ is deduced.

Consider the formula $\mathbf{A8}^\tau$ and the set $\omega \in U$. It follows from the previous paragraph that $\emptyset^t = \emptyset \in \omega$. Let $y \in U$, and let $y \in \omega$. Then as above, it is verified that $Z_3 = \{y\}$, $Z_2 = \{y, \{y\}\}$ and $Z_1 = y \cup \{y\} \in \omega$. By the deduction theorem, the formula $(y \in \omega \Rightarrow Z_1 \in \omega)$ is deduced. Further, by logical means, the formula $(\emptyset^t \in \omega \wedge \forall y \in U(y \in \omega \Rightarrow (y \cup \{y\})^\tau \in \omega))$ and hence the formula $\mathbf{A8}^t$ are deduced.

The regularity axiom $\mathbf{A9}$ transforms into the formula $\mathbf{A9}^t \Leftrightarrow \mathbf{A9}^\tau \equiv \forall X \in U(X \neq \emptyset^t \Rightarrow \exists x \in U(x \in X \wedge (x \cap X)^\tau = \emptyset^t))$, where:

— the set $\emptyset^t$ is defined from the formula $\mathbf{A7}^U$ and, as was proved above, coincides with empty set $\emptyset$,

— the set $Z \equiv (x \cap X)^\tau$ is defined from the formula $\exists Z \in U \forall u \in U(u \in Z \Leftrightarrow u \in x \wedge u \in X)$.

Let us verify that if $X \in U$ and $x \in U$, then $Z = x \cap X$. Let $u \in Z$. Since $Z \in U$ and U is transitive, it follows that $u \in U$. Therefore, the formula for Z implies $u \in x \wedge u \in X$, i.e., $u \in x \cap X$. Therefore, $Z \subset x \cap X$. Conversely, let $u \in x \cap X$, i.e., $u \in x \wedge u \in X$. By the transitivity, $u \in U$. Therefore, the mentioned formula implies $u \in Z$. Therefore, $x \cap X \subset Z$, which proves the required equality. This equality leads to the disappearance of the index τ in the formula for $\mathbf{A9}^\tau$.

Let $X \in U$, and let $X \neq \emptyset^t = \emptyset$. By the regularity axiom, there exists $x \in X$ such that $x \cap X = \emptyset$. By the transitivity, $x \in U$. From this, by logical means, we deduce $\mathbf{9}^t$.

Finally, the axiom of choice $\mathbf{A10}$ translates into the formula $\mathbf{A10}^t \Leftrightarrow \mathbf{A10}^\tau \equiv \forall X \in U(X \neq \emptyset^t \Rightarrow \exists z \in U((z \leftrightharpoons \mathcal{P}(X) \setminus \{\emptyset\} \to X)^\tau \wedge \forall Y \in U(Y \in (\mathcal{P}(X) \setminus \{\emptyset\})^\sigma \Rightarrow z(Y)^\sigma \in Y)))$, where:

— the set $Z_1 \equiv Z_1(X) \equiv (\mathcal{P}(X) \setminus \{\emptyset\})^\sigma$ is defined from the formula $\exists Z_1 \in U \forall u \in U(u \in Z_1 \Leftrightarrow u \in \mathcal{P}(X)^\rho \wedge u \notin \{\emptyset\}^\rho)$,

— the set $Z_2 \equiv z(Y)^\sigma$ is defined from the formula $\langle Y, Z_2 \rangle^\rho \in z$,

— $\eta^\tau \equiv (z \leftrightharpoons \mathcal{P}(X) \setminus \{\emptyset\} \to X)^\tau$ stands for the formula $M \vDash \eta[s^\tau]$, in which s^τ stands for the corresponding change of the sequence s under the transformation of the quantor overformulas $\forall X(\dots)$ and $\exists z(\dots)$ mentioned above.

Fix the conditions $X \in U$ and $X \neq \emptyset^t = \emptyset \in U$. It was stated above that this implies $\mathcal{P}(X)^\rho = \mathcal{P}(X)$ and $\{\emptyset\}^\rho = \{\emptyset\}$. This leads to the disappearance of the index ρ in the formula for Z_1.

Let us verify that $Z_1 = \mathcal{P}(X) \setminus \{\emptyset\} \equiv Z$. Let $u \in Z_1$. Since $Z_1 \in U$ and U is transitive, it follows that $u \in U$. Therefore, the formula for Z_1 implies $y \in Z$. Therefore, $Z_1 \subset Z$. Conversely, let $u \in Z$. Since $\mathcal{P}(X) \in U$, $\mathcal{P}(X) \subset U$ by the transitivity. This implies $u \in U$. Therefore, the mentioned formula implies $u \in Z_1$. Therefore, $Z \subset Z_1$, which proves the required equality. This leads to the replacement of Z_1 by Z in the formula $\mathbf{A10}^\tau$.

Consider the formula $\varphi \equiv (z \leftrightharpoons Z \to X)$. It is the conjunction of the following three formulas: $\varphi_1 \equiv (z \subset Z * X)$, $\varphi_2 \equiv (dom\, z = Z)$ and $\varphi_3 \equiv (\forall x(x \in Z \Rightarrow \forall y(y \in X \Rightarrow \forall y'(y' \in X \Rightarrow (\langle x, y \rangle \in z \wedge \langle x, y' \rangle \in z \Rightarrow y = y')))))$.

Therefore, $\varphi^\tau = \varphi_1^\tau \wedge \varphi_2^\tau \wedge \varphi_3^\tau$. Since $\varphi_1 = (\forall u(u \in z \Rightarrow \exists x \exists y(x \in Z \wedge y \in X \wedge u = \langle x, y \rangle)))$, it follows that $\varphi_1^\tau \Leftrightarrow (\forall u \in U(u \in z \Rightarrow \exists x \in U \exists y \in U(x \in Z_1 \wedge y \in X \wedge u = \langle x, y \rangle^\sigma)))$. Analogously, $\varphi_2 = (\forall x(x \in Z \Rightarrow \exists y(y \in X \wedge \langle x, y \rangle \in z)))$ implies $\varphi_2^\tau \Leftrightarrow (\forall x \in U(x \in Z_1 \Rightarrow \exists y \in U(y \in X \wedge \langle x, y \rangle^\sigma \in z)))$.

Finally, $\varphi_3^\tau \Leftrightarrow (\forall x \in U(x \in Z_1 \Rightarrow \forall y \in U(y \in X \Rightarrow \forall y' \in U(y' \in X \Rightarrow (\langle x, y \rangle^\sigma \in z \wedge \langle x, y' \rangle^\sigma \in z \Rightarrow y = y')))))$.

By the transitivity property, for x, y and y' in the formulas φ_1^τ, φ_2^τ and φ_3^τ, we have $x, y, y' \in U$. Therefore, by what was proved above, in these formulas, the following equality hold: $Z_1 = Z$, $\langle x, y \rangle^\sigma = \langle x, y \rangle$ and $\langle x, y' \rangle^\sigma = \langle x, y' \rangle$. This implies that the formulas φ_1^τ, φ_2^τ, and φ_3^τ differ from the formulas φ_1, φ_2 and φ_3, respectively, by only bounded quantor prefices $\forall \dots \in U$ and $\exists \dots \in U$.

For X, by the axiom of choice $\mathbf{A10}$, there exists z such that $\chi \equiv (z \leftrightharpoons Z \to X) \wedge \forall Y(Y \in Z \Rightarrow z(Y) \in Y)$.

Therefore, the formula $\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is deduced, and hence the formulas φ_1, φ_2 and φ_3 are deduced.

Let $u \in U$, and let $u \in z$. Then from the formula φ_1, it is deduced that there exist $x \in Z$ and $y \in X$ such that $u = \langle x, y \rangle$. Since $x \in Z \in U$ and $y \in X \in U$, it follows that $x, y \in U$ by the transitivity property. This means that for given conditions $u \in U$ and $u \in z$, the formula $\exists x \in U \exists y \in U(x \in Z_1 \wedge y \in X \wedge u = \langle x, y \rangle^\sigma)$ is deduced. Applying the deduction theorem twice and the deduction rule, we deduce the formula φ_1^τ.

Let $x \in U$, and let $x \in Z_1 = Z$. Then from the formula φ_2, it is deduced that for x, there exists $y \in X$ such that $\langle x, y\rangle \in z$. It follows from $y \in X \in U$ that $y \in U$. This means that for given conditions $x \in U$ and $x \in Z_1$, the formula $\exists y \in U(y \in X \wedge \langle x, y\rangle^\sigma \in z)$ is deduced. As in the previous paragraph, we deduce from this the formula φ_2^τ.

Let $x \in U$, $x \in Z_1 = Z$, $y \in U$, $y \in X$, $y' \in U$, $y' \in X$, $\langle x, y\rangle \in z$, and let $\langle x, y'\rangle \in z$. Then $y = y'$ is deduced from the formula φ_3. Applying alternately the deduction theorem and the deductions rules several times, we deduce the formula φ_3^τ.

Therefore, the formula φ^τ is deduced.

Since $z \leftrightharpoons Z \to X$, it follows that $z\langle Y\rangle = \{z(Y)\}$. If $Y \in U$ and $Y \in Z_1 = Z$, then $Z_2 \in U$ implies $\langle Y, Z_2\rangle^\rho = \langle Y, Z_2\rangle$. Then $\langle Y, Z_2\rangle \in z$ implies $Z_2 \in z\langle Y\rangle$, whence $Z_2 = z(Y)$. Therefore, for the function z, the conditions $Y \in U$ and $Y \in Z_1$ imply $Z_2 = z(Y) \in Y$.

Since $Z = Z_1 \in U$ and $X \in U$, it follows that $Z * X \in U$. By Lemma 1 (Sec. 7.2), $z \subset Z * X$ implies $z \in U$.

All this means that from Axiom **A10** is deduced he existence of the object z satisfying the formula χ from which the formula $\xi \equiv (\varphi^\tau \wedge \forall Y \in U(Y \in Z_1 \Rightarrow Z_2 \in Y))$ is deduced. Thus, from the fixed conditions, the formula $\exists z \in U \xi$ is deduced. Applying alternately the deduction theorem and the generalization rule several time, as a result, we deduce the formula $\mathbf{A10}^t$.

Therefore, M is a supertransitive standard model for the ZF theory. □

Corollary. *Any uncountable scheme-inaccessible cumulative set $V_\varkappa$ is a supertransitive standard model for the ZF theory.*

Proof. The assertion follows from Proposition 1 and Theorem 3 (Sec. 7.2). □

Using Theorems 2 and 3 (Sec. 7.2) and Proposition 1, we obtain the following theorem.

Theorem 1. *In the ZF theory, the following assertions are equivalent for a set U:*

(1) *$U = V_\varkappa$ for the scheme-inaccessible cardinal number $\varkappa = |U| = \sup(\mathbf{On} \cap U)$;*

(2) *U is a supertransitive standard model for the ZF theory.*

Proof. (1) $\vdash$ (2). By Theorem 3 (Sec. 7.1), the set $U = V_\varkappa$ is scheme-universal. By Proposition 1, the set U is a supertransitive standard model.

(2) $\vdash$ (1). By Proposition 1, U is quasi-universal. By Theorem 2 (Sec. 7.2), $U = V_\varkappa$ and $\varkappa = \sup(\mathbf{On} \cap U)$. By Corollary 1 of Theorem 2 (Sec. 7.2), $\varkappa = |U|$. □

This theorem yields the *canonical form of supertransitive standard model sets* of the ZF theory. Thus, we have described all natural models of the ZF set theory.

7.4. Tarski scheme sets. Theorem on the characterization of natural model of the ZF theory. A set U in the ZF theory is called a *Tarski scheme set* if

(1) $x \in U \Rightarrow x \subset U$;

(2) $x \in U \Rightarrow \mathcal{P}(x), \cup \in U, \cup x \in U$;

(3) $\forall \vec{p}, \vec{u} \in U(([\varphi(x, y; \vec{p})|U] \leftrightharpoons \langle\sigma(x; \vec{u})|U\langle\rightarrowtail \varepsilon) \wedge \varepsilon \in |U| \Rightarrow \langle\sigma(x; \vec{u})|U\rangle \in U)$, where φ and σ are metavariables for designation on any formulas of the ZF theory;

(4) $\omega \in U$ and $|U| \subset U$.

It follows from Sec. 6.1 that any Tarski set of uncountable cardinality is a scheme Tarski set.

Lemma 1. *If U is a scheme Tarski set and $x \in U$, then $|x| \in |U|$.*

The proof completely coincides with the proof of Lemma 2 of Sec. 6.1.

Lemma 2. *A scheme Tarski set is supertransitive.*

The proof completely coincides with the proof of Lemma 3 in Sec. 6.1.

Lemma 3. *If U is a scheme Tarski set and $x, y \in U$, then $\{x\}, \{x,y\}, \langle x,y\rangle \in U$.*

Proof. Consider the formulas $\sigma_1(s;u) \equiv (s = u)$, $\sigma_2(s;u,v) \equiv (s = u \vee s = v)$, $\varphi_1(s,t;u) \equiv (s = u \Rightarrow t = 0)$ and $\varphi_2(s,t;u,v) \equiv (s = u \Rightarrow t = 0) \wedge (s = v \wedge v = u \Rightarrow t = 0) \wedge (s = v \wedge v \neq u \Rightarrow t = 1)$. Then under the conditions of the lemma, $X_1 \equiv \langle \sigma_1(s;x)|U\rangle = \{x\}$ and $X_2 \equiv \langle \sigma_2(s;x,y)|U\rangle = \{x,y\}$.

Consider the correspondences $f_1 \equiv [\varphi_1(s,t;x)|U]$ and $f_2 \equiv [\varphi_2(s,t;x,y)|U]$. If $s \in X_1$ and $\langle s,t\rangle \in f_1$, then $s = x$ and $t = 0$. Therefore, f_1 is an injective mapping from X_1 into $\{0\} \equiv 1 \in |U|$. By Property 3. $X_1 \in U$.

Now let $s \in X_2$, and let $\langle s,t\rangle \in f_2$. If $s = x$, then $t = 0$. If $s = y \wedge y = x$, then $t = 0$. If $s = y \wedge y \neq x$, then $t = 1$. Therefore, f_2 is an injective mapping from X_2 into $\{0,1\} = 2 \in |U|$. By Property (3), $X_2 \in U$. What was proved above implies $\langle x,y\rangle \in U$. ☐

Corollary 1. *If U is a scheme Tarski set and $x, y \in U$, then $x \cup y \in U$.*

Proof. It follows from Lemma 3 and Property (2) that $x \cup y = \cup\{x,y\} \in U$. ☐

Corollary 2. *If U is a scheme Tarski set and $x, y \in U$, then $x * y \in U$.*

Proof. Since $x * y \subset \mathcal{P}(\mathcal{P}(x \cup y)) \in U$, by Lemma 2, $x * y \in U$. ☐

Lemma 4. *Let U be a scheme Tarski set, $\varphi(a,b;\vec{r})$ be a formula of the ZF theory, and let $x \in U$. If $\vec{r} \in U$ and $[\varphi(a,b;\vec{r})|U] \leftrightharpoons x \to U$, then $rng\,[\varphi(a,b;\vec{r})|U] \in U$.*

Proof. Denote $[\varphi(a,b;\vec{r})|U]$ and $rng\,[\varphi|U]$ for a given $\vec{r} \in U$ from the condition of the lemma by f and R, respectively. Consider the formula $\rho(b;\vec{r},y) \equiv \exists a \in y \varphi(a,b;\vec{r})$. Then $\langle \rho(b;\vec{r},x)|U\rangle = \{b \in U | \exists a \in U(a \in x \wedge \varphi^U(a,b;\vec{r}))\} = R$ for given $\vec{r}, x \in U$.

Consider the formula $\psi(b,c;\vec{r},y) \equiv \forall a \in c(a \in y \wedge \varphi(a,b;\vec{r})) \wedge \forall a \in y(\varphi(a,b;\vec{r}) \Rightarrow a \in c)$ and the correspondence $[\psi(b,c;\vec{r},y)|U] = \{t \in U * U | \exists b, c \in U(t = \langle b,c\rangle \wedge \forall a \in c(a \in y \wedge \varphi^U(a,b;\vec{r})) \wedge \forall a \in y(\varphi^U(a,b;\vec{r}) \Rightarrow a \in c))\}$. It is easy to verify that the correspondence $g \equiv [\psi(b,c;\vec{r},x)|U]$ is an injective mapping from R into $S \equiv \mathcal{P}(x)$ such that $g(b) = f^{-1}(b)$ for any $b \in R$.

By Properties (2) and (4) and Lemma 1, $S \in U$, $|S| \in |U|$ and $|S| \in U$. Consider a certain bijection $h : S \leftrightarrows |S|$. By Corollary 2 to Lemma 3, $S * |S| \in U$. By Lemma 2, $h \subset S * |S|$ implies $h \in U$. Consider the formula $\chi(s,t;e) \equiv (\langle s,t\rangle \in e)$. Then for the value of the parameter e equal to h, we have $[\chi(s,t;h)|U] \equiv \{z \in U * U | \exists s,t \in U(z = \langle s,t\rangle \wedge \langle s,t\rangle \in h)\} = h$. It remains to take the composition of the mappings g and h. For this purpose, consider the formula $\zeta(b,tl\vec{r},y,e) \equiv \exists s \in \mathcal{P}(y)(\forall a \in s(a \in y \wedge \varphi(a,b;\vec{r})) \wedge \forall a \in y(\varphi(a,b;\vec{r}) \Rightarrow a \in s) \wedge \langle s,t\rangle \in e)$ and the correspondence $[\zeta(b,t;\vec{r},y,e)|U] = \{z \in U * U | \exists b,t \in U(z = \langle b,t\rangle \wedge \exists s \in \mathcal{P}(y)(\forall a \in s(a \in y \wedge \varphi^U(a,b;\vec{r})) \wedge \forall a \in y(\varphi^U(a,b;\vec{r}) \Rightarrow a \in s) \wedge \langle s,t\rangle \in e)\}$. Clearly, $F \equiv [\zeta(b,t;\vec{r},x,h)|U] = h \circ g$. Therefore, F is an injective mapping from R into $|S| \in U$. By Property (3), $R \in U$. ☐

Proposition 1. *Any scheme Tarski set is scheme-universal.*

Proof. The assertion follows from Properties (2) and (4), Lemma 3, and Corollaries 1 and 2 of Lemma 3 and Lemma 4. ☐

Now we can prove the theorem on the characterization of natural models of the ZF theory.

Theorem 1. *For a set U, the following assertions are equivalent in the ZF theory*:

(1) *U is a scheme-inaccessible cumulative set, i.e., $U = V_\varkappa$ for a certain scheme-inaccessible cardinal number $\varkappa$*;
(2) *U is a scheme-universal set*;
(3) *U is a supertransitive standard model set for the ZF theory*;
(4) *U is a scheme Tarski set.*

Proof. The equivalence of (1) and (2) was proved in Theorem 3 (Sec. 7.2). The equivalence of (2) and (3) was proved in Theorem 1 (Sec. 7.3).

The deducibility (4) $\vdash$ (2) was proved in Proposition 1.

(1) $\vdash$ (4). Let $U = V_{\varkappa}$ for a certain scheme-inaccessible cardinal number $\varkappa > \omega$. Let us show that U is a scheme Tarski set.

The property $x \in U \Rightarrow x \subset U$ follows from Lemma 4 (Sec. 2.2).

The property $x \in U \Rightarrow \mathcal{P}(x) \in U$ follows from Lemma 7 of (Sec. 2.2).

The property $x \in U \Rightarrow \cup x \in U$ follows from Lemma 5 of (Sec. 7.1).

Property (3) follows from Lemma 6 of Sec. 7.1.

The property $\omega \in U$ follows from Lemma 8 of Sec. 2.2.

Finally, the property $|U| \subset U$ follows from Lemma 1 of Sec. 2.2 and Lemma 3 of Sec. 7.1.

Therefore, U is a scheme Tarski set. □

Theorem 1 belong to the authors and was announced in [1].

Question. Are all properties of the defined scheme Tarski set independent, i.e. is this collection of properties minimal?

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