

LOCAL THEORY OF SETS AS A FOUNDATION FOR CATEGORY THEORY AND ITS CONNECTION WITH THE ZERMELO–FRAENKEL SET THEORY

V. K. Zakharov, E. I. Bunina,
A. V. Mikhalev, and P. V. Andreev

UDC 510.223, 512.581

CONTENTS

Introduction	5764
1. First-Order Theories and LTS	5767
1.1. Language of first-order theories	5767
1.2. Deducibility in a first-order theory	5768
1.3. An interpretation of a first-order theory in a set theory	5770
2. Local Theory of Sets	5772
2.1. Proper axioms and axiom schemes of the local theory of sets	5772
2.2. Some constructions in the local theory of sets	5775
3. Formalization of the Naive Category Theory within the Framework of the Local Theory of Sets	5776
3.1. Definition of a local category in the local theory of sets	5776
3.2. Functors and natural transformations and the “category of categories” and the “category of functors” in the local theory of sets generated by them	5777
4. Zermelo–Fraenkel Set Theory and Mirimanov–von Neumann Sets	5779
4.1. The proper axioms and axiom schemes of the ZF set theory	5779
4.2. Ordinals and cardinals in the ZF set theory	5782
4.3. Mirimanov–von Neumann sets in the ZF set theory	5786
5. Universes, Ordinals, Cardinals, and Mirimanov–von Neumann Classes in the Local Theory of Sets	5791
5.1. The relativization of formulas of the LTS to universal classes. The interpretation of the ZF set theory in universal classes	5791
5.2. The globalization of local constructions	5793
5.3. Ordinals and cardinals in the local theory of sets	5795
5.4. Mirimanov–von Neumann classes in the local theory of sets and their connection with universal classes	5798
5.5. The structure of the assemblies of all universal classes and all inaccessible cardinals in the local theory of sets	5801
6. Relative Consistency between LTS and the ZF Theory	5803
6.1. Additional axioms on inaccessible cardinals in the ZF theory	5803
6.2. “Forks” of relative consistency	5807
7. Proof of Relative Consistency by the Method of Abstract Interpretation	5816
7.1. Abstracts of a set theory	5816
7.2. The abstract interpretation of a first-order theory in a set theory	5817
8. Undeducibility of Some Axioms in LTS	5818
8.1. The undeducibility of the axiom scheme of replacement	5818

Translated from *Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory.*
Vol. 119, Set Theory, 2004.

8.2. The nondependence of axiom ICU on the axioms of the LTS	5826
8.3. Locally-minimal theory of sets	5827
References	5828

Introduction

The crises that arose in naive set theory at the beginning of the 20th century brought to the fore some strict axiomatic constructions of theories of mathematical totalities.

The most widely used of them are the *theory of sets in Zermelo–Fraenkel’s axiomatics* (ZF) (see [23]) and the *theory of classes and sets in von Neumann–Bernays–Gödel’s axiomatics* (NBG) (see [20] and [28]).

These axiomatic theories eliminated all the known paradoxes of naive set theory at the expense of a sharp restriction on possible expressive means. At the same time, they provided an opportunity to include almost all the then existing mathematical objects and constructions within the framework of these theories.

In 1945, a new mathematical notion of *category* was introduced by Eilenberg and MacLane in the initial paper [11]. From that time, category theory became an independent field of mathematics. But from the very beginning of its origin, category theory encountered the unpleasant circumstance that it did not fall not only within the framework of the theory of sets in Zermelo–Fraenkel’s axiomatics but even within the framework of the theory of classes and sets in von Neumann–Bernays–Gödel’s axiomatics (NBG) (see [11]). For this reason, in 1959, in the paper [25], MacLane pose the *general problem of constructing a new and more flexible axiomatic set theory that could serve as an adequate logical foundation for the whole naive category theory*.

Different variants of new axiomatic theories of mathematical totalities, adjusted for one or another need of category theory, were proposed by Ehresmann [10], Dedecker [9], Sonner [29], Grothendieck [14], da Costa ([6] and [7]), Isbell [18], MacLane ([26] and [27]), Feferman [12], Herrlich and Strecker [17], and others.

Ehresmann, Dedecker, Sonner, and Grothendieck introduced an important notion of (*categorical*) *universe* U , i. e., a totality of objects which satisfies the following properties of closedness:

- (1) $X \in U \Rightarrow X \subset U$ (Ehresmann–Dedecker did not propose this property);
- (2) $X \in U \Rightarrow \mathcal{P}(X), \cup X \in U$;
- (3) $X, Y \in U \Rightarrow X \cup Y, \{X, Y\}, \langle X, Y \rangle, X \times Y \in U$;
- (4) $X \in U \wedge (F \in U^X) \Rightarrow \text{rng } F \in U$;
- (5) $\omega \in U$ ($\omega = \{0, 1, 2, \dots\}$ is here the set of all finite ordinal numbers).

Within the framework of such a universe, it is possible to develop a quite rich category theory, in particular, the theory of categories with direct and inverse limits. To satisfy all needs of category theory, these authors proposed to strengthen the ZF or NBG set theories by the strong *universality axiom* postulating that *every set belongs to some universal set*. In MacLane’s axiomatics, the *existence of at least one universal set* is postulated. Similar versions were proposed by Isbell and Feferman. Herrlich–Strecker’s axiomatics dealt with objects of three types: *sets, classes, and conglomerations*.

Within the framework of each of these axiomatics, some definitions of category and functor are given. But the notions of category given in [12, 17, 18, 26, 27] are not closed with respect to such important operations of naive category theory as the “*category of categories*” and the “*category of functors*” (see [16], 8.4).

Within the framework of the axiomatic theories [9, 10, 14, 29] the definition of *\mathcal{U} -category*, *\mathcal{U} -functor*, and *natural \mathcal{U} -transformation* consisting of subsets of a universal set \mathcal{U} is given. This notion of category is closed with respect to such operations as the *\mathcal{V} -category of \mathcal{U} -categories* and the *\mathcal{V} -category of \mathcal{U} -functors*, where \mathcal{V} is some universal set containing the universal set \mathcal{U} as an element.

The axiomatics from [6] and [7] also give definitions of a category and a functor closed under the mentioned operations of naive category theory. But da Costa uses the logic with the nonconstructive rule of deduction $\varphi(V_1), \varphi(V_2), \varphi(V_3), \dots \vdash \forall t \varphi(t)$, where $V_1, V_2, \dots, V_n, \dots$ is an infinite sequence of constants, which denotes universes like the NBG-universe (see [7]).

In connection with the logical difficulties of constructing a set-theoretical foundation for the whole naive category theory, different attempts to construct purely arrow-axiomatic foundations were undertaken (see Lawvere [24], Blanc and Preller [2], and Blanc and Donadieu [3]). But in attempts at an arrow-axiomatic description of indexed categories and fiber categories, their own logical difficulties appeared (see [16], 8.4).

Now let us consider more precisely what the formulation of MacLane’s problem mentioned above means. To do this, we need to have a strict definition of category. This definition became possible after the *elementary* (\equiv *first-order*) *category theories* T_c took shape. There are many such theories, two-sorted and one-sorted (see, for example, [27, I.3, I.8], [16, 8.2], [30, 2.3, 11.1]). But for every set theory S , there exist canonical one-to-one correspondences between the totalities of models of these theories in the theory S . Therefore, for the strict definition of a category one can use any elementary theory T_c . According to the definition of Lawvere [24] (see also [16, 8.2]), a *category in some set theory* S is any model of the theory T_c in the S theory (see 1.3 below). The *complete (partial) formalization of naive category theory* C in the *set theory* S is an adequate translation of all (some) notions and constructions of naive category theory C into strict notions and constructions for categories (as models of the theory T_c) in the set theory S .

Along with the notion of a category in the set theory S , there also exists the notion of *abstract category in* S . An *abstract category in the set theory* S is any abstract model of the theory T_c in the set theory S (see 7.1. below). The *complete (partial) abstract formalization of naive category theory* C in the *set theory* S is an adequate translation of all (some) notions and constructions of naive category theory C into strict notions and constructions for abstract categories (as abstract models of the theory T_c) in the S set theory.

According to these definitions abstract categories in the ZF set theory can be considered also on *classes* (as abstracts of the ZF theory), and abstract categories in the NBG set theory can be considered also on *assemblies* (as abstracts of the NBG theory). However, the complete abstract formalization of naive category theory C in any S set theory is impossible, because it is impossible to take abstracts of abstracts. Only a partial abstract formalization of C in S is possible. This means that the notions of an abstract category in S and a partial abstract formalization of C in S have only an auxiliary significance with respect to the notions of a category in S and the complete formalization of C in S .

Therefore, more precisely MacLane’s problem is understood as *constructing a set theory that admits a complete formalization of naive category theory* C in this set theory.

According to the definitions mentioned above, categories in the ZF set theory can be considered only on sets, but categories in the NBG set theory can be considered also on classes. But a complete formalization of naive category theory C in these set theories is impossible, because it is impossible to define the operations “category of categories” and “category of functors” (at least nowadays, we have no methods of approach to this formalization).

For this reason, Ehresman, Dedecker, Sonner, and Grothendieck proposed the idea of formalization of naive category theory C within the framework of a ZF+U set theory stronger than the ZF set theory with the axiom of universality U. Using the totality of universal sets, we can make a complete and adequate translation of all notions and constructions of naive theory C to the strict notions and constructions for categories in the ZF+U set theory. This also holds for the sets theories of da Costa.

For the other mentioned set theories such a complete translation is impossible. Therefore, the ZF+U set theory and the set theories of da Costa are the most adequate with respect to MacLane’s problem. Moreover, in virtue of the deficiency of da Costa’s set theories mentioned above, the ZF+U set theory is preferred.

However the ZF+U theory is too strong for a complete formalization of category theory in virtue of the redundancy of the totality of all universal sets, because for formalization of the operations “category

of categories” and “category of functors”, it is sufficient to have only a countable totality of universal sets as is done in da Costa’s axiomatics.

The $ZF+U(\omega)$ theory *with the axiom of ω -universality* $U(\omega)$ postulating the existence of an infinite totality of universal sets is weaker than $ZF+U$ and satisfies all the needs of the theory C . But it leaves behind the limits of categorical consideration such mathematical systems that are not elements of universal sets from this infinite totality.

Therefore, there arises a necessity to create a set theory S having an infinite totality U of some objects of the theory S , called *universes* and satisfying the following conditions:

- (1) S must be in some sense weaker than the redundant $ZF+U$ theory;
- (2) S must satisfy all the needs of the theory C to such an extent as the $ZF+U(\omega)$ theory does;
- (3) Contrary to $ZF+U(\omega)$, the theory S must have objects occurring outside of the totality of universes U .

In 2000, V. K. Zakharov proposed the *local* theory of sets (LTS) in the capacity of a more adequate foundation of category theory satisfying all these conditions (see announcements in [30] and [1]), and in 2003, he proposed its equiconsistent strengthening: the *locally*-minimal theory of sets (LMTS). The main idea of LTS and LMTS is that for the construction of a set theory with properties (1)–(3) *it is not necessary to assign a global set-theoretical structure, but it is sufficient to assign only local variants of this structure in each universe U* .

The local theory of sets tries to maintain all positive that is contained in the globally-internal concept of Ehresmann–Dedecker–Sonner–Grothendieck.

The locally-external ideology of LTS is that it is necessary to take the NBG-universe as the basic one and *externally duplicate its local copies*, therefore, to obtain some hierarchy of universes with the following properties:

- (1) every class belongs as a set to some universal class, which is the usual NBG-universe;
- (2) all subclasses of a given universal class are sets of any larger universal class (*property of value change*);
- (3) there exists the least universal class (\equiv the *infra-universe*) belonging to all other universal classes.

The first of these properties is similar to the axiom of universality mentioned above.

Thus, in the LTS, the notion of a large totality becomes relative: totalities that are “*large*” in one universe become “*small*” in any larger universe.

The first section of the given paper describes the structure of an arbitrary first-order theory (including LTS) and contains necessary notions from mathematical logic (see [28]).

In the second section, all proper axioms and axiom schemes of LTS are stated and the important set-theoretical constructions are defined.

In the third section, in LTS such key categorical constructions as the “category of categories” and the “category of functors” is formalized. As in the globally-internal concept here, all categorical notions and constructions are defined only within the framework of local NBG-universes. Therefore, the categories under consideration are said to be *local*.

In the fourth section, the proper axioms and axiom schemes of the ZF theory (Zermelo–Fraenkel with the choice axiom) are cited, the notions of ordinals, cardinals, and inaccessible cardinals are introduced, and properties of Mirimanov–von Neumann sets are studied.

In the fifth section, the notions of ordinals, cardinals, and inaccessible cardinals in LTS are introduced, Mirimanov–von Neumann classes are constructed, and the connection between the universal classes and the Mirimanov–von Neumann classes with indices that are inaccessible cardinals is stated. By means of this, it is proved that the assembly of all universal classes in the LTS is well-ordered with respect to the order by inclusion $U \subset V$, and also its structure is investigated. Here, for the assembly of all classes (\equiv *globalization*), we also define almost all local set-theoretical constructions, except for the most important construction of the full union, which is basic for the construction by transfinite induction.

In the sixth section, the relative consistency between LTS and the ZF theory with some additional axioms is stated. Finally, the independence of the introduced additional axioms and the undeducibility in the LTS of the global axiom scheme of replacement are stated.

For the reader's convenience, this paper contains all the necessary notions. Proofs are given in detail, so that they can be useful for young mathematicians.

The authors are very thankful to S. I. Adyan, N. K. Vereschagin, A. G. Vladimirov, P. V. Golubtsov, V. G. Kanovey, V. A. Lyubetskiy, N. N. Nepejvoda, M. A. Pentus, V. A. Uspensky, and A. D. Yashin for their interest in this work and for a series of valuable consultations.

1. First-Order Theories and LTS

1.1. Language of first-order theories. The proposed theory is a *first-order theory*. We will give the definition of a first-order theory based on [28].

Special symbols of every first-order theory T are the following symbols:

- parentheses $(,)$;
- connectives \Rightarrow ("implies") and \neg ("not");
- quantifier \forall (for all);

a countable set of variables v_i ($i \geq 0$) (in our case, the variables are denoted by the letters $x, X, y, Y, z, Z, u, U, v, V, w, W$, and also these letters with primes);

a nonempty countable set of predicate letters P_i^n ($n \geq 1, i \geq 0$) (LTS contains the unary predicate symbol \bowtie and the binary symbol \in);

a countable set of functional letters F_i^n ($n \geq 1, i \geq 0$) (in LTS, this set is empty);

finally, a countable set of constants a_i ($i \geq 0$) (in LTS, this set consists of symbols \emptyset and \mathbf{a}).

General symbols are symbols that are not special but are often used in mathematics. The special and general symbols compose the *initial alphabet*.

A *symbol-string* is defined by induction in the following way:

- (1) every symbol α of the initial alphabet, except for the blank-symbol, is a *symbol-string*;
- (2) if σ and ρ are symbol-strings, then $\sigma\rho$ and $\rho\sigma$ are *symbol-strings*.

A *designating* (\equiv *shortening*) *symbol-string* σ for a *symbol-string* ρ is introduced in the form of the symbol-string $\sigma \equiv \rho$ or $\rho \equiv \sigma$ (σ is a *designation* for ρ).

If a symbol-string ρ is a part of a symbol-string σ staying at one of the following three positions: $\dots\rho$, $\rho\dots$, $\dots\rho\dots$, then ρ is an *occurrence in* σ (\equiv *occurs in* σ).

A *text* is defined by induction in the following way:

- (1) every symbol-string σ is a *text*;
- (2) if Φ and Ψ are texts, then Φ_Ψ and Ψ_Φ are texts.

If a text Φ is a part of a text Σ staying at one of the following three positions: \dots_Φ , Φ_\dots , \dots_Φ_\dots , then Φ is an *occurrence in* Σ (\equiv Φ *occurs in* Σ).

Some symbol-strings constructed from the special symbols mentioned above are called *terms* and *formulas* of the first-order theory T .

Terms are defined in the following way:

- (1) a variable is a *term*;
- (2) a constant symbol is a *term*;
- (3) if F_i^n is an n -placed functional letter and t_0, \dots, t_{n-1} are terms, then $F_i^n(t_0, \dots, t_{n-1})$ is a *term*;
- (4) a symbol-string is a *term* if and only if it follows from rules (1)–(3).

In the LTS theory, because of lack of functional symbols, terms are constant symbols and variables.

If P_i^n is some n -placed predicate letter and t_0, \dots, t_{n-1} are terms, then the symbol-string

$$P_i^n(t_0, \dots, t_{n-1})$$

is called an *elementary formula*.

Formulas of the first-order theory T are defined in the following way:

- (1) every elementary formula is a *formula*;
- (2) if φ and ψ are formulas and v is a variable, then every symbol-string $(\neg\varphi)$, $(\varphi \Rightarrow \psi)$, and $\forall v(\varphi)$ is a *formula*;
- (3) a symbol-string is a *formula* if and only if it follows from rules (1) and (2).

Let introduce the following abbreviations:

- $(\varphi \wedge \psi)$ for $\neg(\varphi \Rightarrow \neg\psi)$;
- $(\varphi \vee \psi)$ for $(\neg\varphi) \Rightarrow \psi$;
- $(\varphi \equiv \psi)$ for $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$;
- $\exists v \varphi$ is an abbreviation for $(\neg(\forall v (\neg\varphi)))$.

Introduce the notion of *free* and *connected* occurrence of a variable in a formula. An occurrence of a variable v in a given formula is said to be *connected* if v is either a variable of the quantifier $\forall v$ occurring in this formula or is under the action of the quantifier $\forall v$ occurring in this formula; otherwise an occurrence of a variable in a given formula is said to be *free*. Thus, one variable can have free and connected occurrences in the same formula. A variable is said to be a *free (connected) variable* in a given formula if there exist free (connected) occurrences of this variable in this formula, i. e., a variable can be free and connected in one formula at the same time.

A *sentence* is a formula with no free variables.

If ζ is a term or a formula, θ is a term, and v is a variable, then $\zeta(v\|\theta)$ denotes a symbol-string obtained by replacing every free occurrence of the variable v in the symbol-string ζ by the symbol-string θ .

The substitution $v\|\theta$ in ζ is said to be *admissible* if for every free occurrence of a variable w in the symbol-string θ , every free occurrence v in ζ is not a free occurrence in some formula ψ occurring in some formulas $\forall w \psi(w)$ and $\exists w \psi(w)$ occurring in the symbol-string ζ .

In the sequel, if the substitution $v\|\theta$ in ζ is admissible, then, along with $\zeta(v\|\theta)$, we will write $\zeta(\theta)$.

If ζ is a term or a formula, θ is a term, and v is a variable such that the substitution $v\|\theta$ in ζ is admissible, then the substitution $\zeta(v\|\theta)$ is a term or a formula, respectively.

Every free occurrence of some variable u (except for v) in a symbol-string ζ and every free occurrence of some variable w in a symbol-string θ are free occurrences of these variables in the symbol-string $\zeta(v\|\theta)$.

1.2. Deducibility in a first-order theory. A symbol-string γ equipped with some rule is called a *formula scheme of a theory T* , if:

- (1) this rule marks some letters (in particular, free and connected variables) occurring in γ ;
- (2) this rule determines the necessary substitution of these marked letters in γ by some terms (particularly variables);
- (3) after every such a substitution in γ , some propositional formula φ of the theory T is obtained.

Every such a propositional formula φ is called a *propositional formula obtained by the application of the formula scheme γ* .

A text Γ consisting of symbol-strings separated by blank-symbols is called an *axiom text* if every symbol-string γ occurring in Γ is either a formula or a formula scheme of the theory T . If γ is a formula, then γ is called an *explicit axiom of the theory T* . If γ is a formula scheme, then it is called an *axiom scheme of the theory T* . Every formula obtained by the application of the axiom scheme γ is called an *implicit axiom of the theory T* .

Axioms and axiom schemes of every (in particular, the LTS) first-order theory are divided into two classes: *logical* and *proper* (or *mathematical*).

Logical axiom schemes of any first-order theory are cited below:

- LAS1.** $\varphi \Rightarrow (\psi \Rightarrow \varphi)$;
- LAS2.** $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$;
- LAS3.** $(\varphi \wedge \psi) \Rightarrow \varphi$;
- LAS4.** $(\varphi \wedge \psi) \Rightarrow \psi$;

LAS5. $\varphi \Rightarrow (\psi \Rightarrow (\varphi \wedge \psi))$;

LAS6. $\varphi \Rightarrow (\varphi \vee \psi)$;

LAS7. $\psi \Rightarrow (\varphi \vee \psi)$;

LAS8. $(\varphi \Rightarrow \chi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow ((\varphi \vee \psi) \Rightarrow \chi))$;

LAS9. $(\varphi \Rightarrow \psi) \Rightarrow ((\varphi \Rightarrow \neg\psi) \Rightarrow \neg\varphi)$;

LAS10. $(\neg(\neg\varphi)) \Rightarrow \varphi$;

LAS11. $(\forall v\varphi) \Rightarrow \varphi(v\|\theta)$ if v is a variable and θ is a term such that the substitution $v\|\theta$ in θ is admissible.

LAS12. $\varphi(v\|\theta) \Rightarrow (\exists v\varphi)$ under the same conditions as in **LAS11**;

LAS13. $(\forall v(\psi \Rightarrow \varphi(v))) \Rightarrow (\psi \Rightarrow (\forall v\varphi))$ if ψ does not contain a free variable v ;

LAS14. $(\forall v(\varphi(v) \Rightarrow \psi)) \Rightarrow ((\exists v\varphi) \Rightarrow \psi)$ under the same condition as in **LAS13**.

Proper axioms and axiom schemes cannot be formulated in the general case because they depend on a theory. The first-order theory which does not contain any proper axioms is called the *first-order predicate calculus*. Proper axioms and axiom schemes of LTS are stated in the next section.

The deduction rules in the first-order theory are as follows:

- (1) *the implication rule* (\equiv *modus ponens* (MP)): from φ and $\varphi \Rightarrow \psi$, it follows that ψ ;
- (2) *the generalization rule* (Gen): from φ , it follows that $\forall v\varphi$.

Let Φ be a totality of formulas, and let ψ be a formula of the theory T . A sequence $f \equiv (\varphi_i | i \in n+1) \equiv (\varphi_0, \dots, \varphi_n)$ of formulas of the theory T is called a *deduction of the formula ψ from the totality Φ* if $\varphi_n = \psi$, and for any $0 < i \leq n$, one of following conditions holds:

- (1) φ_i belongs to Φ ;
- (2) there exist $0 \leq k < j < i$ such that φ_j is $(\varphi_k \Rightarrow \varphi_i)$, i.e., φ_i is obtained from φ_k and $\varphi_k \Rightarrow \varphi_i$ by the implication rule MP;
- (3) there exists $0 \leq j < i$ such that φ_i is $\forall x\varphi_j$, where x is not a free variable of every formula from Φ , i.e., φ_i is obtained from φ_j by the generalization rule Gen with the given *structural requirement*.

Denote this deduction either by $f \equiv (\varphi_0, \dots, \varphi_n) : \Phi \vdash \psi$, or by $(\varphi_0, \dots, \varphi_n) : \Phi \vdash \psi$, or by $f : \Phi \vdash \psi$.

A totality Φ_a is called a *totality of axioms of the theory T* if Φ_a consists of all explicit proper axioms of the theory T , all implicit proper axioms of the theory T , and all implicit logical axioms of the predicate calculus. If there exists a deduction $f : \Phi_a \vdash \psi$, then the formula ψ is said to be *deducible in the axiomatic theory (T, Φ_a)* , and the deduction f is said to be a *proof of the formula ψ* .

A totality of formulas Φ is said to be *contradictory* (\equiv *non-consistent*) if every formula of the theory T is deducible from it. In the opposite case, Φ is said to be *noncontradictory* (\equiv *consistent*).

An axiomatic theory (T, Φ_a) is said to be *contradictory* (*noncontradictory*) if the totality of its axioms Φ_a is contradictory (noncontradictory).

Lemma 1. *A totality of formulas Φ is contradictory if and only if the formulas φ and $\neg\varphi$ for some sentence φ are deducible from Φ .*

Proof. If the totality Φ is contradictory, then every sentence of the theory T is deducible from it, in particular, φ and $\neg\varphi$ for an arbitrary sentence φ are deducible.

Now suppose that sentences φ and $\neg\varphi$ are deducible from the totality Φ and σ is an arbitrary formula. Let us show that the formula σ can be deduced from φ and $\neg\varphi$.

Actually, this is the following deduction:

1. $\varphi \Rightarrow (\neg\sigma \Rightarrow \varphi)$ (**LAS1**);
2. $\neg\varphi \Rightarrow (\neg\sigma \Rightarrow \neg\varphi)$ (**LAS1**);
3. φ ;
4. $\neg\varphi$;
5. $\neg\sigma \Rightarrow \varphi$ (MP, 1, and 3);
6. $(\neg\sigma \Rightarrow \varphi) \Rightarrow (\neg\sigma \Rightarrow \neg\varphi) \Rightarrow \neg(\neg\sigma)$ (**LAS9**);
7. $(\neg\sigma \Rightarrow \neg\varphi) \Rightarrow \neg(\neg\sigma)$ (MP, 5, and 6);

8. $\neg\sigma \Rightarrow \neg\varphi$ (MP, 2 and 4);
9. $\neg(\neg\sigma)$ (MP, 7, and 8)
10. $(\neg(\neg\sigma)) \Rightarrow \sigma$ (**LAS10**);
11. σ (MP, 9, and 10).

□

1.3. An interpretation of a first-order theory in a set theory. The consistency of first-order theories is often proved by the method of interpretations going back to Tarski (see [28]).

A first-order theory S is called a *set theory* if the binary predicate symbol \in belongs to the set of its predicates symbols. This symbol denotes *belonging* ($\in(x, y)$ is read as “ x belongs to y ,” “ x is an element of y ,” and so on.)

Let some object D be chosen by means of the set theory S . This chosen object D of the set theory S is said to be *equipped* if in S , for all $n \geq 1$, the notions of n -finite sequence $(x_i \in D | i \in n)$ of elements of the object D , n -placed relation $R \subset D^n$, n -placed operation $O : D^n \rightarrow D$, and also the notion of an infinite sequence x_0, \dots, x_q, \dots of elements of the object D are defined.

Let S be some fixed set theory with some fixed equipped object D .

An *interpretation of a first-order theory T in the set theory S with the equipped object D* is a pair M consisting of the object D and some correspondence I that assigns some n -placed operation to every predicate letter P_i^n relation $I(P_i^n)$ in D , to every functional letter F_i^n some n -placed operation $I(F_i^n)$ in D , and to every constant symbol a_i some element $I(a_i)$ of D .

Let s be an infinite sequence x_0, \dots, x_q, \dots of elements of the object D .

Define the *value of a term t of the theory T on the sequence s under the interpretation M of the theory T in the set theory S* (denoted by $t_M[s]$) by induction in the following way:

- if $t \equiv v_i$, then $t_M[s] \equiv x_i$;
- if $t \equiv a_i$, then $t_M[s] \equiv I(a_i)$;
- if $t \equiv F(t_0, \dots, t_{n-1})$, where F is an n -placed functional symbol and t_0, \dots, t_{n-1} are terms, then $t_M[s] \equiv I(F)(t_{0M}[s], \dots, t_{(n-1)M}[s])$.

Define the *translation of a formula φ on the sequence s under the interpretation M of the theory T in the set theory S* (denoted by $M \vDash \varphi[s]$) by induction in the following way:

- if $\varphi \equiv (P(t_0, \dots, t_{n-1}))$, where P is an n -placed predicate symbol and t_0, \dots, t_{n-1} are terms, then $M \vDash \varphi[s] \equiv ((t_{0M}[s], \dots, t_{(n-1)M}[s]) \in I(P))$;
- if $\varphi \equiv (\neg\theta)$, then $M \vDash \varphi[s] \equiv (\neg M \vDash \theta[s])$;
- if $\varphi \equiv (\theta_1 \Rightarrow \theta_2)$, then $M \vDash \varphi[s] \equiv (M \vDash \theta_1[s] \Rightarrow M \vDash \theta_2[s])$;
- if $\varphi \equiv (\forall v_i \theta)$, then $M \vDash \varphi[s] \equiv (\forall x(x \in D \Rightarrow M \vDash \theta[x_0, \dots, x_{i-1}, x, x_{i+1}, \dots, x_q, \dots]))$.

Using the abbreviations cited above, we also have the following:

- if $\varphi \equiv (\theta_1 \wedge \theta_2)$, then $M \vDash \varphi[s] \equiv (M \vDash \theta_1[s] \wedge M \vDash \theta_2[s])$;
- if $\varphi \equiv (\theta_1 \vee \theta_2)$, then $M \vDash \varphi[s] \equiv (M \vDash \theta_1[s] \vee M \vDash \theta_2[s])$;
- if $\varphi \equiv (\exists v_i \theta)$, then $M \vDash \varphi[s] \equiv (\exists x(x \in D \wedge M \vDash \theta[x_0, \dots, x_{i-1}, x, x_{i+1}, \dots, x_q, \dots]))$;
- if $\varphi \equiv (\theta_1 \Leftrightarrow \theta_2)$, then $M \vDash \varphi[s] \equiv (M \vDash \theta_1[s] \Leftrightarrow M \vDash \theta_2[s])$.

If, in the theory S , the symbol-string $\sigma(s) \equiv ((t_{0M}[s], \dots, t_{(n-1)M}[s]) \in I(P))$ is a formula of the theory S , then this definition implies that $M \vDash \varphi[s]$ is always a formula of the theory S .

Further, in this section we will consider the set theory S for which all symbol-strings $\sigma(s)$, are formulas of the theory S for every sequence s from D . All concrete set theories considered later in this paper will possess this property.

An interpretation M is called a *model of the axiomatic theory (T, Φ_a) in the axiomatic set theory (S, Ξ_a) with the chosen equipped object D* if for every sequence s from D , the translation $M \vDash \varphi[s]$ of every axiom φ of the theory T is a deducible formula in the theory (S, Ξ_a) .

Define now the *translation of the deduction $f \equiv (\varphi_0, \dots, \varphi_n) : \Phi \vdash \psi$ of the formula ψ from the totality Φ of formulas of the theory T on the sequence s under the interpretation M of the theory T in*

the set theory S in the form of the sequence $g \equiv (M \vDash \varphi_0[s], \dots, M \vDash \varphi_n[s])$, which is a D -bounded deduction of the formula $M \vDash \psi[s]$ from the totality $M \vDash \Phi[s] \equiv \{M \vDash \varphi[s] \mid \varphi \in \Phi\}$ in such a sense that the generalization rule $Gen \equiv \frac{\sigma}{\forall x \sigma}$ is used in the following D -bounded form:

$$Gen_D \equiv \frac{\sigma}{\forall x(x \in D \Rightarrow \sigma)},$$

where x and σ are a variable and a formula of the theory T , respectively.

Lemma 2. *A sequence $g \equiv (M \vDash \varphi_0[s], \dots, M \vDash \varphi_n[s])$ can be canonically extended to some sequence $g_{ext} \equiv (M \vDash \varphi_0[s], \dots, M \vDash \varphi_n[s])_{ext}$ so that $g_{ext} : M \vDash \Phi[s] \vdash M \vDash \psi[s]$, i.e., g_{ext} is a usual deduction of the formula $M \vDash \psi[s]$ from the totality $M \vDash \Phi[s]$.*

Proof. We will look through all j from 1 to n . Let φ_j be $\forall y \varphi_i$ for some $i < j$, where y is not a free variable of every formula φ of the totality Φ . The parameters of every formula $M \vDash \varphi[s]$ from $M \vDash \Phi[s]$ are only some members of the sequence s . Then $M \vDash \varphi_j[s]$ is $\forall x(x \in D \Rightarrow M \vDash \varphi_i[s])$, where x differs from all the parameters of the totality $M \vDash \Phi[s]$. To the right after the formula $M \vDash \varphi_i[s]$, we insert in g the explicit axiom

$$\xi \equiv (M \vDash \varphi_i(s) \Rightarrow (x \in D \Rightarrow M \vDash \varphi_i[s]))$$

obtained from the logical axiom scheme **LAS1**. Then applying the rule MP to the two previous formulas $M \vDash \varphi_i[s]$ and ξ , after ξ , we can insert in g the formula $\chi \equiv (x \in D \Rightarrow M \vDash \varphi_i[s])$. Then the formula $M \vDash \varphi_j[s] \equiv \forall x \chi$ is obtained as the usual application of the rule Gen to the formula χ . \square

This lemma implies that the translation of a deduction $f \equiv (\varphi_0, \dots, \varphi_n) : \Phi \vdash \psi$ leads to the deduction $g_{ext} \equiv (M \vDash \varphi_0[s], \dots, M \vDash \varphi_n[s])_{ext} : M \vDash \Phi(s) \vdash M \vDash \psi[s]$. This procedure is always used without any stipulations.

Lemma 3. *Let every formula from the translation $M \vDash \Phi[s]$ of a totality Φ of formulas of the theory T be deducible in the axiomatic theory (S, Ξ_a) from the totality Ξ_a of axioms of the theory S . Moreover, let $f \equiv (\varphi_0, \dots, \varphi_n) : \Phi \vdash \psi$. Then the deduction $g_{ext} : M \vDash \Phi(s) \vdash M \vDash \psi[s]$ can be extended to some deduction $h : \Xi_a \vdash M \vDash \psi[s]$ in the theory (S, Ξ_a) .*

Proof. Consider the deduction $g_{ext} : M \vDash \Phi[s] \vdash M \vDash \psi[s]$. Let $g_{ext} = (\xi_0, \dots, \xi_m)$. Every formula ξ_i of the theory S in this deduction is either one of the formulas of the totality $M \vDash \Phi[s]$ or follows from the previous formulas of this sequence as a result of application of one of the rules of deduction. First, we consider only ξ_i such that they do not follow from previous formulas of the deduction. These formulas belong to the totality $M \vDash \Phi[s]$. By the condition of the lemma, each of these formulas ξ_i is deduced in the theory (S, Ξ_a) from the totality Ξ_a of axioms of this theory, i.e., for each ξ_i , there exists a deduction $g_i \equiv (\eta_i^0, \dots, \eta_i^{k_i}) : (\Xi_a)_i \vDash \xi_i$, where $(\Xi_a)_i$ is some finite subtotaity of the totality Ξ_a . In the finite subtotaity $(\Xi_a)_0, \dots, (\Xi_a)_m$, change all free variables in such a way that they become different from those variables which were touched by the application of the generalization rule in the deduction g_{ext} . For $i \in m + 1$ such that ξ_i is a consequence of the previous formulas, we set $k_i \equiv 0$, $\eta_i^0 \equiv \xi_i$. Then $h \equiv (\eta_0^0, \dots, \eta_0^{k_0}, \eta_1^0, \dots, \eta_1^{k_1}, \dots, \eta_m^0, \dots, \eta_m^{k_m})$ is a deduction of the formula $M \vDash \psi[s]$ from the totality Ξ_a in the theory (S, Ξ_a) . \square

Lemma 4. *Let, for every sequence s from D , every formula from the translation $M \vDash \Phi_a[s]$ of axioms of the theory (T, Φ_a) be deduced in the theory (S, Ξ_a) from the totality Ξ_a of axioms of the theory (S, Ξ_a) , i.e., M is a model of the theory (T, Φ_a) in the theory (S, Ξ_a) . Under this condition, if the theory (S, Ξ_a) is consistent, then the theory (T, Φ_a) is also consistent.*

Proof. Suppose that the theory T is contradictory, i.e., there exist some formula ψ of the theory T and some deduction $f \equiv (\varphi_0, \dots, \varphi_n) : \Phi_a \vdash \psi \wedge \neg\psi$. On an arbitrary sequence s from D , consider its translation $g \equiv (M \vDash \varphi_0[s], \dots, M \vDash \varphi_n[s])$ and its canonical extension $g_{ext} : M \vDash \Phi_a[s] \vdash (M \vDash \psi[s] \wedge \neg M \vDash \psi[s])$ from Lemma 2. Then according to Lemma 3, there exists a deduction $h : \Xi_a \vdash (M \vDash \psi[s] \wedge \neg M \vDash \psi[s])$.

However, by the consistency of the theory S , such a deduction is impossible. Therefore, the theory T is consistent. \square

2. Local Theory of Sets

2.1. Proper axioms and axiom schemes of the local theory of sets. The local theory of sets is a first-order theory with two predicate symbols: the binary predicate symbol of *belonging* \in (write $A \in B$) and the unary predicate symbol of *universality* \bowtie (write $A \bowtie$), and also with two constants \emptyset (*empty class*) and \mathbf{a} (*infra-universe*).

Objects of the LTS are called *classes*.

The notation $\varphi(\vec{u})$ is used for the formula $\varphi(u_0, \dots, u_{n-1})$, where u_0, \dots, u_{n-1} are free variables of the formula φ .

For technical reasons, it is useful to consider the totality \mathbf{C} of all classes A satisfying a given formula $\varphi(x)$. This totality \mathbf{C} is called the *assembly defined by the formula* φ . The totality \mathbf{C} of all classes A satisfying a formula $\varphi(x, \vec{u})$, is called the *assembly defined by the formula* φ *through the parameter* \vec{u} . Along with these words, we will use the notation

$$A \in \mathbf{C} \equiv \varphi(A), \quad A \in \mathbf{C} \equiv \varphi(A, \vec{u})$$

and

$$\mathbf{C} \equiv \{x | \varphi(x)\}, \quad \mathbf{C} \equiv \{x | \varphi(x, \vec{u})\}.$$

If $\mathbf{C} \equiv \{x | \varphi(x)\}$ and φ contains only one free variable x , then the assembly \mathbf{C} is said to be *well-defined by the formula* φ . Assemblies will be usually denoted by semibold Latin letters.

Every class A can be considered as the assembly $\{x | x \in A\}$.

The *universal assembly* is the assembly of all classes $\overline{\mathbf{V}} \equiv \{x | x = x\}$.

An assembly $\mathbf{C} \equiv \{x | \varphi(x)\}$ is called a *subassembly of an assembly* $\mathbf{D} \equiv \{x | \psi(x)\}$ (denoted by $\mathbf{C} \subset \mathbf{D}$) if $\forall x(\varphi(x) \Rightarrow \psi(x))$. Assemblies \mathbf{C} and \mathbf{D} are said to be *equal* if $(\mathbf{C} \subset \mathbf{D}) \wedge (\mathbf{D} \subset \mathbf{C})$ (denoted by $\mathbf{C} = \mathbf{D}$).

We will use the notation $\{x \in A | \varphi(x)\} \equiv \{x | x \in A \wedge \varphi(x)\}$.

A1 (*extensionality axiom*).

$$\forall y \forall z ((y = z) \Rightarrow (\forall X (y \in X \Leftrightarrow z \in X))).$$

Let α be some fixed class. A class A will be called a *class of class* α ($\equiv \alpha$ -class) if $A \subset \alpha$. A class A is called a *set of class* α ($\equiv \alpha$ -set) if $A \in \alpha$.

A formula φ is said to be α -*predicative* (see [28], ch. 4, § 1) if for all variables x , all symbol-strings $\forall x$ and $\exists x$, occurring in the formula φ stay only at the following positions: $\forall x(x \in \alpha \Rightarrow \dots)$ and $\exists x(x \in \alpha \wedge \dots)$.

AS2 (*full comprehension axiom scheme*). Let $\varphi(x)$ be an X -predicative formula such that the substitution $\varphi(x|y)$ is admissible and φ does not contain Y as a free variable. Then

$$\forall X (\exists Y (\forall y ((y \in Y) \Leftrightarrow (y \in X \wedge \varphi(y))))).$$

This axiom scheme postulates the existence of α -classes, which will be denoted as the assemblies $\{x \in \alpha | \varphi(x)\}$.

Let A be a class, and let the assembly $\mathbf{C} \equiv \{x | \varphi(x)\}$ be defined by an A -predicative formula φ . If $\mathbf{C} \subset A$, then the assembly \mathbf{C} is a class. Actually, by **AS2**, there exists a class $B \equiv \{x \in A | \varphi(x)\}$. If $x \in \mathbf{C}$, then $\forall x(\varphi(x) \Rightarrow x \in A)$ implies $x \in B$. Therefore, $\mathbf{C} \subset B$. Conversely, if $x \in B$, then, by **AS2**, $x \in A \wedge \varphi(x)$, i.e., $x \in \mathbf{C}$. Hence $\mathbf{C} = B$.

A3 (*empty class axiom*).

$$\forall Z ((\forall x (x \notin Z)) \Leftrightarrow Z = \emptyset).$$

Lemma 1. $\forall X (\emptyset \subset X)$.

Proof. Denote the formulas $x \in \emptyset$ and $x \in X$ by φ and ψ , respectively. Apply the theorem $\neg\varphi \Rightarrow (\varphi \Rightarrow \psi)$. From **A3**, it follows that $\neg\varphi$. Hence, by the implication rule, it follows that $\varphi \Rightarrow \psi$. Applying the generalization rule, we obtain $\forall x (x \in \emptyset \Rightarrow x \in X)$. \square

A class α is said to be *universal* if $\alpha \bowtie$.

A4 (*equiuniversality axiom*).

$$\forall U \forall V ((U = V) \Rightarrow (U \bowtie \Leftrightarrow V \bowtie)).$$

This axiom postulates that equal classes are simultaneously universal or not universal.

A5 (*infra-universality axiom*).

$$\mathbf{a} \bowtie \wedge \forall U (U \bowtie \Rightarrow \mathbf{a} \subset U).$$

This axiom postulates that the class \mathbf{a} is the “smallest” universal class. We will call it *infra-universal* or the *infra-universe*.

A6 (*universality axiom*).

$$\forall X \exists U (U \bowtie \wedge X \in U).$$

This axiom postulates that every class A is an element of some universal class.

The following axioms explain what the notion “universality” means.

A7 (*transitivity axiom*).

$$\forall U (U \bowtie \Rightarrow \forall X (X \in U \Rightarrow X \subset U)).$$

This axiom postulates that if α is a universal class, then every α -set is an α -class.

A8 (*quasitransitivity axiom*).

$$\forall U (U \bowtie \Rightarrow \forall X \forall Y (X \in U \wedge Y \subset X \Rightarrow Y \in U)).$$

This axiom postulates that if α is a universal class, then every subclass of every α -set is an α -set.

Within the framework of every class α , we can define all basic set-theoretical constructions.

For every class A , the α -class $\mathcal{P}_\alpha(A) \equiv \{x \in \alpha \mid x \subset A\}$ is called the *full α -ensemble of the class A* .

A9 (*full ensemble axiom*).

$$\forall U (U \bowtie \Rightarrow \forall X (X \in U \Rightarrow \mathcal{P}_U(X) \in U)).$$

This axiom postulates that if α is a universal class and A is an α -set, then $\mathcal{P}_\alpha(A)$ is an α -set.

For classes A and B , the α -class $A \bigcup_\alpha B \equiv \{x \in \alpha \mid x \in A \vee x \in B\}$ is called the *α -union of the classes A and B* ; the α -class $A \bigcap_\alpha B \equiv \{x \in \alpha \mid x \in A \wedge x \in B\}$ is called the *α -intersection of the classes A and B* .

A10 (*binary union axiom*).

$$\forall U (U \bowtie \Rightarrow \forall X \forall Y (X \in U \wedge Y \in U \Rightarrow X \cup_U Y \in U)).$$

This axiom postulates that if α is a universal class, then the binary α -union of α -sets is an α -set. **A10** and **A8** imply that the same also holds for the binary α -intersection.

For a class A , consider the *solitary α -class* $\{A\}_\alpha \equiv \{x \in \alpha \mid x = A\}$.

Call the α -class

$$\{A, B\}_\alpha \equiv \{A\}_\alpha \bigcup_\alpha \{B\}_\alpha$$

the *unordered α -pair*, and the α -class $\langle A, B \rangle_\alpha \equiv \{\{A\}_\alpha, \{A, B\}_\alpha\}_\alpha$ the *coordinate α -pair of the classes A and B* .

Lemma 2. *Let α be a universal class, and let $a, b \in \alpha$. Then $\{a\}_\alpha$, $\{a, b\}_\alpha$, and $\langle a, b \rangle_\alpha$ are α -sets.*

Proof. If a is an α -set, then by **A9**, $\mathcal{P}_\alpha(a) \in \alpha$. From $\{a\}_\alpha \subset \mathcal{P}_\alpha(a)$, by **A8**, it follows that $\{a\}_\alpha$ is an α -set.

From **A10**, we now have $\{a, b\}_\alpha \in \alpha$. According to the proved property, this fact, together with $\{a\}_\alpha \in \alpha$, implies $\langle a, b \rangle_\alpha \in \alpha$. \square

Corollary. *Let α be a universal class, and let $a, a', b, b' \in \alpha$ and $\langle a, b \rangle_\alpha = \langle a', b' \rangle_\alpha$. Then $a = a'$ and $b = b'$.*

For classes A and B , the α -class

$$A *_{\alpha} B \equiv \{x \in \alpha \mid \exists y \exists z (y \in A \wedge z \in B \wedge x = \langle y, z \rangle_{\alpha})\}$$

will be called the *coordinate α -product of classes A and B* .

Lemma 3. *Let α be a universal class and $A, B \in \alpha$. Then $A *_{\alpha} B \in \alpha$.*

Proof. Let $a \in A$ and $b \in B$. Then $\{a\}_{\alpha} \subset A \cup_{\alpha} B$ and $\{b\}_{\alpha} \subset A \cup_{\alpha} B$ implies $\{a, b\}_{\alpha} \subset A \cup_{\alpha} B$. By **A10**, $A \cup_{\alpha} B \in \alpha$. According to Lemma 2, $\{a\}_{\alpha} \in \mathcal{P}_{\alpha}(A \cup_{\alpha} B)$ and $\{a, b\}_{\alpha} \in \mathcal{P}_{\alpha}(A \cup_{\alpha} B)$. For the same reason, $\langle a, b \rangle_{\alpha} = \{\{a\}_{\alpha}, \{a, b\}_{\alpha}\}_{\alpha} \subset \mathcal{P}_{\alpha}(A \cup_{\alpha} B)$. Hence $\langle a, b \rangle_{\alpha} \in \mathcal{P}_{\alpha}(\mathcal{P}_{\alpha}(A \cup_{\alpha} B))$. Therefore, $A *_{\alpha} B \subset \mathcal{P}_{\alpha}(\mathcal{P}_{\alpha}(A \cup_{\alpha} B)) \in \alpha$. By **A8**, we have $A *_{\alpha} B \in \alpha$. \square

Further, A and B will denote some fixed α -classes.

An α -subclass u of the α -class $A *_{\alpha} B$ will be called an α -correspondence from the α -class A into the α -class B and will be also denoted by $u : A \prec_{\alpha} B$. The formula $u \subset A *_{\alpha} B$ will be denoted also by $u \rightleftharpoons A \prec_{\alpha} B$. For the α -correspondence $u : A \prec_{\alpha} B$, consider the α -classes

$$\text{dom}_{\alpha} u \equiv \{x \in \alpha \mid x \in A \wedge ((\exists y (y \in B \wedge \langle x, y \rangle_{\alpha} \in u))\}$$

and

$$\text{rng}_{\alpha} u \equiv \{y \in \alpha \mid y \in B \wedge ((\exists x (x \in A \wedge \langle x, y \rangle_{\alpha} \in u))\}$$

The α -subclass $B_a \equiv u \langle a \rangle \equiv \{y \in \alpha \mid y \in B \wedge \langle a, y \rangle_{\alpha} \in u\}$ of the α -class B will be called the α -class of values of the α -correspondence u on the element $a \in A$, and the α -subclass $u[A'] \equiv \{y \in \alpha \mid y \in B \wedge (\exists x (x \in A' \wedge \langle x, y \rangle_{\alpha} \in u))\}$ of the class B the image of the subclass A' of the class A with respect to the α -correspondence u . It is clear that $u[\{a\}_{\alpha}] = u \langle a \rangle$ for each $a \in A$ and $u[A] = \text{rng}_{\alpha} u$.

If $u \langle a \rangle$ contains a single element $b \in B$ (in such a sense that $\exists y (y \in B \wedge u \langle a \rangle = \{y\}_{\alpha})$), then this single element b is called the *value of the α -correspondence u on the element $a \in A$* and is denoted by $u(a)$ or by b_a .

An α -correspondence u is said to be *total* if $\text{dom}_{\alpha} u = A$ and *single-valued* if $u \langle a \rangle = \{u(a)\}_{\alpha}$ for every $a \in \text{dom}_{\alpha} u$. A single-valued α -correspondence is also called an α -mapping ($\equiv \alpha$ -function).

A total single-valued α -correspondence $u : A \prec_{\alpha} B$ is called an α -mapping ($\equiv \alpha$ -function) from the α -class A into the α -class B and is denoted by $u : A \rightarrow_{\alpha} B$. The formula expressing the property for the α -class u to be an α -mapping from the α -class A into the α -class B will be denoted by $u \rightleftharpoons A \rightarrow_{\alpha} B$.

An α -mapping $u : A \rightarrow_{\alpha} B$ is said to be

- *injective* if $\forall x, y \in A (u(x) = u(y) \Rightarrow x = y)$ (this is denoted by $u : A \rightarrow_{\alpha} B$);
- *surjective* if $\text{rng}_{\alpha} u = B$ (this is denoted by $u : A \rightarrow_{\alpha} B$);
- *bijective* (\equiv *one-to-one*) if it is injective and surjective (this is denoted by $u : A \rightleftharpoons_{\alpha} B$).

The α -class $\{x \in \alpha \mid x \rightleftharpoons A \rightarrow_{\alpha} B\}$ of all α -mappings from the α -class A into the α -class B which are α -sets will be denoted by $B_{(\alpha)}^A$ or by $\text{Map}_{\alpha}(A, B)$.

A11 (*full union axiom*).

$$\begin{aligned} \forall U (U \bowtie \Rightarrow \forall X \forall Y \forall z (X \in U \wedge Y \subset U \wedge (z \subset X *_{U} Y) \\ \wedge (\forall x (x \in X \Rightarrow z \langle x \rangle \in U))) \Rightarrow (\text{rng}_U z \in U)). \end{aligned}$$

An α -correspondence u from α -class A into α -class B will be also called a (*multivalued*) α -collection of α -subclasses B_a of the α -class B , indexed by the α -class A . In this case, the class u and the formula $u \rightleftharpoons A \prec_{\alpha} B$ will be also denoted by $\langle B_a \subset B \mid a \in A \rangle_{\alpha}$ and $u \rightleftharpoons \langle B_a \subset B \mid a \in A \rangle_{\alpha}$, respectively. An α -mapping u from A into B will also be called a *simple α -collection of the elements b_{α} of the α -class B indexed by the α -class A* . In this case, the class u , the class rng_u and the formula $u \rightleftharpoons A \rightarrow_{\alpha} B$ are also denoted by $\langle b_a \in B \mid a \in A \rangle_{\alpha}$, $\{b_a \in B \mid a \in A\}$, and $u \rightleftharpoons \langle b_a \in B \mid a \in A \rangle_{\alpha}$, respectively.

The α -class $\{y \in \alpha \mid \exists x \in A (y \in B_x)\}$ is called the α -union of α -collection $\langle B_a \subset B \mid a \in A \rangle_{\alpha}$ and is denoted by $\cup_{\alpha} \langle B_a \subset B \mid a \in A \rangle_{\alpha}$. The α -class $\{y \in \alpha \mid \forall x \in A (y \in B_x)\}$ is called the α -intersection of α -collection $\langle B_a \subset B \mid a \in A \rangle_{\alpha}$ and is denoted by $\cap_{\alpha} \langle B_a \subset B \mid a \in A \rangle_{\alpha}$.

In these terms and notation, the axiom of full union means that if α is a universal class and $\langle B_a \subset B | a \in A \rangle_\alpha$ is an α -collection of α -subsets B_a of the α -class B , indexed by the α -set A , then its α -union $\cup_\alpha \langle B_a \subset B | a \in A \rangle_\alpha$ is an α -set.

With α -class A , in the canonical way, one associates the α -collection $\langle a \subset \alpha | a \in A \rangle_\alpha$ of one-element α -sets of the α -class A (according to Axiom **A7**, $a \in A \subset \alpha$ implies $a \subset \alpha$). The α -union of this α -collection $\langle a \subset \alpha | a \in A \rangle_\alpha$ is called the α -union ($\equiv \alpha$ -sum) of the α -class A and is denoted by $\cup_\alpha A$. If α is a universal class and A is an α -set, then $\cup_\alpha A$ is also an α -set. With every α -class, A , in the canonical way, one associates the simple α -collection $\langle a \in A | a \in A \rangle_\alpha$ of elements of the α -class A . It is clear that $\{a \in A | a \in A\}_\alpha = A$.

The next axiom serves, in particular, for exclusion of the possibility for a set to be its own element.

A12 (regularity axiom).

$$\forall U (U \bowtie \Rightarrow \forall X (X \subset U \wedge X \neq \emptyset \Rightarrow \exists x (x \in X \wedge x \cap_U X = \emptyset))).$$

The next axiom postulates the existence of an infinite set.

A13 (infra-infinity axiom).

$$\exists X (X \in \mathfrak{a} \wedge \emptyset \in X \wedge \forall x (x \in X \Rightarrow (x \cup_{\mathfrak{a}} \{x\}_{\mathfrak{a}} \in X))).$$

Denote the postulated \mathfrak{a} -set by π .

This axiom implies that the class \emptyset is an \mathfrak{a} -set. By Axiom **A5**, \emptyset is an α -set for every universe α .

Consider the \mathfrak{a} -class

$$\varkappa \equiv \{Y \in \mathfrak{a} \mid Y \subset \pi \wedge \emptyset \in Y \wedge \forall y (y \in Y \Rightarrow (y \subset \mathfrak{a} \wedge y \cup_{\mathfrak{a}} \{y\}_{\mathfrak{a}} \in Y))\}.$$

Since $\varkappa \subset \mathcal{P}_{\mathfrak{a}}(\pi)$, Axioms **A9** and **A8** imply that \varkappa is an α -set.

Consider the \mathfrak{a} -class $\omega \equiv \{y \in \mathfrak{a} \mid \forall Y (Y \in \varkappa \Rightarrow y \in Y)\}$. Since $\omega \subset \pi$, by Axiom **A8**, we infer that ω is an \mathfrak{a} -set. Call it the \mathfrak{a} -set of natural numbers. By Axiom **A5**, ω is an α -set for every universe α .

Consider the initial natural numbers $0 \equiv \emptyset$, $1 \equiv 0 \cup_{\mathfrak{a}} \{0\}_{\mathfrak{a}}$, $2 \equiv 1 \cup_{\mathfrak{a}} \{1\}_{\mathfrak{a}}$, \dots . From the definitions of \varkappa and ω , it follows that $0, 1, 2, \dots \in \omega$. By Axiom **A7**, $0, 1, 2, \dots \in \alpha$ for every universe α .

The last axiom postulates the existence of a choice function.

A14 (axiom of choice).

$$\forall U (U \bowtie \Rightarrow \forall X (X \in U \wedge X \neq \emptyset \Rightarrow \exists z ((z \Leftarrow \mathcal{P}_U(X) \setminus \{\emptyset\}_U \rightarrow_U X) \wedge \forall Y (Y \in \mathcal{P}_U(X) \setminus \{\emptyset\}_U \Rightarrow z(Y) \in Y))).$$

The description of mathematical axioms and axiom schemes is completed.

2.2. Some constructions in the local theory of sets. Almost all modern mathematics (except for the naive category theory and the naive theory of mathematical systems) can be formalized within the framework of the infra-universe \mathfrak{a} . Only the mentioned naive theories require the use of other higher universes.

To show that all naive category theories can be formalized within the framework of the local theory of sets, we need to introduce an analog of the coordinate α -pair $\langle A, B \rangle_\alpha$ also working with improper α -classes A and B (see the corollary of Lemma 2).

Now, let α be some fixed universal class.

Let A, A', A'', \dots be α -classes, where the prime symbol ($'$) is used only for the sake of uniformity of the notation.

The α -collection $\langle \alpha_i \subset \alpha \mid i \in 2 \rangle_\alpha$ such that $\alpha_0 \equiv A$ and $\alpha_1 \equiv A'$ will be called the (multivalued) sequential α -pair of α -classes A and A' and will be denoted by $\langle A, A' \rangle_\alpha$. The α -collection $\langle \alpha_i \subset \alpha \mid i \in 3 \rangle_\alpha$ such that $\alpha_0 \equiv A$, $\alpha_1 \equiv A'$, and $\alpha_2 \equiv A''$ will be called the (multivalued) sequential α -triplet of α -classes $A, A',$ and A'' and will be denoted by $\langle A, A', A'' \rangle_\alpha$. And so on.

Now let a, a', a'', \dots be α -sets.

The simple α -collection $\langle a_i \in \alpha \mid i \in 2 \rangle_\alpha$ such that $a_0 \equiv a$ and $a_1 \equiv a'$ will be called the simple sequential α -pair of α -sets a and a' and will be denoted by $\langle a, a' \rangle_\alpha$. The simple α -collection $\langle a_i \in \alpha \mid i \in 3 \rangle_\alpha$ such

that $a_0 \equiv a$, $a_1 \equiv a'$, and $a_2 \equiv a''$ will be called the *simple sequential α -triplet of α -sets a , a' , and a''* and will be denoted by $(a, a', a'')_\alpha$. And so on.

If A, A', B , and B' are α -classes and $\langle A, A' \rangle_\alpha = \langle B, B' \rangle_\alpha$, then $A = B$ and $A' = B'$. If a, a', b , and b' are α -sets and $(a, a')_\alpha = (b, b')_\alpha$, then $a = b$ and $a' = b'$. Similar properties also hold for every finite α -collections. Thus, the α -pairs $\langle A, A' \rangle_\alpha$ and $(a, a')_\alpha$ possess the mentioned property of the Kuratowski α -pair $\langle a, a' \rangle_\alpha$ (see the corollary of Lemma 2). However, in contrast to the latter, the α -pair $\langle A, A' \rangle_\alpha$ works also for improper α -classes. Multivalued sets were introduced in [31] and [32].

Let some α -collection $u \equiv \langle A_i \subset \alpha \mid i \in I \rangle_\alpha$ be indexed by α -class $I \neq \emptyset$. The α -class $\prod_\alpha \langle A_i \subset \alpha \mid i \in I \rangle_\alpha \equiv \{z \in \alpha \mid (z : I \rightarrow_\alpha \alpha) \wedge (\forall x (x \in I \Rightarrow z(x) \in A_x))\}$ will be called the *α -product of the α -collection u* . In a particular case, if A, A', A'', \dots are α -classes, then the α -classes $\prod_\alpha \langle A, A' \rangle_\alpha$, $\prod_\alpha \langle A, A', A'' \rangle_\alpha, \dots$ will be called the *α -product of the α -pair $\langle A, A' \rangle_\alpha$, the α -triplet $\langle A, A', A'' \rangle_\alpha, \dots$* and will be denoted by $A \times_\alpha A', A \times_\alpha A' \times_\alpha A'', \dots$

One can verify that $A \times_\alpha A' = \{x \in \alpha \mid (\exists y \exists y' (y \in A \wedge y' \in A' \wedge x = (y, y')_\alpha))\}$. It is seen from this equality that the α -product $A \times_\alpha A'$ is similar to the coordinate α -product $A *_\alpha A'$, but in contrast to the latter, it is a particular case of the general product $\prod_\alpha \langle A_i \subset \alpha \mid i \in I \rangle_\alpha$.

If $A = A' = A'' = \dots$, then $A \times_\alpha A = A_{(\alpha)}^2 \equiv \text{Map}_\alpha(2, A)$, $A \times_\alpha A \times_\alpha A = A_{(\alpha)}^3 \equiv \text{Map}_\alpha(3, A), \dots$ At the same time, $A *_\alpha A \neq A_{(\alpha)}^2$, and between the α -classes $A *_\alpha A$ and $A_{(\alpha)}^2$, there exists only a bijective α -mapping of the canonical form $\langle a, a' \rangle \mapsto (a, a')$. Precisely this stipulates the necessity of introducing the noncoordinate α -product $A \times_\alpha A', A \times_\alpha A' \times_\alpha A'', \dots$

If $n \in \omega$, then an α -subclass R of the α -class $A_{(\alpha)}^n \equiv \text{Map}_\alpha(n, A)$ is called *n -placed α -correspondence on α -class A* . The α -mapping $O : A_{(\alpha)}^n \rightarrow_\alpha A$ is called *n -placed α -operation on α -class A* . Note that $O \subset A_{(\alpha)}^n *_\alpha A \neq A_{(\alpha)}^{n+1}$. Therefore an n -placed operation O cannot be considered as an $(n + 1)$ -placed correspondence.

3. Formalization of the Naive Category Theory within the Framework of the Local Theory of Sets

3.1. Definition of a local category in the local theory of sets. By the naive notion of a category we mean the notion of *metacategory* given by MacLane in [27]. According to [27], a *metacategory* consists of *objects* a, b, c, \dots , *arrows* f, g, h, \dots , and four *operations* $f \mapsto \text{dom } f$, $f \mapsto \text{codom } f$, $a \mapsto i_a$, and $f, g \mapsto g \circ f$ satisfying some additional conditions. Unfortunately, even such a pathological object as the *metacategory of all metacategories* satisfies this definition.

The *problem of formalization of the naive category theory* appeared because of the internal contradictoriness of the notion of metacategory. The aim of such a formalization consists in that in some precisely described axiomatics (set-theoretical, arrow or mixed), it is necessary to give some strict definitions of some notion that, first, corresponds to the naive understanding of a category, second, is closed with respect to all operations and constructions of naive category theory, third, includes in itself all known important concrete examples of categories, and fourth, cuts off such naive pathological examples as the metacategory of all metacategories.

Using the notation of the previous section we can formalize the naive notion of a category in the following way. For a universe α , we define an α -category as a large two-sorted algebraic system with two relations and one operation (see [32] and [4]). For this purpose, we will use the notion of α -collection of α -classes and α -pair of α -classes introduced in the previous sections.

Consider a fixed α -set Ω_c , consisting of three elements of class α denoted by the signs $\#, \circ$, and \leftrightarrow and called the *symbol of partition*, the *symbol of composition*, and the *symbol of identification*, respectively. The set Ω_c is called the *signature of the category*. Since $\mathbf{a} \subset \alpha$, we infer that Ω_c is also an α -set for every universe α .

Fix some universe α .

Consider an α -pair $A \equiv \langle Obj, Arr \rangle_\alpha$ containing two α -classes Obj and Arr and the α -collection $s_c \equiv \langle \omega_A \subset \alpha \mid \omega \in \Omega_c \rangle_\alpha$ with the three components $\zeta \equiv s_c \langle \# \rangle \equiv \#_A$, $\eta \equiv s_c \langle \circ \rangle \equiv \circ_A$, and $\vartheta \equiv s_c \langle \leftrightarrow \rangle \equiv \leftrightarrow_A$.

An α -class $\mathcal{C} \equiv \langle A, s_c \rangle_\alpha$ will be called an α -category (\equiv a category of the class α) if the α -classes Obj , Arr , ζ , η , and ϑ occur in the sequential conjunction of the following formulas (written in an informal way):

P_c1. $(\zeta \subset (Obj \times_\alpha Obj) *_\alpha Arr) \wedge (\eta \subset (Arr \times_\alpha Arr) *_\alpha Arr) \wedge (\vartheta \subset Obj *_\alpha Arr)$;

This formula postulates that to every pair of elements of Obj , the partition assigns some α -class of elements of Arr ; to every pair of elements of Arr , the composition assigns some third element Arr , and to every element of Obj , the identification assigns some element of Arr .

P_c2. $(rng_\alpha \zeta = Arr) \wedge (\forall x, y \in Obj \times_\alpha Obj (x \neq y \Rightarrow \zeta \langle x \rangle \cap_\alpha \zeta \langle y \rangle = \emptyset))$; ζ is usually written in the form of the α -collection $\zeta \equiv \langle Arr(\pi, \varkappa) \subset Arr \mid (\pi, \varkappa)_\alpha \in Obj \times_\alpha Obj \rangle_\alpha$; in this notation, the indicated property means that the α -class Arr is equal to the α -union of this pairwise disjoint α -collection.

P_c3. $(dom_\alpha \eta = \{x \in \alpha \mid \exists u \exists v \exists w \exists v' \exists w' ((u, v, w \in Obj) \wedge (v', w' \in Arr) \wedge (v' \in Arr(u, v)) \wedge (w' \in Arr(v, w)) \wedge (x = (v', w')_\alpha))\}) \wedge (\eta : dom_\alpha \eta \rightarrow Arr)$;

P_c4. $(\vartheta : Obj \rightarrow_\alpha Arr) \wedge (\forall X \in Obj (\vartheta(X) \in Arr(X, X))) \wedge (\forall f \in Obj (\vartheta(f) \in Arr(f, f))$; ϑ is usually written in the form of the simple α -collection $\vartheta \equiv \langle i_\pi \in Arr \mid \pi \in Obj \rangle_\alpha$, where $i_\pi \equiv \vartheta(\pi)$;

P_c5. $\eta \langle Arr(\pi, \varkappa) \times_\alpha Arr(\varkappa, \rho) \rangle \subset Arr(\pi, \rho)$ for every elements $\pi, \varkappa, \rho \in Obj$;

P_c6. $\eta(\eta(F, G), H) = \eta(F, \eta(G, H))$ for every element $\pi, \varkappa, \rho, \sigma \in Obj$ and every element $F \in Arr(\pi, \varkappa)$, $G \in Arr(\varkappa, \rho)$ and $H \in Arr(\rho, \sigma)$;

P_c7. $\eta(F, i_\pi) = F$ and $\eta(i_\pi, G) = G$ for every element $\pi, \varkappa, \rho \in Obj$ and every element $F \in Arr(\varkappa, \pi)$ and $G \in Arr(\pi, \rho)$.

α -Categories defined in such a way are said to be *local*.

Elements F of the class $Arr(\pi, \varkappa)$ are called *arrows from the object π to the object \varkappa* . The formula $F \in Arr(\pi, \varkappa)$ is also denoted by $F : \pi \dashrightarrow \varkappa$. The correspondence η is called the *composition* and is usually denoted simply by \circ ; in this case, along with $\circ(F, G)$, we also write $G \circ F$.

An α -category \mathcal{C} is said to be *small* if \mathcal{C} is an α -set. An α -category \mathcal{C} is said to be *locally small* if every α -class $Arr(\pi, \varkappa)$ is an α -set.

Remark. In category theory the rejection of the term a “*morphism F from an object π to an object \varkappa* ” was due to the following reason. For algebraic systems U and V , the notion of a *homomorphism f from U to V* is usual; for smooth manifolds U and V the notion of a *diffeomorphism f from U to V* is usual and so on; the generalization of all these notions is the notion of a *morphism of mathematical systems* (see [4, 31, 32]). However mathematical systems U, V, \dots of type \mathfrak{C} and morphisms f from U to V of status \mathfrak{S} do not form a category, because the morphism f does not uniquely define the system V ; therefore, for some systems, the property $\langle U, V \rangle \neq \langle U', V' \rangle \Rightarrow Mor(U, V) \cap Mor(U', V') \neq \emptyset$ is possible. But this property contradicts the property P_c2. To form the corresponding category, it is necessary to take not morphisms f from U into V but triplets $\langle f, U, V \rangle$, which can be naturally called *arrows from the system U into the system V , defined by the morphisms f* (see. [27], I.8).

3.2. Functors and natural transformations and the “category of categories” and the “category of functors” in the local theory of sets generated by them. Let \mathcal{C} and \mathcal{D} be α -categories. An α -class $\Phi \equiv \langle \Phi_O, \Phi_T \rangle_\alpha$ is called a (*covariant*) α -*functor* (\equiv a functor of class α) from the α -category \mathcal{C} to the α -category \mathcal{D} if:

- (1) Φ_O is an α -mapping from the α -class $Obj\mathcal{C}$ into the α -class $Obj\mathcal{D}$;
- (2) Φ_T is an α -mapping from the α -class $Arr\mathcal{C}$ into the α -class $Arr\mathcal{D}$;
- (3) $\Phi_T(F) \in Arr_{\mathcal{D}}(\Phi_O(\pi), \Phi_O(\varkappa))$ for every object $\pi, \varkappa \in Obj\mathcal{C}$ and every arrow $F \in Arr_{\mathcal{C}}(\pi, \varkappa)$;
- (4) $\Phi_T(G \circ F) = \Phi_T(G) \circ \Phi_T(F)$ for every object $\pi, \varkappa, \rho \in Obj\mathcal{C}$ and every arrow $F \in Arr_{\mathcal{C}}(\pi, \varkappa)$ and $G \in Arr_{\mathcal{C}}(\varkappa, \rho)$;
- (5) $\Phi_T(i_\pi) = i_{\Phi_O(\pi)}$ for every object $\pi \in Obj\mathcal{C}$.

Usually α -mappings Φ_O and Φ_T are denoted by the same symbol Φ .

α -Functors are exactly homomorphisms between α -categories considered as algebraic systems.

The composition $\Psi \circ \Phi$ of an α -functor $\Phi \equiv \langle \Phi_O, \Phi_T \rangle_\alpha$ from \mathcal{C} to \mathcal{D} and an α -functor $\Psi \equiv \langle \Psi_O, \Psi_T \rangle_\alpha$ from \mathcal{D} to \mathcal{E} is the α -functor $\langle \Psi_O \circ \Phi_O, \Psi_T \circ \Phi_T \rangle_\alpha$ from \mathcal{C} to \mathcal{E} .

The identity α -functor $I_{\mathcal{C}}$ for the α -category \mathcal{C} is the α -functor $\langle id_{Obj\mathcal{C}}, id_{Arr\mathcal{C}} \rangle_\alpha$ containing the two identical mappings for the α -classes $Obj\mathcal{C}$ and $Arr\mathcal{C}$, respectively.

Now we will formalize the operation of the naive category theory known as the “category of categories.”

Take any universe β such that $\alpha \in \beta$.

Consider the β -class $Cat_\alpha^\beta \equiv \{X \in \beta \mid X \text{ is a } \alpha\text{-category}\}$ of all α -categories \mathcal{C} .

Also, consider the β -class ${}^f Arr_\alpha^\beta \equiv \{X \in \beta \mid \exists x \exists Y \exists Z ((Y, Z \text{ are } \alpha\text{-categories}) \wedge (x \text{ is an } \alpha\text{-functor from } Y \text{ to } Z) \wedge (X = \langle x, Y, Z \rangle_\alpha))\}$ of all α -functorial arrows $F \equiv \langle \Phi, \mathcal{C}, \mathcal{D} \rangle_\alpha$.

For every simple β -pair $(\mathcal{C}, \mathcal{D})_\beta$ of α -categories \mathcal{C} and \mathcal{D} , consider the β -class ${}^f Arr_\alpha^\beta(\mathcal{C}, \mathcal{D}) \equiv \{X \in \beta \mid \exists x \exists Y \exists Z ((Y, Z \text{ are } \alpha\text{-categories}) \wedge (x \text{ is an } \alpha\text{-functor from } Y \text{ to } Z) \wedge (X = \langle x, Y, Z \rangle_\alpha))\}$. Also, consider the β -collection $\zeta' \equiv \langle {}^f Arr_\alpha^\beta(\mathcal{C}, \mathcal{D}) \subset {}^f Arr_\alpha^\beta \mid (\mathcal{C}, \mathcal{D})_\beta \in Cat_\alpha^\beta \times_\beta Cat_\alpha^\beta \rangle_\beta$.

Consider the β -correspondence η' from ${}^f Arr_\alpha^\beta \times_\beta {}^f Arr_\alpha^\beta$ to ${}^f Arr_\alpha^\beta$ generated by the composition of α -functors and consider the β -mapping ϑ' from Cat_α^β to ${}^f Arr_\alpha^\beta$ such that $\vartheta'(\mathcal{C}) = \langle I_{\mathcal{C}}, \mathcal{C}, \mathcal{C} \rangle_\alpha$.

These β -classes give the opportunity to consider the β -correspondence s'_c from Ω_c into β such that $s'_c\langle \# \rangle \equiv \zeta'$, $s'_c\langle \circ \rangle \equiv \eta'$, and $s'_c\langle \leftrightarrow \rangle \equiv \vartheta'$.

As a result, we obtain the β -category $\mathbb{C}_\alpha^\beta \equiv \langle \langle Cat_\alpha^\beta, {}^f Arr_\alpha^\beta \rangle_\beta, s'_c \rangle_\beta$. It will be called the β -category of all α -categories and all α -functorial arrows between them.

Now let \mathcal{C} and \mathcal{D} be fixed α -categories. Suppose that Φ and Ψ are α -functors from \mathcal{C} to \mathcal{D} . A simple α -collection $T = (t_\pi \in Arr\mathcal{D} \mid \pi \in Obj\mathcal{C})_\alpha$ will be called a (natural) α -transformation from the α -functor Φ to the α -functor Ψ if:

- (1) $t_\pi \in Arr_{\mathcal{D}}(\Phi(\pi), \Psi(\pi))$ for every object π from $Obj\mathcal{C}$;
- (2) $\Psi(F) \circ t_\pi = t_{\mathcal{Z}} \circ \Phi(F)$ for every objects π, \mathcal{Z} from $Obj\mathcal{C}$ and every arrow F from $Arr_{\mathcal{C}}(\pi, \mathcal{Z})$.

The composition $U \circ T$ of an α -transformation $T = (t_\pi \in Arr\mathcal{D} \mid \pi \in Obj\mathcal{C})_\alpha$ from Φ to Ψ and an α -transformation $U = (u_\pi \in Arr\mathcal{D} \mid \pi \in Obj\mathcal{C})_\alpha$ from Ψ to Ω is the α -transformation $(u_\pi \circ t_\pi \in Arr\mathcal{D} \mid \pi \in Obj\mathcal{C})_\alpha$ from Φ to Ω .

The identity α -transformation I_Φ from the α -functor Φ to the α -functor Φ is the α -transformation $(I_{\Phi(\pi)} \in Arr\mathcal{D} \mid \pi \in Obj\mathcal{C})_\alpha$ from Φ to Φ .

Finally, we will formalize the operation of naive category theory known as the “category of functors.”

Consider the β -class $Funct_\alpha^\beta(\mathcal{C}, \mathcal{D})$ of all α -functors from the α -category \mathcal{C} to the α -category \mathcal{D} . Also, consider the β -class ${}^c Arr_\alpha^\beta \equiv \{X \in \beta \mid \exists x \exists Y \exists Z ((Y, Z \text{ are } \alpha\text{-functors from } \mathcal{C} \text{ to } \mathcal{D}) \wedge (x \text{ is an } \alpha\text{-transformation from } Y \text{ to } Z) \wedge (X = \langle x, Y, Z \rangle_\alpha))\}$ of all α -transformational arrows $F \equiv \langle T, \Phi, \Psi \rangle_\alpha$.

For every simple β -pair $(\Phi, \Psi)_\beta$ of α -functors Φ and Ψ from \mathcal{C} to \mathcal{D} , consider the β -class ${}^c Arr_\alpha^\beta(\Phi, \Psi) \equiv \{X \in \beta \mid \exists x ((x \text{ is an } \alpha\text{-transformation from } \Phi \text{ to } \Psi) \wedge (X = \langle x, \Phi, \Psi \rangle_\alpha))\}$. Consider the β -collection $\zeta'' \equiv \langle {}^c Arr_\alpha^\beta(\Phi, \Psi) \subset {}^c Arr_\alpha^\beta \mid (\Phi, \Psi)_\beta \in Funct_\alpha^\beta(\mathcal{C}, \mathcal{D}) \times_\beta Funct_\alpha^\beta(\mathcal{C}, \mathcal{D}) \rangle_\beta$.

Consider the β -correspondence η'' from ${}^c Arr_\alpha^\beta \times_\beta {}^c Arr_\alpha^\beta$ to ${}^c Arr_\alpha^\beta$ generated by the composition of α -transformations.

Also, consider the β -mapping ϑ'' from $Funct_\alpha^\beta(\mathcal{C}, \mathcal{D})$ to ${}^c Arr_\alpha^\beta$ such that $\vartheta''(\Phi) = \langle I_\Phi, \Phi, \Phi \rangle_\alpha$.

These β -classes give the opportunity to consider the β -correspondence s''_c from Ω_c into β such that $s''_c\langle \# \rangle \equiv \zeta''$, $s''_c\langle \circ \rangle \equiv \eta''$, and $s''_c\langle \leftrightarrow \rangle \equiv \vartheta''$.

As result, we obtain the β -category $\mathbb{F}_\alpha^\beta(\mathcal{C}, \mathcal{D}) \equiv \langle \langle Funct_\alpha^\beta(\mathcal{C}, \mathcal{D}), {}^c Arr_\alpha^\beta \rangle_\beta, s''_c \rangle_\beta$. It will be called the β -category of all α -functors from the α -category \mathcal{C} to the α -category \mathcal{D} and all α -transformational arrows between α -functors.

The constructions \mathbb{C}_α^β and $\mathbb{F}_\alpha^\beta(\mathcal{C}, \mathcal{D})$ show that the notion of α -category is closed with respect to such important operations of naive category theory as the “category of categories” and the “category of functors.” Thus, the notion of α -category has all the good properties of the notion of \mathcal{U} -category.

4. Zermelo–Fraenkel Set Theory and Mirimanov–von Neumann Sets

4.1. The proper axioms and axiom schemes of the ZF set theory. To begin with, we will cite the list of proper axioms and axiom schemes of the ZF theory (Zermelo-Fraenkel’s set theory with the axiom of choice; see [13, 19, 21, 22, 33]).

This theory is a first-order theory with two binary predicate symbols of *belonging* \in (we write $A \in B$) and *equality* $=$ (we write $A = B$).

The predicate of equality $=$ satisfies the following axiom and axiom scheme:

- (1) $\forall x(x = x)$ (*reflexivity of equality*);
- (2) $(x = y) \Rightarrow (\varphi(x, x) \Rightarrow \varphi(x, y))$ (*replacement of equals*), where x and y are variables, $\varphi(x, x)$ is an arbitrary formula, and $\varphi(x, y)$ is constructed from $\varphi(x, x)$ by changing some (not necessarily all) free occurrences of x by occurrences of y with the condition that y is free for occurrences of x that are changed.

Objects of the given theory are called *sets*.

As above, it is useful to consider the totality \mathbf{C} of all sets A satisfying a given formula $\varphi(x)$. The totality \mathbf{C} is called the *class (ZF), defined by the formula φ* . The totality $\mathbf{C}(\vec{u})$ of all sets A satisfying a formula $\varphi(x, \vec{u})$ is called the *class (ZF) defined by the formula φ through the parameter \vec{u}* . Along with these words, we will use the notation

$$A \in \mathbf{C} \equiv \varphi(A), \quad A \in \mathbf{C}(\vec{u}) \equiv \varphi(A, \vec{u})$$

and

$$\mathbf{C} \equiv \{x | \varphi(x)\}, \quad \mathbf{C}(\vec{u}) \equiv \{x | \varphi(x, \vec{u})\}.$$

If $\mathbf{C} \equiv \{x | \varphi(x)\}$ and φ contains only one free variable x , then the class \mathbf{C} is called *well-defined by the formula φ* .

Every set A can be considered as the class $\{x | x \in A\}$.

A class $\mathbf{C} \equiv \{x | \varphi(x)\}$ is called a *subclass of a class $\mathbf{D} \equiv \{x | \psi(x)\}$* (denoted by $\mathbf{C} \subset \mathbf{D}$) if $\forall x(\varphi(x) \Rightarrow \psi(x))$. Classes \mathbf{C} and \mathbf{D} are called *equal* if $(\mathbf{C} \subset \mathbf{D}) \wedge (\mathbf{D} \subset \mathbf{C})$.

Further, we will use the notation $\{x \in A | \varphi(x)\} \equiv \{x | x \in A \wedge \varphi(x)\}$.

If a class \mathbf{C} is not equal to any set, then \mathbf{C} is called a *proper class*. Not every class is a set: the class $\{x | x \notin x\}$ is a proper class.

The *universal class* is the class of all sets $\mathbf{V} \equiv \{x | x = x\}$.

For classes $\mathbf{C} \equiv \{x | \varphi(x)\}$ and $\mathbf{D} \equiv \{x | \psi(x)\}$ define the *binary union* $\mathbf{C} \cup \mathbf{D}$ and the *binary intersection* $\mathbf{C} \cap \mathbf{D}$ as the classes

$$\mathbf{C} \cup \mathbf{D} \equiv \{x | \varphi(x) \vee \psi(x)\} \text{ and } \mathbf{C} \cap \mathbf{D} \equiv \{x | \varphi(x) \wedge \psi(x)\}.$$

A1 (*extensionality axiom*).

$$\forall X \forall Y (\forall u (u \in X \Leftrightarrow u \in Y) \Rightarrow X = Y).$$

This axiom postulates that if two sets consist of the same elements, then they are equal.

For sets A and B define the *unordered pair* $\{A, B\}$ as the class

$$\{A, B\} \equiv \{z | z = A \vee z = B\}.$$

A2 (*pair axiom*).

$$\forall u \forall v \exists x \forall z (z \in x \Leftrightarrow z = u \vee z = v).$$

A2 and **A1** imply that an unordered pair of sets is a set.

For sets A and B define:

- the *solitary set* $\{A\} \equiv \{A, A\}$;
- the *ordered pair* $\langle A, B \rangle \equiv \{\{A\}, \{A, B\}\}$;

From the previous assertions, we infer that $\{A\}$ and $\langle A, B \rangle$ are sets.

Lemma 1. $\langle A, B \rangle = \langle A', B' \rangle$ if and only if $A = A'$ and $B = B'$.

AS3 (separation axiom scheme).

$$\forall X \exists Y \forall u (u \in Y \Leftrightarrow u \in X \wedge \varphi(u, \vec{p})),$$

where the formula $\varphi(u, \vec{p})$ does not contain Y as a free variable.

This axiom scheme postulates that the class $\{u | u \in X \wedge \varphi(u, \vec{p})\}$ is a set. This set is unique by **A1**. Actually, suppose that there exist some sets Y and Y' such that $\forall u (u \in Y \Leftrightarrow u \in X \wedge \varphi(u, \vec{p}))$ and $\forall u (u \in Y' \Leftrightarrow u \in X \wedge \varphi(u, \vec{p}))$. Then, by **LAS11**, $u \in Y \Leftrightarrow u \in X \wedge \varphi(u, \vec{p})$ and $u \in Y' \Leftrightarrow u \in X \wedge \varphi(u, \vec{p})$. Whence, by **LAS3** and **LAS4**, $u \in Y \Rightarrow u \in X \wedge \varphi(u, \vec{p})$, $u \in Y' \Rightarrow u \in X \wedge \varphi(u, \vec{p})$, $\varphi(u, \vec{p}) \wedge u \in X \Rightarrow u \in Y$, and $u \in X \wedge \varphi(u, \vec{p}) \Rightarrow u \in Y'$. Consequently, $u \in Y \Leftrightarrow u \in Y'$, and by the generalization Gen rule, $\forall u (u \in Y \Leftrightarrow u \in Y')$, whence, by **A1**, $Y = Y'$.

Consider the class $\mathbf{C} = \{u | \varphi(u, \vec{p})\}$. Then the scheme **AS3** can be expressed in the following form:

$$\forall X \exists Y (Y = \mathbf{C}(\vec{p}) \cap X).$$

For classes **A** and **B**, define the *difference* $A \setminus B$ as the class $A \setminus B \equiv \{x \in A | x \notin B\}$.

If A is a set, then, by **AS3**, the difference $A \setminus \mathbf{B}$ is a set.

Since $A \cap B = \{x \in A | x \in B\} \subset A$, by **AS3**, we infer that for any sets A and B , the binary intersection $A \cap B$ is a set.

For a class $\mathbf{C} \equiv \{x | \varphi(x)\}$ define the *union* $\cup \mathbf{C}$ as the class

$$\cup \mathbf{C} \equiv \{z | \exists x (\varphi(x) \wedge z \in x)\}.$$

A4 (union axiom).

$$\forall X \exists Y \forall u (u \in Y \Leftrightarrow \exists z (u \in z \wedge z \in X)) \wedge z \in X).$$

We can deduce from **A4** and **AS3** that for every set A , its union $\cup A$ is a set.

We have the equality $A \cup B = \cup \{A, B\}$. Therefore, for every set A and B , their binary union $A \cup B$ is a set.

We will call the *full ensemble* of a class \mathbf{C} the class $\mathcal{P}(\mathbf{C}) \equiv \{u | u \subseteq \mathbf{C}\}$.

A5 (power set (\equiv full ensemble) axiom).

$$\forall X \exists Y \forall u (u \in Y \Rightarrow u \subset X).$$

If A is a set, then, according to **A5** and **A1**, $\mathcal{P}(A)$ is a set.

For classes **A** and **B**, define the (*coordinate*) *product*

$$\mathbf{A} * \mathbf{B} \equiv \{x | \exists u \exists v (u \in \mathbf{A} \wedge v \in \mathbf{B} \wedge x = \langle u, v \rangle)\}.$$

The fact that $A * B$ is a set for sets A and B follows from **AS3**, because $A * B \subseteq \mathcal{P}\mathcal{P}(A \cup B)$.

A class (in particular, a set) \mathbf{C} is called a *correspondence* if

$$\forall u (u \in \mathbf{C} \Rightarrow \exists x \exists y (u = \langle x, y \rangle)).$$

For a correspondence \mathbf{C} , consider the following classes:

$$\text{dom } \mathbf{C} \equiv \{u | \exists v (\langle u, v \rangle \in \mathbf{C})\};$$

$$\text{rng } \mathbf{C} \equiv \{v | \exists u (\langle u, v \rangle \in \mathbf{C})\}.$$

If \mathbf{C} is a set, then $\text{dom } \mathbf{C} \subset \cup \cup \mathbf{C}$, by **A4** and **AS3**, implies that $\text{dom } \mathbf{C}$ also is a set.

A correspondence \mathbf{F} is called a *function* (\equiv a *mapping*) if

$$\forall x \forall y \forall y' (\langle x, y \rangle \in \mathbf{F} \wedge \langle x, y' \rangle \in \mathbf{F} \Rightarrow y = y').$$

For a class \mathbf{F} , the formula expressing the property to be a mapping will be denoted by $\text{func}(\mathbf{F})$. For the expression $\langle x, y \rangle \in \mathbf{F}$, we also use the notations $y = \mathbf{F}(x)$, $\mathbf{F} : x \mapsto y$, etc.

A correspondence \mathbf{C} is called a *correspondence from a class A into a class B* if $\text{dom } \mathbf{C} \subset \mathbf{A}$ and $\text{rng } \mathbf{C} \subset \mathbf{B}$ (denoted by $\mathbf{C} : \mathbf{A} \prec \mathbf{B}$). A function \mathbf{F} is called a *function from a class A into a class B* if $\text{dom } \mathbf{F} = \mathbf{A}$ and $\text{rng } \mathbf{F} \subset \mathbf{B}$ (denoted by $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$).

The class

$$\{f | \text{func}(f) \wedge \text{dom } f = A \wedge \text{rng } f \subseteq B\}$$

of all functions from a class \mathbf{A} into a class \mathbf{B} which are sets is denoted by $\mathbf{B}^{\mathbf{A}}$ or by $Map(\mathbf{A}, \mathbf{B})$. Since $B^A \subset \mathcal{P}(A * B)$, we infer that the class B^A is a set for any sets A and B .

The *restriction of the function \mathbf{F} to the class \mathbf{A}* is defined as the class

$$\mathbf{F}|\mathbf{A} \equiv \{x|\exists u\exists v(x = \langle u, v \rangle \wedge x \in \mathbf{F} \wedge u \in \mathbf{A})\}.$$

The *image* and the *inverse image* of the class \mathbf{A} with respect to the function \mathbf{F} are defined as the classes $\mathbf{F}[\mathbf{A}] \equiv \{v|\exists u \in \mathbf{A}(v = \mathbf{F}(u))\}$ and $\mathbf{F}^{-1}[\mathbf{A}] \equiv \{u|\mathbf{F}(u) \in \mathbf{A}\}$.

A correspondence \mathbf{C} from a class \mathbf{A} into a class \mathbf{B} is also called a (*multivalued*) *collection of subclasses* $\mathbf{B}_a \equiv \mathbf{C}(a) \equiv \{y|y \in \mathbf{B} \wedge \langle a, y \rangle \in \mathbf{C}\}$ of the class \mathbf{B} indexed by the class \mathbf{A} . In this case, the correspondence \mathbf{C} and the class *rng* \mathbf{C} are also denoted by $\langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$ and $\cup \langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$, respectively. The class $\cup \langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$ is also called the *union of the collection* $\langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$. The class $\{y|\forall x \in \mathbf{A}(y \in \mathbf{B}_x)\}$ is called the *intersection of the collection* $\langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$ and is denoted by $\cap \langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$. With every class \mathbf{A} , in a canonical way, one associates the *collection of element sets of the class \mathbf{A}* . For this collection, the equality $\cup \mathbf{A} = \cup \langle a \subset \mathbf{V} | a \in \mathbf{A} \rangle$ is holds.

A function \mathbf{F} from a class \mathbf{A} into the class \mathbf{B} is also called the *simple collection of elements* $b_a \equiv \mathbf{F}(a)$ of the class \mathbf{B} indexed by the class \mathbf{A} . In this case, the function \mathbf{F} and the class *rng* \mathbf{F} are also denoted by $\langle b_a \in \mathbf{B} | a \in \mathbf{A} \rangle$ and $\{b_a \in \mathbf{B} | a \in \mathbf{A}\}$, respectively. The collection $\langle b_a \in \mathbf{V} | a \in \mathbf{A} \rangle$ is also denoted by $\langle b_a | a \in \mathbf{A} \rangle$. With every class \mathbf{A} , in a canonical way, one associates the *simple collection of elements of the class \mathbf{A}* . It is clear that $\{a \in \mathbf{A} | a \in \mathbf{A}\} = \mathbf{A}$.

AS6 (*replacement axiom scheme*).

$$\begin{aligned} \forall x \forall y \forall y' (\varphi(x, y, \vec{p}) \wedge \varphi(x, y', \vec{p}) \Rightarrow y = y') \Rightarrow \\ \Rightarrow \forall X \exists Y \forall x \in X \forall y (\varphi(x, y, \vec{p}) \Rightarrow y \in Y), \end{aligned}$$

where the formula $\varphi(x, y, \vec{p})$ does not contain Y as a free variable.

To explain the essence of this axiom scheme, consider the class

$$\mathbf{F} \equiv \{u|\exists x \exists y (u = \langle x, y \rangle \wedge \varphi(x, y, \vec{p}))\}.$$

The premise in **AS6** states that \mathbf{F} is a function. Therefore, Scheme **AS6** can be expressed in the following way:

$$func(\mathbf{F}) \Rightarrow \forall X \exists Y (\mathbf{F}[X] \subseteq Y).$$

In other words, if \mathbf{F} is a function, then for every set X , the class $\mathbf{F}[X]$ is a set.

If A is a set, then by **AS6**, we infer that the class *rng* $\mathbf{F} \equiv \{b_a \in \mathbf{B} | a \in \mathbf{A}\}$ is a set. Then $\mathbf{F} \subset A \times \text{rng } \mathbf{F}$ implies that the class $\mathbf{F} \equiv \langle b_a \in \mathbf{B} | a \in \mathbf{A} \rangle$ is also a set. Therefore, if A is a set, then we use the notation $F : A \rightarrow \mathbf{B}$ and $F \equiv \langle b_a \in \mathbf{B} | a \in \mathbf{A} \rangle$.

A7 (*empty set axiom*).

$$\exists x \forall z (\neg(z \in x)).$$

Axiom **A1** implies that the set containing no elements is unique. It is denoted by \emptyset .

A8 (*infinity axiom*).

$$\exists Y (\emptyset \in Y \wedge \forall u (u \in Y \Rightarrow u \cup \{u\} \in Y)).$$

According to this axiom, there exists a set I containing $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$, and so on.

A9 (*regularity axiom*).

$$\forall X (X \neq \emptyset \Rightarrow \exists x (x \in X \wedge x \cap X = \emptyset)).$$

A function $f : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A$ is called a *choice function for the set A* , if $f(X) \in X$ for every $X \in \mathcal{P}(A) \setminus \{\emptyset\}$.

The last axiom postulates the existence of a choice function for every nonempty set.

A10 (*axiom of choice* (AC)).

$$\forall X(X \neq \emptyset \Rightarrow \exists z((z \rightleftharpoons \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X) \wedge \\ \wedge \forall Y(Y \in \mathcal{P}(X) \setminus \{\emptyset\} \Rightarrow z(Y) \in Y))).$$

The described first-order theory is called the *axiomatic set theory of Zermelo–Fraenkel* ZF (*with the choice axiom*) (see Zermelo [4] and Fraenkel [33]).

4.2. Ordinals and cardinals in the ZF set theory. In the same manner as was done in LTS, changing the word “ α -class” to the word “class” and the word “ α -set” to the word “set” in ZF, we define the (*multivalued*) *sequential pair* $\langle \mathbf{A}, \mathbf{A}' \rangle$, the *triplet* $\langle \mathbf{A}, \mathbf{A}', \mathbf{A}'' \rangle$, ... of *classes* $\mathbf{A}, \mathbf{A}', \mathbf{A}'', \dots$, the *simple sequential pair* (a, a') , the *triplet* (a, a', a'') , ... of the *sets* a, a', a'', \dots , the *product* $\prod \langle \mathbf{A}_i \subset A | i \in I \rangle$ of a *collection* $\langle \mathbf{A}_i \subset A | i \in I \rangle$, the *product* $\mathbf{A} \times \mathbf{A}', \mathbf{A} \times \mathbf{A}' \times \mathbf{A}'', \dots$ of the *pair* $\langle \mathbf{A}, \mathbf{A}' \rangle$, the *triplet* $\langle \mathbf{A}, \mathbf{A}', \mathbf{A}'' \rangle$, ..., of *classes* $\mathbf{A}, \mathbf{A}', \mathbf{A}'', \dots$, an *n-placed relation* $\mathbf{R} \subset \mathbf{A}^n \equiv \text{Map}(n, \mathbf{A})$ on the class \mathbf{A} , an *n-placed operation* $\mathbf{O} : \mathbf{A}^n \rightarrow \mathbf{A}$ on a class \mathbf{A} , and so on.

It can be verified that $\mathbf{A} \times \mathbf{A}' = \{x | \exists y \exists y' (y \in \mathbf{A} \wedge y' \in \mathbf{A}' \wedge x = (y, y'))\}$ and $\mathbf{A}^2 \neq \mathbf{A} * \mathbf{A}$.

A class \mathbf{P} is said to be *ordered by a binary relation \leq on \mathbf{P}* if:

- (1) $\forall p \in \mathbf{P} (p \leq p)$;
- (2) $\forall p, q \in \mathbf{P} (p \leq q \wedge q \leq p \Rightarrow p = q)$;
- (3) $\forall p, q, r \in \mathbf{P} (p \leq q \wedge q \leq r \Rightarrow p \leq r)$.

If, in addition,

- (4) $\forall p, q \in \mathbf{P} (p \leq q \vee q \leq p)$,

then the relation \leq is called the *linear order* on the class \mathbf{P} .

An ordered class \mathbf{P} is said to be *well-ordered* if

- (5) $\forall Q (\emptyset \neq Q \subseteq \mathbf{P} \Rightarrow \exists x \in Q (\forall y \in Q (x \leq y)))$, i.e., every nonempty subset of the class \mathbf{P} has the smallest element.

Let \mathbf{P} and \mathbf{Q} be ordered classes. A mapping $\mathbf{F} : \mathbf{P} \rightarrow \mathbf{Q}$ is said to be *monotone* (\equiv *increasing, order preserving*) if $p \leq p'$ implies $\mathbf{F}(p) \leq \mathbf{F}(p')$. The mapping \mathbf{F} is said to be *strictly monotone* if $p < p'$ implies $\mathbf{F}(p) < \mathbf{F}(p')$. The mapping \mathbf{F} is said to be *isotone* if it is monotone and $\mathbf{F}(p) \leq \mathbf{F}(p')$ implies $p \leq p'$. It can be verified that:

- (1) if \mathbf{F} is isotone, then \mathbf{F} is injective and strictly monotone;
- (2) if \mathbf{F} is isotone and surjective, then \mathbf{F} is bijective and the inverse mapping $\mathbf{F}^{-1} : \mathbf{Q} \rightarrow \mathbf{P}$ is also isotone.

Ordered classes $\langle \mathbf{P}, \leq \rangle$ and $\langle \mathbf{Q}, \leq \rangle$ are said to be *order equivalent* (write $\langle \mathbf{P}, \leq \rangle \approx \langle \mathbf{Q}, \leq \rangle$) if there exists some isotone bijective mapping $\mathbf{F} : \mathbf{P} \rightarrow \mathbf{Q}$.

If a class \mathbf{P} is ordered by a relation \leq and \mathbf{A} is a nonempty subclass of the class \mathbf{P} , then an element $p \in \mathbf{P}$ is said to be the *smallest upper bound* or the *supremum of the subclass \mathbf{A}* if $\forall x \in \mathbf{A} (x \leq p) \wedge \forall y \in \mathbf{P} ((\forall x' \in \mathbf{A} (x' \leq y)) \Rightarrow p \leq y)$. This formula is denoted by $p = \sup \mathbf{A}$.

A class \mathbf{S} is said to be *transitive* if $\forall x (x \in \mathbf{S} \Rightarrow x \subseteq \mathbf{S})$.

A class (a set) \mathbf{S} is called an *ordinal* (*ordinal number*) if \mathbf{S} is transitive and well-ordered by the relation $\in \cup =$ on \mathbf{S} . The property of a class \mathbf{S} to be an ordinal will be denoted by $\text{On}(\mathbf{S})$.

In the form of the formula, this is

$$\text{On}(\mathbf{S}) \equiv \forall x (x \in \mathbf{S} \Rightarrow x \subseteq \mathbf{S}) \wedge \forall x, y, z (x \in \mathbf{S} \wedge y \in \mathbf{S} \wedge z \in \mathbf{S} \wedge x \in y \wedge y \in z \Rightarrow x \in z) \\ \wedge \forall x, y (x \in \mathbf{S} \wedge y \in \mathbf{S} \Rightarrow x \in y \vee x = y \vee y \in x) \wedge \forall T (\emptyset \neq T \subseteq \mathbf{S} \Rightarrow \exists x (x \in T \wedge \forall y (y \in T \Rightarrow x \in y))).$$

Ordinal numbers are usually denoted by Greek letters α, β, γ , and so on. The class of all ordinal numbers is denoted by \mathbf{On} . The natural ordering of the class of ordinal numbers is the relation $\alpha \leq \beta \equiv \alpha = \beta \vee \alpha \in \beta$. The class \mathbf{On} is transitive and linearly ordered by the relation $\in \cup =$.

There are some simple assertions about ordinal numbers:

- (1) if α is an ordinal number, A is a set, and $A \in \alpha$, then A is an ordinal number;

- (2) $\alpha = \{\beta | \beta \in \alpha\}$ for every ordinal number α ;
- (3) $\alpha + 1 \equiv \alpha \cup \{\alpha\}$ is the smallest of all ordinal numbers that are greater than α ;
- (4) every nonempty set of ordinal numbers has the smallest element.

Therefore, the ordered class **On** is well-ordered. Thus, **On** is an ordinal.

Lemma 2. *Let **A** be a nonempty subclass of the class **On**. Then **A** has the smallest element.*

Proof. By condition, there exists some ordinal number $\alpha \in \mathbf{A}$. Consider the class $\mathbf{B} \equiv \{x | x \in \mathbf{A} \wedge x \in \alpha + 1\}$. By separation axiom Scheme **AS3** (ZF) this class is a set. Since $\alpha \in \mathbf{B} \subset \mathbf{On}$ and the class **On** is well-ordered, the set **B** has the smallest element β . Take an arbitrary element $\gamma \in \mathbf{A}$. If $\gamma \leq \alpha$, then $\gamma \in \mathbf{B}$, and, therefore, $\gamma \geq \beta$. If $\gamma > \alpha$, then $\gamma > \beta$. Thus, β is the smallest element of the class **A**. \square

Lemma 3. *If **A** is a nonempty set of ordinal numbers, then:*

- (1) *the class $\cup A$ is an ordinal number;*
- (2) *$\cup A = \sup A$ in the ordered class **On**.*

Proof. (1) The set $\cup A$ is transitive. In fact, if $x \in y \in \cup A$, then $y \in \alpha \in A$ for some ordinal α . By the transitivity of α , we obtain $x \in \alpha$, whence $x \in \cup A$. It is clear that the set $\cup A$ is well-ordered by the relation $\in \cup =$.

- (2) Let us show that $p \equiv \cup A$ satisfies the formula $p = \sup A$. First, p is an ordinal number. Second, assume that there is $x \in A$ such that $p < x$, i.e., $p \in x$. Since $p \in x$ and $x \in A$, we infer that $p \in \cup A \equiv p$, but this is impossible. Therefore, $\forall x \in A(x \leq p)$. Third, let $\exists y \in \mathbf{On}((\forall x' \in A(x' \leq y)) \wedge y \in p)$. Since $y \in p$, we infer that $\exists x \in A(y \in x)$, but it contradicts $\forall x' \in A(x' \leq y)$. Therefore, $p = \sup A$. \square

Corollary. *The class **On** is a proper class.*

An ordinal number α is said to be *successive* if $\alpha = \beta + 1$ for some ordinal number β . In the opposite case α is said to be *limiting*. This unique number β is denoted by $\alpha - 1$. For an ordinal number α , the formula expressing the property of being successive (limiting) will be denoted by $Son(\alpha)$ [$Lon(\alpha)$].

Lemma 4. *An ordinal number α is limiting if and only if $\alpha = \sup \alpha$.*

The smallest (in the class **On**) nonzero limiting ordinal is denoted by ω . The existence of such an ordinal follows from **A7**, **AS6**, and **AS3**. Ordinals which are smaller than ω are called *natural numbers*.

Collections $(\mathbf{B}_n \subset \mathbf{B} | n \in N \subset \omega)$ and $(b_n \in \mathbf{B} | n \in N \subset \omega)$, where N is an arbitrary subset of ω , are called *sequences*. If $N \subset n \in \omega$, then these collections are said to be *finite*; otherwise, they are said to be *infinite*.

Theorem 1 (principle of transfinite induction). *Let **C** be a class of ordinal numbers such that:*

- (1) $\emptyset \in \mathbf{C}$;
- (2) $\alpha \in \mathbf{C} \Rightarrow \alpha + 1 \in \mathbf{C}$;
- (3) $(\alpha \text{ is a limiting ordinal number} \wedge \alpha \subset \mathbf{C}) \Rightarrow S \in \mathbf{C}$.

Then $\mathbf{C} = \mathbf{On}$.

Proof. Let this be not true. Consider the subclass $\mathbf{D} \equiv \mathbf{On} \setminus \mathbf{C}$. The class **D** is nonempty, and, therefore, according to Lemma 2, it has the smallest element γ . Now $\gamma \neq \emptyset$, because $\emptyset \in \mathbf{C}$. Thus, γ is either a successive ordinal number, or limiting. Suppose that $\gamma = \gamma' + 1$. Since $\gamma' \in \gamma$, it follows that $\gamma' \notin \mathbf{D}$ and, therefore, $\gamma' \in \mathbf{C}$. Then, by condition (2) of the theorem, $\gamma = \gamma' + 1 \in \mathbf{C}$. Suppose that γ is a limiting ordinal number. Then $\gamma \subset \mathbf{C}$ and, by condition (3) of the theorem, $\gamma \in \mathbf{C}$. In both cases, we come to the contradiction with the fact that $\gamma \notin \mathbf{C}$. Therefore, $\mathbf{C} = \mathbf{On}$. \square

Theorem 2 (construction by transfinite induction). *For every function $\mathbf{G} : \mathbf{V} \rightarrow \mathbf{V}$, there exists a unique function $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{V}$ such that for every $\alpha \in \mathbf{On}$ the equality*

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|_\alpha)$$

holds.

Proof. Consider the class $\mathbf{C} \equiv \{f | \text{func}(f) \wedge \text{On}(\text{dom } f) \wedge \forall x \in \text{dom } f (f(x) = \mathbf{G}(f|x))\}$.

Take any function $f, g \in \mathbf{C}$ and consider the numbers $\alpha \equiv \text{dom } f$ and $\beta \equiv \text{dom } g$. Let $\alpha \subset \beta$. If $\alpha = 0$, then $f = \emptyset \subset g$. If $\alpha \neq 0$, then consider the set $P \equiv \{x \in \alpha | f(x) \neq g(x)\}$. Suppose that the set P is nonempty. Then P contains the smallest element π . Since $f(0) = \mathbf{G}(f|\emptyset) = \mathbf{G}(\emptyset) = \mathbf{G}(g|\emptyset) = g(0)$, we infer that $\pi \neq 0$. By definition, for every $x \in \pi$, we have $f(x) = g(x)$, whence $f|_\pi = g|_\pi$. This implies $f(\pi) = \mathbf{G}(f|\pi) = \mathbf{G}(g|\pi) = g(\pi)$. But it follows from $\pi \in P$ that $f(\pi) \neq g(\pi)$. From this contradiction, we infer that the set P is empty. Therefore, $f \subset g$. Thus, we have proved that $\alpha \subset \beta$ implies $f \subset g$. It follows from this property that $\alpha = \beta$ implies $f = g$.

Consider the correspondence $\mathbf{E} \equiv \{z | \exists \alpha \in \mathbf{On} \exists f \in \mathbf{C} (\alpha = \text{dom } f \wedge z = \langle \alpha, f \rangle)\}$. From what was proved above, we infer that \mathbf{E} is a mapping from the class $\mathbf{D} \equiv \text{dom } \mathbf{E}$ into the class \mathbf{V} . We will consider this mapping in the form of a simple collection $\mathbf{E} \equiv (f_\alpha \in \mathbf{C} | \alpha \in \mathbf{D})$. As was proved above, $\alpha \subset \beta$ implies $f_\alpha \subset f_\beta$.

Let us prove that $\mathbf{D} = \mathbf{On}$ by transfinite induction.

Since $\emptyset \in \mathbf{C}$ and $\langle 0, \emptyset \rangle \in \mathbf{E}$, we infer that $0 \in \mathbf{D}$.

Let $\alpha \in \mathbf{D}$. Using the function f_α , we define the function $g : \alpha + 1 \rightarrow \mathbf{V}$ setting $g|_\alpha \equiv f_\alpha$ and $f(\alpha) \equiv \mathbf{G}(f_\alpha)$. From $\alpha \cap \{\alpha\} = \emptyset$, $\alpha + 1 = \alpha \cup \{\alpha\}$, $g = f_\alpha \cup \{\langle \alpha, \mathbf{G}(f_\alpha) \rangle\}$, and the union axiom **A4**, it follows that this definition is correct. Let $x \in \alpha + 1$. If $x \in \alpha$, then $g(x) = f_\alpha(x) = \mathbf{G}(f_\alpha|x)$. By the transitivity, $x \subset \alpha = \text{dom } f_\alpha$. Therefore, $f_\alpha|x = g|x$ implies $g(x) = \mathbf{G}(g|x)$. If $x \in \{\alpha\}$, then $g(x) \equiv \mathbf{G}(f_\alpha) = \mathbf{G}(g|x)$. Consequently, $g \in \mathbf{C}$ and $\langle \alpha + 1, g \rangle \in \mathbf{E}$ implies $\alpha + 1 \in \mathbf{D}$.

Let α be a limiting ordinal number, and let $\alpha \subset \mathbf{D}$. By Lemmas 4 and 3, $\alpha = \sup \alpha = \cup \alpha$. Let $x \in \alpha$. Then $x \in y$ for some $y \in \alpha \subset \mathbf{D}$. Let $x \in z \in \alpha$. If $y = z$, then $f_y(x) = f_z(x)$. If $y \in z$, then by of the imbedding $f_y \subset f_z$, $y \subset z$ implies $f_y(x) = f_z(x)$. If $z \in y$, then in a similar way, $f_z(x) = f_y(x)$.

Define the function $g : \alpha \rightarrow \mathbf{V}$ setting $g(x) \equiv f_y(x)$ for any $y \in \alpha$ such that $x \in y$. It is clear that $g|_y = f_y$. From $\alpha = \{y | y \in \alpha\}$, $g = \cup \{f_y | y \in \alpha\}$, the replacement axiom scheme **AS6** and the union axiom **A4**, it follows that this definition is correct. Let us verify that $g \in \mathbf{C}$.

Let $x \in \text{dom } g = \alpha$. Then $x \in y \in \alpha$ implies $g(x) = f_y(x) = \mathbf{G}(f_y|x) = \mathbf{G}(g|x)$, since $x \subset y \subset \alpha$. Now from $g \in \mathbf{C}$ and $\alpha = \text{dom } g$, it follows that $\alpha \in \mathbf{D}$.

By the principle of transfinite induction, we conclude that $\mathbf{D} = \mathbf{On}$.

Let $x \in \mathbf{On}$. Then $x \in x + 1 \equiv \alpha$. Let $x \in \beta$ and $x \in \gamma$ for some $\beta, \gamma \in \mathbf{On}$. Since $\alpha \subset \beta$, $\alpha \subset \gamma$, $f_\alpha \subset f_\beta$, and $f_\alpha \subset f_\gamma$, we infer that $f_\beta(x) = f_\alpha(x) = f_\gamma(x)$. It follows that we can correctly define a function $\mathbf{F} : \mathbf{On} \rightarrow \mathbf{V}$ setting $\mathbf{F}(x) \equiv f_\beta(x)$ for every $\beta \in \mathbf{On}$ such that $x \in \beta$. It is clear that $\mathbf{F}|_\beta = f_\beta \in \mathbf{V}$ for every $\beta \in \mathbf{On}$.

If $x \in \mathbf{On}$, then $x \in x + 1 \equiv \alpha$ and $x \subset \alpha$ implies $\mathbf{F}(x) = f_\alpha(x) = \mathbf{G}(f_\alpha|x) = \mathbf{G}(\mathbf{F}|_x)$.

It remains to show that the function \mathbf{F} is unique. Assume that there is a function $\mathbf{F}' : \mathbf{On} \rightarrow \mathbf{V}$ such that $\mathbf{F}'(\alpha) = \mathbf{G}(\mathbf{F}'|_\alpha)$ for any $\alpha \in \mathbf{On}$. Note that by the replacement axiom scheme **AS6**, $\mathbf{F}'|_\alpha \in \mathbf{V}$ for any $\alpha \in \mathbf{On}$. Consider the class $\mathbf{A} \equiv \{\alpha \in \mathbf{On} | \mathbf{F}(\alpha) = \mathbf{F}'(\alpha)\}$. Since $\mathbf{F}(0) = \mathbf{G}(\mathbf{F}|\emptyset) = \mathbf{G}(\emptyset) = \mathbf{G}(\mathbf{F}'|\emptyset) = \mathbf{F}'(0)$, we infer that $0 \in \mathbf{A}$, i.e., this class is nonempty. Assume that the class $\mathbf{B} \equiv \mathbf{On} \setminus \mathbf{A}$ is nonempty. Then it contains the smallest element β .

If $\alpha \in \beta$, then $\alpha \in \mathbf{A}$ implies $\mathbf{F}(\alpha) = \mathbf{F}'(\alpha)$. So $\mathbf{F}|_\beta = \mathbf{F}'|_\beta$. Hence we obtain $\mathbf{F}(\beta) = \mathbf{G}(\mathbf{F}|_\beta) = \mathbf{G}(\mathbf{F}'|_\beta) = \mathbf{F}'(\beta)$, but this contradicts the inequality $\mathbf{F}(\beta) \neq \mathbf{F}'(\beta)$. It follows from the obtained contradiction that $\mathbf{A} = \mathbf{On}$. \square

In ZF, there exists the following principle of \in -induction.

Lemma 5. *If a class \mathbf{C} satisfies the condition*

$$\forall x \subseteq \mathbf{C} \Rightarrow x \in \mathbf{C},$$

then $\mathbf{C} = \mathbf{V}$.

Proof. Suppose that $\mathbf{C} \neq \mathbf{D}$, i. e., $\mathbf{D} \equiv \mathbf{V} \setminus \mathbf{C} \neq \emptyset$. Then there exists $P \in \mathbf{D}$. If $P \cap \mathbf{D} = \emptyset$, then we set $X \equiv P$.

Let $P \cap \mathbf{D} \neq \emptyset$. Consider the set N containing all $n \in \omega$ satisfying the condition that there exists a unique sequence $u \equiv u(n) \equiv (R_k | k \in n + 1)$ of sets R_k such that $R_0 = P$ and $R_{k+1} = \cup R_k$ for every $k \in n$. Since the sequence $(R_k | k \in 1)$ such that $R_0 \equiv X$ satisfies this property, we infer that $0 \in N$. Let $n \in N$, i. e., for n , there exists a unique sequence $u \equiv (R_k | k \in n + 1)$. Define the sequence $v \equiv (S_k | k \in n + 2)$, setting $S_k \equiv R_k$ for every $k \in n + 1$ and $S_{n+1} \equiv \cup R_n = \cup S_n$, i. e., $v = u \cup \{ \langle n + 1, \cup R_n \rangle \}$. It is clear that the sequence v possesses all the necessary properties. Let us verify its uniqueness. Suppose that there exists a sequence $w \equiv (T_k | k \in n + 2)$ such that $T_0 = P$ and $\forall k \in n + 1 (T_{k+1} = \cup T_k)$. Consider the set M' consisting of all $m \in n + 2$ such that $S_m = T_m$. Let $M'' \equiv \omega \setminus (n + 2)$ and $M \equiv M' \cup M''$. Since $S_0 = P = T_0$, we infer that $0 \in M' \subset M$.

Let $m \in M'$. If $m = n + 1$, then $m + 1 = n + 2 \in M'' \subset M$. If $m < n + 1$, then $m + 1 \in n + 2$ and $S_{m+1} = \cup S_m = \cup T_m = T_{m+1}$ implies $m + 1 \in M' \subset M$. If $m \in M'$, then $m + 1 \in M'' \subset M$. Therefore, $m \in M$ implies $m + 1 \in M$. By the principle of transfinite induction, $M = \omega$. Hence $M' = n + 2$ and $v = w$, i. e., the sequence v is unique. Therefore, $n + 1 \in N$. By the principle of natural induction, $N = \omega$. Thus, for every $n \in \omega$, there exists a unique sequence $u(n)$. By the uniqueness, we can denote it by $(R_k^n | k \in n + 1)$.

Consider the following formula of the ZF theory: $\varphi(x, y) \equiv (x \in \omega \Rightarrow y = R_x^x) \wedge (x \notin \omega \Rightarrow y = \emptyset)$. By the axiom scheme of replacement **AS6**, for ω , there exists a set Y such that $\forall x \in \omega (\forall y (\varphi(x, y) \Rightarrow y \in Y))$. If $n \in \omega$, then $\varphi(n, R_n^n)$ implies $R_n^n \in Y$. Therefore, in the set $\omega \times Y$, we can define an infinite sequence $u \equiv (R_n \in Y | n \in \omega)$ setting $u \equiv \{ z \in \omega \times Y | \exists x \in \omega (z = \langle x, R_x^x \rangle) \}$. The uniqueness property mentioned above implies $u(m) = u(n) | (m + 1)$ for all $m \leq n$. Thus $u | (n + 1) = u(n)$. Hence the sequence u satisfies the following properties: $R_0 = P$ and $R_{k+1} = \cup R_k$ for every $k \in \omega$. Having the set u , we can take the set $A \equiv \text{rng } u \equiv \{ R_n | n \in \omega \}$ and the set $Q \equiv \cup A = \{ y | \exists x \in \omega (y \in R_x) \} = \cup \{ R_n | n \in \omega \}$. It is clear that $R_n \subset Q$ for every $n \in \omega$ and, therefore, $P = R_0 \subset Q$.

Since $P \cap \mathbf{D} \neq \emptyset$, we infer that $R \equiv Q \cap \mathbf{D} \neq \emptyset$. By the regularity axiom **A8**, there exists $X \in R$ such that $X \cap R = \emptyset$. Let us verify that $X \cap \mathbf{D} = \emptyset$. Actually, suppose that there exists $x \in X \cap \mathbf{D}$. Since $X \in Q$, we infer that $X \in R_n$ for some $n \in \omega$. Therefore, $x \in X \in R_n$ implies $x \in \cup R_n = R_{n+1} \subset Q$. Thus, $x \in R$. As a result, we have $x \in X \cap R = \emptyset$, but this is impossible. It follows from this contradiction that $X \in \mathbf{D}$ and $X \cap \mathbf{D} = \emptyset$.

Thus, in both cases, $X \subset \mathbf{C}$. By the condition, $X \in \mathbf{C}$, but this is impossible, because $X \in \mathbf{D}$. From this contradiction, we infer that $\mathbf{C} = \mathbf{V}$. \square

Sets A and B are said to be *equivalent* ($A \sim B$) if there exists a bijective function $u : A \rightleftharpoons B$.

An ordinal number α is said to be *cardinal* if for every ordinal number β , the conditions $\beta \leq \alpha$ and $\beta \sim \alpha$ imply $\beta = \alpha$. The class of all cardinal numbers will be denoted by **Cn**. The class **Cn** with the order induced from the class **On** is well-ordered.

Lemma 6. *For every set A , there exists an ordinal number α such that $A \sim \alpha$.*

Now, for a set A , consider the class $\{ \beta | \beta \in \mathbf{On} \wedge \beta \sim A \}$. By Lemma 6, this class is nonempty, and therefore it contains the smallest element α . It is clear that α is a cardinal number. Moreover, this class contains only one cardinal number α . This number α is said to be the *power of the set A* (it is denoted by $|A|$ or *card A*). Two sets having the same power are said to be *equivalent* (it is denoted by $|A| = |B|$). A set of the power ω is said to be *denumerable*. Sets of the power $n \in \omega$ are said to be *finite*. A set is called *countable* if it is finite or denumerable. A set is said to be *infinite* if it is not finite.

Note that if \varkappa is an infinite cardinal number, then \varkappa is a limiting ordinal number.

Actually, if $\varkappa = \alpha + 1$, then $\varkappa = \text{card } \varkappa = \text{card } (\alpha + 1) = \text{card } \alpha \leq \alpha < \varkappa$, but this is impossible.

Let α be an ordinal. A *cofinality* of α is the ordinal number $cf(\alpha)$, which is equal to the smallest ordinal number β for which there exists a function f from β into α such that $\cup \text{rng } f = \alpha$.

A cardinal \varkappa is said to be *regular* if $cf(\varkappa) = \varkappa$, i.e., for every ordinal number β , for which there exists a function $f : \beta \rightarrow \varkappa$ such that $\bigcup rng f = \varkappa$, $\varkappa \leq \beta$ holds.

A cardinal $\varkappa > \omega$ is said to be (*strongly*) *inaccessible* if \varkappa is regular and $card \mathcal{P}(\lambda) < \varkappa$ for all ordinal numbers $\lambda < \varkappa$. The property of a cardinal number \varkappa to be inaccessible will be denoted by $Icn(\varkappa)$. The class of all inaccessible cardinal numbers will be denoted by **In**.

The existence of inaccessible cardinals cannot be proved in ZF (see [19], § 13).

4.3. Mirimanov–von Neumann sets in the ZF set theory. Now we will apply the construction by transfinite induction in the following situation. Consider the class

$$\begin{aligned} \mathbf{G} \equiv \{ & Z | \exists X \exists Y (Z = \langle X, Y \rangle) \wedge ((X = \emptyset \Rightarrow Y = \emptyset) \\ & \vee (X \neq \emptyset \Rightarrow (\neg func(X) \Rightarrow Y = \emptyset) \vee (func(X) \Rightarrow (\neg On(dom X) \Rightarrow Y = \emptyset) \\ & \vee (On(dom X) \Rightarrow (Son(dom X) \Rightarrow Y = X(dom X - 1) \cup \mathcal{P}(X(dom X - 1))) \\ & \vee (Lon(dom X) \Rightarrow Y = \bigcup rng X)))))) \}. \end{aligned}$$

If we express the definition of the class **G** less formally, then **G** consists of all pairs $\langle X, Y \rangle$ for which there are the following five disjunctive cases:

- (1) if $X = \emptyset$, then $Y = \emptyset$;
- (2) if $X \neq \emptyset$ and X is not a function, then $Y = \emptyset$;
- (3) if $X \neq \emptyset$, X is a function, and $dom X$ is not an ordinal number, then $Y = \emptyset$;
- (4) if $X \neq \emptyset$, X is a function, $dom X$ is an ordinal number, and $dom X = \alpha + 1$, then $Y = X(\alpha) \cup \mathcal{P}(X(\alpha))$;
- (5) if $X \neq \emptyset$, X is a function, and $dom X$ is a limiting ordinal number, then $Y = \bigcup rng X$.

By definition, **G** is a correspondence. Since any set X possesses one of these properties, it follows that $dom \mathbf{G} = \mathbf{V}$. Since in each of these five cases the set Y is defined by a set X in a unique way, using the property of an unordered pair from Lemma 1, we infer that **G** is a function from **V** into **V**.

According to Theorem 2, for the function **G**, there exists a function **F** : **On** \rightarrow **V** for which for any $\alpha \in \mathbf{On}$, we have

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|_\alpha).$$

From case (1), for the function **G**, it follows that $\mathbf{F}(\emptyset) = \mathbf{G}(\mathbf{F}|_\emptyset) = \mathbf{G}(\emptyset) = \emptyset$.

From case (4), it follows that if β is a successive ordinal number and $\beta = \alpha + 1$, then $\mathbf{F}(\beta) = \mathbf{G}(\mathbf{F}|_\beta) = (\mathbf{F}|_\beta)(\alpha) \cup \mathcal{P}((\mathbf{F}|_\beta)(\alpha)) = \mathbf{F}(\alpha) \cup \mathcal{P}(\mathbf{F}(\alpha))$.

Finally, from case (5), it follows that if α is a limiting ordinal number, then $\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|_\alpha) = \bigcup rng (\mathbf{F}|_\alpha) = \bigcup \{ \mathbf{F}(\beta) | \beta \in \alpha \}$.

Denote $\mathbf{F}(\alpha)$ by V_α . Thus, we obtained the collection $\{ V_\alpha \subset \mathbf{V} | \alpha \in \mathbf{On} \}$, satisfying the following conditions:

- (1) $V_0 = \emptyset$;
- (2) $V_\alpha = \bigcup \{ V_\beta | \beta \in \alpha \}$, if α is a limiting ordinal number;
- (3) $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$.

This collection is called the *Mirimanov–von Neumann collection*, and its elements V_α are called *Mirimanov–von Neumann sets*.

Let us prove now some lemmas on the sets V_α , which we will need later.

Lemma 7. *If α and β are ordinal numbers then:*

- (1) $\alpha < \beta \Leftrightarrow V_\alpha \in V_\beta$;
- (2) $\alpha = \beta \Leftrightarrow V_\alpha = V_\beta$;
- (3) $\alpha \subset V_\alpha$ and $\alpha \in V_{\alpha+1}$.

Proof. (1) and (2). By the transfinite induction, we will prove that for any ordinal number β ($\alpha \in \beta \Rightarrow V_\alpha \in V_\beta$).

If $\beta = \emptyset$, then this is clear, because $\forall \alpha \neg (\alpha \in \beta)$.

If for some ordinal number β ($\alpha \in \beta \Rightarrow V_\alpha \in V_\beta$), then consider the ordinal number $\beta + 1$. From $\alpha \in \beta + 1$, we infer that $\alpha \in \beta \vee \alpha = \beta$. If $\alpha \in \beta$, then, by the inductive assumption, $V_\alpha \in V_\beta$, and since $V_{\beta+1} = V_\beta \cup \mathcal{P}(V_\beta)$, we infer that $V_\alpha \in V_{\beta+1}$. If $\alpha = \beta$, then $V_\alpha = V_\beta \in V_{\beta+1}$, because $V_\beta \in \mathcal{P}(V_\beta)$. Therefore, for $\beta + 1$, the property $\alpha \in \beta + 1 \Rightarrow V_\alpha \in V_{\beta+1}$ holds.

Suppose now that β is a limiting ordinal number and $\forall \gamma \in \beta \forall \alpha (\alpha \in \gamma \Rightarrow V_\alpha \in V_\gamma)$. Let α belong to β . Since β is a limiting ordinal number, it follows that $\alpha + 1 \in \beta$. From $V_\beta = \cup \{V_\gamma \mid \gamma \in \beta\}$, it follows that $V_{\alpha+1} \subset V_\beta$. In this case, $V_\alpha \in V_{\alpha+1}$ implies $V_\alpha \in V_\beta$.

It is clear that $\alpha = \beta \Rightarrow V_\alpha = V_\beta$. If $V_\alpha = V_\beta$, then either $\alpha < \beta$, or $\alpha = \beta$, or $\beta < \alpha$. If $\alpha < \beta$, then $V_\alpha \in V_\beta$; if $\beta < \alpha$, then $V_\beta \in V_\alpha$. Therefore, $\beta = \alpha$.

If $V_\alpha \in V_\beta$, then $\alpha < \beta$, because for $\alpha = \beta$, $V_\alpha = V_\beta$, and for $\beta < \alpha$, $V_\beta \in V_\alpha$.

(3) Consider the class $\mathbf{C} \equiv \{x \mid x \in \mathbf{On} \wedge x \subset V_x\}$. Since $0 \subset \emptyset = V_0$, we infer that $0 \in \mathbf{C}$. If $\alpha \in \mathbf{C}$, then $\alpha \subset V_\alpha$ implies $\alpha + 1 \equiv \alpha \cup \{\alpha\} \subset V_\alpha \subset V_{\alpha+1}$. Let α be a limiting ordinal number, and let $\alpha \subset \mathbf{C}$. By construction, $V_\alpha = \cup \{V_\beta \mid \beta \in \alpha\}$. If $x \in \alpha$, then $x \in \mathbf{C}$ means that $x \subset V_x$. Therefore, $x \in \mathcal{P}(V_x) \subset V_{x+1}$. Since α is a limiting ordinal number, $x + 1 \in \alpha$ implies $x \in V_\alpha$. Thus, $\alpha \subset V_\alpha$, and, therefore, $\alpha \in \mathbf{C}$. By Theorem 1, $\mathbf{C} = \mathbf{On}$.

The lemma is proved. \square

Lemma 8. For every ordinal number α , the condition $z \subset x \in V_\alpha$ implies $z \in V_\alpha$.

Proof. We will prove this assertion by transfinite induction.

Precisely, let $\mathbf{C} = \{\alpha \mid \alpha \in \mathbf{On} \wedge \forall x \forall z (z \subset x \in V_\alpha \Rightarrow z \in V_\alpha)\}$. Let us show that $\mathbf{C} = \mathbf{On}$.

If $\alpha = \emptyset$, then it is clear that $\alpha \in \mathbf{C}$.

Suppose that $\alpha \in \mathbf{C}$. We prove that in this case, $\alpha + 1 \in \mathbf{C}$. Let $z \subset x \in V_{\alpha+1}$. Since $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$, we infer that $x \in V_\alpha$ or $x \subset V_\alpha$. If $z \subset x \in V_\alpha$, then $z \in V_\alpha$ by the inductive assumption, and, therefore, $z \in V_{\alpha+1}$. If $x \subset V_\alpha$ and $z \subset x$, then $z \subset V_\alpha$, and, therefore, $z \in V_{\alpha+1}$. Thus, $\alpha + 1 \in \mathbf{C}$.

If α is a limiting ordinal number and $\forall \beta \in \alpha (\beta \in \mathbf{C})$, then from $z \subset x \in V_\alpha$, we infer that $\exists \beta \in \alpha (z \subset x \in V_\beta)$, and, by the inductive assumption, we conclude that $\exists \beta \in \alpha (z \in V_\beta)$. From $V_\alpha = \cup \{V_\beta \mid \beta \in \alpha\}$, it follows now that $z \in V_\alpha$.

Therefore, by the transfinite induction, $\mathbf{C} = \mathbf{On}$. \square

This property is similar to the axiom of subset **A8** from LTS.

Lemma 9. For any ordinal number α ,

$$\forall x (x \in V_\alpha \Rightarrow x \subset V_\alpha).$$

Proof. This lemma will also be proved by the transfinite induction.

For $\alpha = \emptyset$, the given formula holds, because $\forall x \neg (x \in V_\emptyset)$.

Let for some ordinal number α ,

$$\forall x (x \in V_\alpha \Rightarrow x \subset V_\alpha).$$

Consider the ordinal number $\alpha + 1$. If $x \in V_{\alpha+1}$, then $x \in V_\alpha \vee x \in \mathcal{P}(V_\alpha)$, or, more precisely, $x \in V_\alpha \vee x \subset V_\alpha$. In the case $x \in V_\alpha$, by the inductive assumption, $x \subset V_\alpha$, and $V_\alpha \subset V_{\alpha+1}$ implies $x \subset V_{\alpha+1}$. If $x \subset V_\alpha$, then we infer that $x \subset V_{\alpha+1}$ from $V_\alpha \subset V_{\alpha+1}$.

Now let α be a limiting ordinal number, and let $\forall \beta \in \alpha \forall x (x \in V_\beta \Rightarrow x \subset V_\beta)$. Then from $x \in V_\alpha$, we infer that $\exists \beta \in \alpha (x \in V_\beta)$. By the inductive assumption, $\exists \beta \in \alpha (x \subset V_\beta)$, and, therefore, $x \subset V_\alpha$.

The lemma is proved. \square

This property is similar to the transitivity axiom **A7** from the LTS.

Corollary 1. If α and β are ordinal numbers and $\alpha \leq \beta$, then $V_\alpha \subset V_\beta$.

Corollary 2. For every ordinal number α the inclusion $V_\alpha \subset \mathcal{P}(V_\alpha)$ and the equality $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ are valid.

Proof. If $x \in V_\alpha$, then by the given lemma, $x \subset V_\alpha$, i.e., $x \in \mathcal{P}(V_\alpha)$. Thus, $V_\alpha \subset \mathcal{P}(V_\alpha)$. Therefore, $V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha) = \mathcal{P}(V_\alpha)$. \square

Corollary 3. *If α and β are ordinal numbers and $\alpha < \beta$, then $|V_\alpha| < |V_\beta|$.*

Proof. By the previous two corollaries, $V_\alpha \subset \mathcal{P}(V_\alpha) = V_{\alpha+1} \subset V_\beta$. Using Cantor's theorem, we infer that $|V_\alpha| < |\mathcal{P}(V_\alpha)| = |V_{\alpha+1}| \leq |V_\beta|$. \square

Lemma 10. *For every ordinal number α , if $x \in V_{\alpha+1}$, then $x \subset V_\alpha$.*

Proof. Suppose that $x \in V_{\alpha+1}$. This means that $x \in V_\alpha \vee x \subset V_\alpha$. If $x \subset V_\alpha$, then everything is proved. If $x \in V_\alpha$, then by the previous lemma, $x \subset V_\alpha$. \square

Lemma 11. *For every ordinal number α ,*

$$\forall x \forall y (x \in V_\alpha \wedge y \in V_\alpha \Rightarrow x \cup y \in V_\alpha).$$

Proof. We will again use the principle of transfinite induction.

If $\alpha = \emptyset$, then the conclusion of lemma is valid, because $\forall x \neg(x \in V_\emptyset)$.

Now let $\alpha = \beta + 1$ for some ordinal number β . Then from the formula $x \in V_\alpha \wedge y \in V_\alpha$, by Lemma 10, we infer that $x \subset V_\beta \wedge y \subset V_\beta$. Therefore, $x \cup y \subset V_\beta$, whence $x \cup y \in V_{\beta+1}$, i.e., $x \cup y \in V_\alpha$.

Now suppose that α is a limiting ordinal number and $\forall \beta \in \alpha \forall x \forall y (x \in V_\beta \wedge y \in V_\beta \Rightarrow x \cup y \in V_\beta)$. Then $x, y \in V_\alpha$ implies $\exists \beta \in \alpha (x, y \in V_\beta)$. Therefore, by the inductive assumption, $\exists \beta \in \alpha (x \cup y \in V_\beta)$, and, therefore, $x \cup y \in V_\alpha$. \square

This property is similar to the axiom of binary union **A10** from the LTS.

Lemma 12. *For every limiting ordinal number α , the condition $x \in V_\alpha$ implies $\mathcal{P}(x) \in V_\alpha$.*

Proof. Suppose that α is some limiting ordinal number and $x \in V_\alpha$. Then there exists $\beta \in \alpha$ such that $x \in V_\beta$. Let us show that in this case, $\mathcal{P}(x) \subset V_\beta$. Actually, by Lemma 8, from $x \in V_\beta$ and $z \subset x$, we infer $z \in V_\beta$, whence $\forall z (z \in \mathcal{P}(x) \Rightarrow z \in V_\beta)$, and this means that $\mathcal{P}(x) \subset V_\beta$. If $\mathcal{P}(x) \subset V_\beta$, then $\mathcal{P}(x) \in V_{\beta+1} \subset V_\alpha$. \square

Corollary 1. *For every limiting ordinal number α , the condition $x, y \in V_\alpha$ implies $\{x\}, \{x, y\}, \langle x, y \rangle \in V_\alpha$.*

Proof. By Lemma 12, $\mathcal{P}(x) \in V_\alpha$. By Lemma 8, $\{x\} \subset \mathcal{P}(x)$ implies $\{x\} \in V_\alpha$. Now, by Lemma 11, $\{x, y\} \in V_\alpha$. It follows from the proved properties that $\langle x, y \rangle \in V_\alpha$. \square

Corollary 2. *For every limiting ordinal number α the condition $X, Y \in V_\alpha$ implies $X * Y \in V_\alpha$.*

Proof. Let $x \in X$ and $y \in Y$. Then $\{x\} \subset X \cup Y$ and $\{y\} \subset X \cup Y$ imply $\{x, y\} \subset X \cup Y$. By Lemma 11, $X \cup Y \in V_\alpha$. Since $\{x\} \in \mathcal{P}(X \cup Y)$ and $\{x, y\} \in \mathcal{P}(X \cup Y)$, we infer that $\langle x, y \rangle \equiv \{\{x\}, \{x, y\}\} \subset \mathcal{P}(X \cup Y)$. Hence $\langle x, y \rangle \in \mathcal{P}(\mathcal{P}(X \cup Y))$. Therefore, $X * Y \subset \mathcal{P}(\mathcal{P}(X \cup Y))$. By Lemmas 11, 12, and 8, $X * Y \in V_\alpha$. \square

This property is similar to the axiom of full ensemble **A9** from LTS.

Lemma 13. *If $\alpha \geq \omega$, then $\omega \subset V_\alpha$. If $\alpha > \omega$, then $\omega \in V_\alpha$.*

Proof. By Lemma 7, $\omega \subset V_\omega \subset V_\alpha$. If $\alpha > \omega$, then by Lemma 8, $\omega \subset V_\omega \in V_{\omega+1}$, implies $\omega \in V_{\omega+1} \subset V_\alpha$. \square

Now we will adduce some assertions about sets V_\varkappa for inaccessible cardinal numbers \varkappa , which we will need later.

Let λ be an ordinal number. Consider a collection $K(\lambda) \equiv \langle M_\beta | \beta \in \lambda + 1 \rangle$ of the sets $M_\beta \equiv \{x | x \rightleftharpoons \mathcal{P}(|V_\beta|) \rightleftharpoons |\mathcal{P}(|V_\beta|)|\}$ of all corresponding bijective mappings for all $\beta \in \lambda + 1$ and the set $M(\lambda) \equiv \cup \langle M_\beta | \beta \in \lambda + 1 \rangle$. By the axiom of choice there exists a choice function $ch(\lambda) : \mathcal{P}(M(\lambda)) \setminus \{\emptyset\} \rightarrow M(\lambda)$ such that $ch(\lambda)(P) \in P$ for every $P \in \mathcal{P}(M(\lambda)) \setminus \{\emptyset\}$. Since $M_\beta \subset M(\lambda)$ for $\beta \in \lambda + 1$, we infer that $c_\beta(\lambda) \equiv ch(\lambda)(M_\beta) \in M_\beta$, i.e., $c_\beta(\lambda)$ is a bijection from $\mathcal{P}(|V_\beta|)$ onto $|\mathcal{P}(|V_\beta|)|$.

The following assertion can be called the *theorem on initial synchronization of powers of Mirimanov–von Neumann sets*.

Theorem 3. Let λ be an ordinal number. Then for every ordinal number $\alpha \leq \lambda$, there exists a unique collection $u(\alpha) \equiv u(\lambda)(\alpha) \equiv (f_\beta | \beta \in \alpha + 1)$ of bijective functions $f_\beta : V_\beta \xrightarrow{\cong} |V_\beta|$ such that:

- (1) $f_0 \equiv \emptyset$;
- (2) if $\gamma < \beta \in \alpha + 1$, then $f_\gamma = f_\beta | V_\gamma$;
- (3) if $\beta \in \alpha + 1$ and $\beta = \gamma + 1$, then $f_\beta | V_\gamma = f_\gamma$ and $f_\beta(x) = c_\gamma(\lambda)(f_\gamma[x])$ for every $x \in V_\beta \setminus V_\gamma = \mathcal{P}(V_\gamma) \setminus V_\gamma$;
- (4) if $\beta \in \alpha + 1$ and β is a limiting ordinal number, then $f_\beta = \cup \{ f_\gamma | \gamma \in \beta \}$.

It follows from the uniqueness property that $u(\alpha) | \delta + 1 = u(\delta)$ for every $\delta \leq \alpha$, i.e., these collections continue each other.

Proof. First, let us verify the uniqueness of the collection $u \equiv u(\alpha)$. For α , let there exist a collection $v \equiv (g_\beta | \beta \in \alpha + 1)$ of bijective functions $g_\beta : V_\beta \xrightarrow{\cong} |V_\beta|$ possessing properties (1)–(4). Consider the set $D' \equiv \{ \beta \in \alpha + 1 | f_\beta = g_\beta \}$, the class $\mathbf{D}'' \equiv \mathbf{On} \setminus (\alpha + 1)$, and the class $\mathbf{D} \equiv D' \cup \mathbf{D}''$. It is clear that $0 \in D' \subset \mathbf{D}$.

Let $\beta \in \mathbf{D}$. If $\beta \geq \alpha$, then $\beta + 1 \in \mathbf{D}'' \subset \mathbf{D}$. Let $\beta < \alpha$. Then $\beta \in D'$ and $\beta + 1 \in \alpha + 1$. Therefore, by property (3), $f_{\beta+1}(x) = f_\beta(x) = g_\beta(x) = g_{\beta+1}(x)$ for every $x \in V_\beta$ and $f_{\beta+1}(x) = c_\beta(\lambda)(f_\beta[x]) = c_\beta(\lambda)(g_\beta[x]) = g_{\beta+1}(x)$ for every $x \in V_{\beta+1} \setminus V_\beta$, i.e., $f_{\beta+1} = g_{\beta+1}$. Therefore, $\beta + 1 \in D' \subset \mathbf{D}$. Thus, $\beta \in \mathbf{D}$ implies $\beta + 1 \in \mathbf{D}$.

Let β be a limiting ordinal number, and let $\beta \in \mathbf{D}$. If $\beta \cap \mathbf{D}'' \neq \emptyset$, then there exists $\gamma \in \beta$ such that $\gamma \geq \alpha + 1$. Therefore, $\beta > \gamma \geq \alpha + 1$ implies $\beta \in \mathbf{D}'' \subset \mathbf{D}$. Let $\beta \cap \mathbf{D}'' = \emptyset$, i.e., $\beta \in D'$. Then for every $\gamma \in \beta$, $f_\gamma = g_\gamma$ holds. Since $\beta \in \alpha + 1$, then $\beta \leq \alpha + 1$. If $\beta = \alpha + 1$, we infer that $\beta \in \mathbf{D}'' \subset \mathbf{D}$. Let $\beta \in \alpha + 1$. If $x \in V_\beta = \cup \{ V_\gamma | \gamma \in \beta \}$, then $x \in V_\gamma$ for some $\gamma \in \beta$. Therefore, by property (2), $f_\beta(x) = f_\gamma(x) = g_\gamma(x) = g_\beta(x)$ for every $x \in V_\beta$, i.e., $f_\beta = g_\beta$. So $\beta \in D' \subset \mathbf{D}$. Thus, the properties $Lon(\beta)$ and $\beta \in \mathbf{D}$ implies $\beta \in \mathbf{D}$.

By the principle of transfinite induction, $\mathbf{D} = \mathbf{On}$. Consequently, $D' = \alpha + 1$. Therefore, $u = v$.

Now we will write c_γ instead of $c_\gamma(\lambda)$.

Consider the set C' consisting of all ordinal numbers $\alpha \leq \lambda$ for which there exists a collection $u(\alpha)$ with properties (1)–(4). Also, consider the classes $\mathbf{C}'' \equiv \mathbf{On} \setminus (\lambda + 1)$ and $\mathbf{C} \equiv C' \cup \mathbf{C}''$. Since $V_0 = \emptyset$ and $|V_0| = 0$, we infer that the collection $u(0) \equiv (f_\beta | \beta \in 1)$ with the bijective function $f_0 = \emptyset : V_0 \xrightarrow{\cong} |V_0|$ possesses all properties (1)–(4), and, therefore, $0 \in \mathbf{C}$.

Let $\alpha \in \mathbf{C}$. If $\alpha \geq \lambda$, then $\alpha + 1 \in \mathbf{C}'' \subset \mathbf{C}$. Now let $\alpha < \lambda$. Then $\alpha + 1 \in \lambda + 1$ means that we can use the function c_α . Since $\alpha \in C'$, we infer that for α , there exists a unique collection $u \equiv (f_\beta | \beta \in \alpha + 1)$. Define the collection $v \equiv (g_\beta | \beta \in \alpha + 2)$ of bijective functions $g_\beta : V_\beta \xrightarrow{\cong} |V_\beta|$ setting $g_\beta \equiv f_\beta$ for every $x \in V_\alpha$ and $g_{\alpha+1}(x) \equiv c_\alpha(f_\alpha[x])$ for every $x \in V_{\alpha+1} \setminus V_\alpha = \mathcal{P}(V_\alpha) \setminus V_\alpha$.

Let us verify that v possesses properties (1)–(4). Let $\beta \in \alpha + 2$. If $\beta \in \alpha + 1$, then properties (1)–(4) are obviously true. Let $\beta = \alpha + 1$. Then $g_\beta(x) = g_{\alpha+1}(x) = f_\alpha(x) = g_\alpha(x)$ for every $x \in V_\alpha$ and $g_\beta(x) = g_{\alpha+1}(x) = c_\alpha(f_\alpha[x]) = c_\alpha(g_\alpha[x])$ for every $x \in V_\beta \setminus V_\alpha$. Moreover, $g_\beta | V_\alpha = f_\alpha = g_\alpha$. Therefore, $\gamma < \beta$ implies $g_\beta | V_\gamma = g_\alpha | V_\gamma = f_\alpha | V_\gamma = f_\gamma = g_\gamma$. Therefore, $\alpha + 1 \in C' \subset \mathbf{C}$.

Let α be a limiting ordinal number, and let $\alpha \in \mathbf{C}$. If $\alpha \cap \mathbf{C}'' \neq \emptyset$, then there exists $\beta \in \alpha$ such that $\beta \geq \alpha + 1$. Consequently, $\alpha > \beta \leq \lambda + 1$ implies $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Let $\alpha \cap \mathbf{C}'' = \emptyset$, i.e., $\alpha \in C'$. Then for every $\beta \in \alpha$, there exists a unique collection $u_\beta \equiv (f_\gamma^\beta | \gamma \in \beta + 1)$ of bijective functions $f_\gamma^\beta : V_\gamma \xrightarrow{\cong} |V_\gamma|$ with properties (1)–(4). Since $\alpha \in \lambda + 1$, it follows that $\alpha \leq \lambda + 1$. If $\alpha = \lambda + 1$, then $\alpha \in \mathbf{C}'' \subset \mathbf{C}$. Now let $\alpha \in \lambda + 1$. For every $\delta \leq \beta \in \alpha$, consider the collection $w \equiv u_\beta | \delta + 1 \equiv (f_\gamma^\beta | \gamma \in \delta + 1)$. The collection w possesses properties (1)–(4). By the uniqueness, which was proved above, $w = u_\delta$. Therefore, $u_\delta = u_\beta | \delta + 1$, i.e., $f_\gamma^\delta = f_\gamma^\beta$ for every $\gamma \in \delta + 1$. In particular, $f_\delta^\delta = f_\delta^\beta$ for every $\delta \leq \beta$.

Define the collection $v \equiv (g_\beta | \beta \in \alpha + 1)$ of functions g_β setting $g_\beta \equiv f_\beta^\beta$ for every $\beta \in \alpha$ and $g_\alpha(x) \equiv f_\gamma^\beta(x)$ for every $x \in V_\alpha = \cup \{ V_\gamma | \gamma \in \alpha \}$ and every $\gamma \leq \beta \in \alpha$ such that $x \in V_\gamma$. It is clear that $g_\beta \xrightarrow{\cong} V_\beta \xrightarrow{\cong} |V_\beta|$ for every $\beta \in \alpha$. Let us verify that $g \xrightarrow{\cong} V_\alpha \rightarrow |V_\alpha|$. By Corollary 1 of Lemma 9, $V_\gamma \subset V_\alpha$. Consequently, $|V_\gamma| \subset |V_\alpha|$. Therefore, for every $x \in V_\alpha$, $g_\alpha(x) \equiv f_\gamma^\beta(x) \in |V_\gamma| \subset \aleph \equiv$

$\cup(|V_\gamma| \subset |V_\alpha| \mid \gamma \in \alpha) \subset |V_\alpha|$. Let $x, y \in V_\alpha$, and let $g_\alpha(x) = g_\alpha(y)$. Then $x \in V_\gamma$ and $y \in V_\delta$ for some $\gamma, \delta \in \alpha$. Consider the number β , which is the greatest of the numbers γ and δ . By definition, $f_\beta^\beta(x) = g_\alpha(x) = g_\alpha(y) = f_\beta^\beta(y)$. From the injectivity of this function, we infer that $x = y$. Therefore, the function g_α is surjective. Let $z \in \aleph$. Then $z \in |V_\gamma|$ for some $\gamma \in \alpha$. Since the function $f_\gamma^\gamma : V_\gamma \rightarrow |V_\gamma|$ is injective, we infer that $z = f_\gamma^\gamma(x)$ for some $x \in V_\gamma \subset V_\alpha$. Consequently, $z = g_\alpha(x)$. Thus, g_α is a bijective function from V_α onto \aleph , i.e., $V_\alpha \sim \aleph$.

By Corollary 3 of Lemma 9, $|V_\gamma| \in |V_\alpha|$. Therefore, there exists a set $A \equiv \{x \in |V_\alpha| \mid \exists y \in \alpha (x = |V_y|)\} = \{|V_\gamma| \mid \gamma \in \alpha\}$ of all ordinal numbers $|V_\gamma|$. Since α is a limiting ordinal number, we infer that $A \neq \emptyset$. Therefore, by Lemma 3, the set $\cup A = \sup A$ is an ordinal number. If $z \in \cup A = \{z \mid \exists x \in A (z \in x)\}$, then $z \in |V_\gamma| \subset \aleph$ for some $\gamma \in \alpha$. Conversely, if $z \in \aleph$, then $z \in |V_\gamma| \in A$ for some $\gamma \in \alpha$. Therefore, $z \in \cup A$. Consequently, $\aleph = \cup A$, i.e., \aleph is an ordinal number.

Prove that \aleph is a cardinal number. Let β be an ordinal number, $\beta \leq \aleph$, and let $\beta \sim \aleph$. Suppose that $\beta < \aleph$. Then $\beta \in \aleph$ implies $\beta \in |V_\gamma|$ for some $\gamma \in \alpha$. Consequently, $\beta < |V_\gamma| = \text{card } |V_\gamma| \leq |\aleph| = |\beta|$. Since β is an ordinal number, we infer that $|\beta| \leq \beta$. As a result, we come to the inequality $\beta < \beta$, which is impossible. It follows from this contradiction that $\beta = \aleph$. This means that \aleph is a cardinal number.

Since \aleph is a cardinal number and $\aleph \sim V_\alpha$, we infer that $\aleph = |V_\alpha|$. Therefore, $g_\alpha \equiv V_\alpha \rightarrow |V_\alpha|$.

Let us verify that the collection v possesses properties (1)–(4). By the definition of this collection, $g_0 \equiv f_0^0 = \emptyset$. Let $\gamma < \beta \in \alpha + 1$. If $\beta \in \alpha$, then the equality $f_\gamma^\gamma = f_\beta^\beta$, which was proved above, implies $g_\beta|V_\gamma = f_\beta^\beta|V_\gamma = f_\gamma^\gamma = f_\gamma^\gamma \equiv g_\gamma$. If $\beta = \alpha$, then, by construction, $g_\beta|V_\gamma = g_\alpha|V_\gamma = f_\gamma^\gamma \equiv g_\gamma$. Thus, property (2) holds for v .

Let $\beta \in \alpha + 1$, $\beta = \gamma + 1$, and let $x \in V_\beta = \mathcal{P}(V_\gamma)$. If $\beta \in \alpha$, then the equality $f_\gamma^\gamma = f_\beta^\beta$, which was proved above, implies $g_\beta(x) = f_\beta^\beta(x) = f_\gamma^\gamma(x) = g_\gamma(x)$ for every $x \in V_\gamma$ and $g_\beta(x) = f_\beta^\beta(x) = c_\gamma(f_\gamma^\beta[x]) = c_\gamma(f_\gamma^\gamma[x]) = c_\gamma(g_\gamma[x])$ for every $x \in V_\beta \setminus V_\gamma$. Therefore, property (3) holds for v .

Property (4) follows from property (2). From the properties, which were already verified, we infer that $\alpha \in C' \subset C$.

By the principle of transfinite induction, $C = \mathbf{On}$, and, therefore, $C' = \lambda + 1$. □

Note that since the functions $c_\gamma(\lambda)$ depend on the number λ , we cannot compose the (continuing each other) collections $u(\lambda)(\alpha)$ into one global collection indexed by all ordinal numbers.

Corollary. For every limiting ordinal number α , the equalities $|V_\alpha| = \cup(|V_\beta| \mid \beta \in \alpha) = \cup\{|V_\beta| \mid \beta \in \alpha\} = \sup\{|V_\beta| \mid \beta \in \alpha\}$ hold.

Proof. Consider the number $\lambda \equiv \alpha$. By Theorem 3, there exists the corresponding collection $u(\alpha) \equiv (f_\beta \mid \beta \in \alpha + 1)$. Since α is a limiting ordinal number and $\alpha \in \alpha + 1$, by property (4), it follows that $f_\alpha = \cup(f_\beta \mid \beta \in \alpha)$. Therefore, $|V_\alpha| = \text{rng } f_\alpha = \cup(\text{rng } f_\beta \mid \beta \in \alpha) = \cup(|V_\beta| \mid \beta \in \alpha) = \cup\{|V_\beta| \mid \beta \in \alpha\} = \sup\{|V_\beta| \mid \beta \in \alpha\}$, where the latter equality follows from Lemma 3. □

Lemma 14. For every inaccessible cardinal number \aleph and every ordinal number $\alpha \in \aleph$, the property $|V_\alpha| < \aleph$ holds.

Proof. Consider the set $C' \equiv \{x \in \aleph \mid |V_x| < \aleph\}$ and the classes $C'' \equiv \mathbf{On} \setminus \aleph$ and $C \equiv C' \cup C''$. Since $V_0 = \emptyset$, we have $|V_0| = 0 < \aleph$. Therefore, $0 \in C$.

Let $\alpha \in C$. If $\alpha \geq \aleph$, then $\alpha + 1 \in C'' \subset C$. Let $\alpha < \aleph$. Then $\alpha \in C'$. If $\alpha + 1 = \aleph$, then $\alpha + 1 \in C'' \subset C$. Let $\alpha + 1 < \aleph$. Since $V_\alpha \sim |V_\alpha|$, we have $\mathcal{P}(V_\alpha) \sim \mathcal{P}(|V_\alpha|)$. Therefore, $|\mathcal{P}(V_\alpha)| = |\mathcal{P}(|V_\alpha|)|$. By Corollary 2 of Lemma 9, $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = |\mathcal{P}(|V_\alpha|)|$. Since $|V_\alpha| < \aleph$ and the cardinal number \aleph is inaccessible, we obtain $|\mathcal{P}(|V_\alpha|)| < \aleph$. Hence $|V_{\alpha+1}| < \aleph$. Thus, $\alpha + 1 \in C' \subset C$.

Let α be a limiting ordinal number, and let $\alpha \in C$. If $\alpha \cap C'' \neq \emptyset$, then there exists $\beta \in \alpha$ such that $\beta \geq \aleph$. Therefore, $\alpha > \beta \geq \aleph$ implies $\alpha \in C'' \subset C$. Let $\alpha \cap C'' = \emptyset$, i.e., $\alpha \subset C' \subset \aleph$. If $\alpha = \aleph$, then $\alpha \in C'' \subset C$. Let $\alpha < \aleph$. By $\alpha \in C'$, for every $\beta \in \alpha$, we have $|V_\beta| < \aleph$. Therefore, $\sup\{|V_\beta| \mid \beta \in \alpha\} \leq \aleph$.

Using the property $|V_\beta| \in \aleph$, we can correctly define the function $f : \alpha \rightarrow \aleph$, setting $f(\beta) \equiv |V_\beta|$. It is clear that $\text{rng } f = \{|V_\beta| \mid \beta \in \alpha\}$. By Corollary of Theorem 3, $\cup \text{rng } f = \cup\{|V_\beta| \mid \beta \in \alpha\} =$

$\sup\{|V_\beta| \mid \beta \in \alpha\} = |V_\alpha|$. By the inequality which was proved above, we infer that $|V_\alpha| \leq \varkappa$. Suppose that $|V_\alpha| = \varkappa$. Then by the regularity of the number \varkappa , $\varkappa = \text{Urng } f$ implies $\varkappa \leq \alpha$, but this contradicts the initial inequality $\alpha < \varkappa$. Therefore, $|V_\alpha| < \varkappa$. Consequently, $\alpha \in C' \subset \mathbf{C}$.

By the principle of transfinite induction, $\mathbf{C} = \mathbf{On}$. Thus, $C' = \varkappa$. □

Lemma 15. *If \varkappa is an inaccessible cardinal, then $\varkappa = |V_\varkappa|$.*

Proof. By Lemma 7, $\varkappa \subset V_\varkappa$. Therefore, $\varkappa = |\varkappa| \leq |V_\varkappa|$. By Corollary 1 of Theorem 3, $|V_\varkappa| = \sup(|V_\beta| \mid \beta \in \varkappa)$. Since $|V_\beta| < \varkappa$, by Lemma 14 we have $|V_\varkappa| \leq \varkappa$. As a result, we have $\varkappa = |V_\varkappa|$. □

Lemma 16. *If \varkappa is an inaccessible cardinal number, α is an ordinal number such that $\alpha < \varkappa$, and f is a correspondence from V_α into V_\varkappa such that $\text{dom } f = V_\alpha$ and $f\langle x \rangle \in V_\varkappa$ for every $x \in V_\alpha$, then $\text{rng } f \in V_\varkappa$.*

Proof. Since \varkappa is a limiting ordinal number, we have $V_\varkappa = \cup\{V_\delta \mid \delta \in \varkappa\}$. For $x \in V_\alpha$, there exists $\delta \in \varkappa$ such that $f\langle x \rangle \in V_\delta$. Therefore, the nonempty set $\{y \in \varkappa \mid f\langle x \rangle \in V_y\}$ contains the smallest element z . By the uniqueness of the element z , we can correctly define the function $g : V_\alpha \rightarrow \varkappa$ setting $g(x) \equiv z$. Consider the ordinal number $\beta \equiv |V_\alpha|$ and take some bijective mapping $h : \beta \xrightarrow{\cong} V_\alpha$. Consider the mapping $\varphi \equiv g \circ h : \beta \rightarrow \varkappa$ and the ordinal number $\gamma \equiv \text{Urng } \varphi = \sup \text{rng } \varphi \leq \varkappa$.

Suppose that $\gamma = \varkappa$. Since the cardinal \varkappa is regular, the supposition $\text{Urng } \varphi = \varkappa$ implies $\varkappa \leq \beta \equiv |V_\alpha|$. But, by Lemma 16, $|V_\alpha| < \varkappa$. It follows from this contradiction that $\gamma < \varkappa$.

Since h is bijective, $\text{rng } \varphi = \text{rng } g$. Therefore, $\gamma = \sup \text{rng } g$. If $x \in V_\alpha$, then $f\langle x \rangle \in V_z = V_{g(x)}$. From $g(x) \leq \gamma$, by Lemma 7, we infer that $V_{g(x)} \subset V_\gamma$. Consequently, by Lemma 9, $f\langle x \rangle \in V_\gamma$ implies $f\langle x \rangle \in V_\gamma$. Therefore, $\text{rng } f \subset V_\gamma$. By Lemma 7, $\text{rng } f \in V_{\gamma+1} \subset V_\varkappa$. □

Lemma 17. *If \varkappa is an inaccessible cardinal number, $A \in V_\varkappa$, and f is a correspondence from A into V_\varkappa such that $f\langle x \rangle \in V_\varkappa$ for every $x \in A$, then $\text{rng } f \in V_\varkappa$.*

Proof. Since \varkappa is a limiting ordinal number, $V_\varkappa = \cup\{V_\alpha \mid \alpha \in \varkappa\}$. Therefore, $A \in V_\alpha$ for some $\alpha \in \varkappa$. By Lemma 9, $A \subset V_\alpha$. Define the correspondence g from V_α into V_\varkappa setting $g|A \equiv f$ and $g\langle x \rangle \equiv \emptyset \subset V_\varkappa$ for every $x \in V_\alpha \setminus A$. Then $\text{dom } g = V_\alpha$ and $\text{rng } g = \text{rng } f$. If $x \in A$, then $g\langle x \rangle = f\langle x \rangle \in V_\varkappa$, and if $x \in V_\alpha \setminus A$, then $g\langle x \rangle = \emptyset \in V_\varkappa$. Therefore, by Lemma 16, we obtain $\text{rng } f = \text{rng } g \in V_\varkappa$. □

Consider the class $\mathbf{\Pi} \equiv \{x \mid \exists \alpha \in \mathbf{On}(x \in V_\alpha)\} \equiv \cup\{V_\alpha \mid \alpha \in \mathbf{On}\}$.

Lemma 18. $\mathbf{\Pi} = \mathbf{V}$.

Proof. Let us show that $\mathbf{\Pi}$ satisfies the principle of \in -induction.

Introduce the function $\text{ran} : \mathbf{\Pi} \rightarrow \mathbf{On}$ setting $\text{ran}(x) \equiv$ the smallest ordinal α such that $x \in V_{\alpha+1}$.

It follows from Lemma 7 that all ordinal numbers are contained in $\mathbf{\Pi}$. Let us verify that $x \subset \mathbf{\Pi}$ implies $x \in \mathbf{\Pi}$ for every set x . If $x = \emptyset$, then, by Lemma 7, $x = 0 \in V_1 \subset \mathbf{\Pi}$. Let $x \neq \emptyset$. Consider the following formula of the ZF theory: $\varphi(y, z) \equiv (y \in \mathbf{\Pi} \Rightarrow z = \text{ran}(y)) \wedge (y \notin \mathbf{\Pi} \Rightarrow z = \emptyset)$. By the axiom scheme of replacement **AS6**, for the set x , there exists a set B such that $\forall y \in x \forall z(\varphi(y, z) \Rightarrow z \in B)$. If $y \in x$, then $\varphi(y, \text{ran}(y))$ implies $\text{ran}(y) \in B$. Therefore, $A \equiv \{z \in B \mid \exists y \in x(z = \text{ran}(y))\} \subset B$. By the axiom scheme of separation **AS3**, A is a set. By Lemma 3, $\alpha \equiv \cup A = \sup A$ is an ordinal number.

If $y \in x$, then $z \equiv \text{ran}(y) \in A$ implies $z \leq \alpha$. Therefore, by Lemma 7, $y \in V_{z+1} \subset V_{\alpha+1}$. So $x \subset V_{\alpha+1}$. Whence $x \in \mathcal{P}(V_{\alpha+1}) \subset V_{\alpha+2} \subset \mathbf{\Pi}$.

By the principle of \in -induction, we now infer that $\mathbf{\Pi} = \mathbf{V}$. □

5. Universes, Ordinals, Cardinals, and Mirimanov–von Neumann Classes in the Local Theory of Sets

5.1. The relativization of formulas of the LTS to universal classes. The interpretation of the ZF set theory in universal classes. It is convenient to denote by \vec{x} a line of variables x_0, \dots, x_{n-1} . Let all free variables of a formula φ be among y, \vec{x} . Denote by $\vec{x} \in U$ the formula $x_0 \in U \wedge \dots \wedge x_{n-1} \in U$. We use the abbreviations $\forall x \in X(\varphi)$ for $\forall x(x \in X \Rightarrow \varphi)$ and $\exists x \in X(\varphi)$ for $\exists x(x \in X \wedge \varphi)$. By φ^U we

will denote the formula (the *relativization of the formula φ to the class U*) obtained by changing in φ all subformulas of the form $\forall x\varphi'$ and $\exists x\varphi'$ by $\forall x \in U\varphi'$ and $\exists x \in U\varphi'$, respectively.

Statement 1. *Let U be a universal class in the LTS. Consider the interpretation $M \equiv (U, I)$ of the ZF theory in the LTS in which to the predicate symbols \in_{ZF} and $=_{ZF}$ the correspondence I assigns the binary U -correspondences $B \equiv \{z \in U | \exists x \in U \exists y \in U (z = (x, y)_U \wedge x \in_{LTS} y)\}$ and $E \equiv \{z \in U | \exists x \in U \exists y \in U (z = (x, y)_U \wedge x =_{LTS} y)\}$ on the class U . The interpretation M is a model of ZF in the LTS.*

Proof. We need to verify that in the LTS there exists a deduction of the formula or the scheme of formulas $M \models \varphi[s]$ for every proper axiom or axiom scheme φ of the ZF theory and every sequence $s \equiv x_0, \dots, x_q, \dots$ of elements of the class U .

On s , Axiom **A1** is translated into the formula

$$M \models \mathbf{A1}[s] = \mathbf{A1}^U \equiv \forall X \in U \forall Y \in U (\forall u \in U (u \in X \Leftrightarrow u \in Y) \Rightarrow X = Y).$$

By the definition of equality in the LTS, this formula is evidently deducible.

On s , Axiom **A2** is translated into the formula

$$M \models \mathbf{A2}[s] = \mathbf{A2}^U \equiv \forall u \in U (\forall v \in U \exists x \in U \forall z \in U (z \in x \Leftrightarrow z = u \vee z = v)).$$

For the U -sets u and v , consider the unordered U -pair $x \equiv \{u, v\}_U$. By Lemma 2 of Sec. 2, $x \in U$. From the corresponding definitions, we infer that $x = \{u\}_U \cup \{v\}_U = \{y \in U | y \in \{u\}_U \vee y \in \{v\}_U\} = \{y \in U | y = u \vee y = v\}$. By the axiom scheme **AS2** (LTS), we have $\forall z \in U (z \in x \Leftrightarrow z = u \vee z = v)$ and obtain the desired deducibility.

On s , the axiom scheme of separation **AS3** is translated into the scheme $M \models \mathbf{AS3}[s] \equiv \forall X \in U \exists Y \in U \forall u \in U (u \in Y \Leftrightarrow u \in X \wedge \varphi^U(u, \vec{p}_M[s]))$, where Y is not a free variable of the formula $\varphi(u, \vec{p})$.

By **AS2** (LTS), for the U -predicative formula $\varphi^U(u, \vec{p}_M[s])$ and U -set X , there exists an U -class $Y \equiv \{u \in U | \varphi^U(u, \vec{p}_M[s]) \wedge u \in X\}$ such that $u \in Y \Leftrightarrow (u \in U \wedge u \in X \wedge \varphi^U(u, \vec{p}_M[s]))$. Since $Y \subset X \in U$, by the axiom of subset **A8** (LTS) we have $Y \in U$, and this gives us the desired deducibility.

On s , the axiom of union **A4** is translated into the formula $M \models \mathbf{A4}[s] = \mathbf{A4}^U \equiv \forall X \in U \exists Y \in U \forall z \in U \forall u \in U (u \in z \wedge z \in X \Rightarrow u \in Y)$.

For the U -set X and the corresponding U -predicative formula, by **AS2** (LTS), there exists the U -class $Z \equiv \{w \in U | \exists x, y \in U (x \in X \wedge y \in x \wedge w = \langle x, y \rangle_U)\} \subset X *_U U$. Since $Z \langle x \rangle = x \in U$ for any $x \in X$, by the axiom of full union **A11** (LTS), $Y \equiv \cup_U \{x \subset U | x \in X\}_U \equiv \text{rng}_U Z \in U$. If $u \in z \in X$, then $u \in Y$, and we obtain the desired deducibility.

On s , the axiom of power set **A5** is translated into the formula $M \models \mathbf{A5}[s] = \mathbf{A5}^U \equiv \forall X \in U \exists Y \in U \forall u \in U (u \subset X \Rightarrow u \in Y)$.

For the U -set X , by **AS2** (LTS), there exists an U -class $Y \equiv \mathcal{P}_U(X) \equiv \{x \in U | x \subset X\}$. By Axiom **A8** (LTS), $Y \in U$. If $u \in U$ and $u \subset X$, then, by **AS2**, $u \in Y$, and we obtain the desired deducibility.

On the sequence s , the axiom scheme of replacement **AS6** is translated into the scheme

$$\begin{aligned} M \models \mathbf{AS6}[s] &\equiv \forall x \in U \forall y \in U \forall y' \in U (\varphi^U(x, y, \vec{p}_M[s]) \\ &\wedge \varphi^U(x, y', \vec{p}_M[s]) \Rightarrow y = y') \Rightarrow \forall X \in U \exists Y \in U (\forall x \in U (x \in X \\ &\Rightarrow \forall y \in U (\varphi^U(x, y, \vec{p}_M[s]) \Rightarrow y \in Y)))) \end{aligned}$$

where $\vec{p}_M[s]$ denotes the line of values of the terms p_0, \dots, p_{m-1} on s under the interpretation M .

By **AS2** (LTS), for the U -predicative formula $\varphi^U(x, y, \vec{p}_M[s])$, there exists the U -class $F \equiv \{z \in U | \exists x, y \in U (z = \langle x, y \rangle_U \wedge \varphi^U(x, y, \vec{p}_M[s]))\}$. From the formula scheme cited above, we infer that the U -class F is an U -function.

Consider any U -set X and the U -class $Y \equiv F[X]$. Consider the U -class $G \equiv \{z \in U | \exists x, y \in U (z = \langle x, y \rangle_U \wedge \varphi^U(x, y, \vec{p}_M[s]) \wedge x \in X)\} = F|X \subset X *_U Y$. If $x \in X$, then $G \langle x \rangle = \emptyset \in U$ for $x \notin \text{dom}_U F$ and $G \langle x \rangle = \{F(x)\}_U$ for $x \in \text{dom}_U F$. Therefore, by the axiom of full union **A11** (LTS), $Y = \text{rng}_U G \in U$.

If $x \in X$, $y \in U$, and $\varphi^U(x, y, \vec{p}_M[s])$, then $\langle x, y \rangle_U \in F$. Thus, $y \in F[X] \equiv Y$. This proves the formula scheme $M \models \mathbf{AS6}[s]$.

On s , the axiom of empty set **A7** is translated into the formula $M \models \mathbf{A7}[s] = \mathbf{A7}^U \equiv \exists x \in U \forall z \in U (\neg(z \in x))$.

By axiom **A3** (LTS), the empty U -set \emptyset_{LTS} possesses the necessary property $\forall z \in U (z \notin \emptyset_{LTS})$.

On s , the axiom of infinity **A8** is translated into the formula $M \models \mathbf{A8}[s] \equiv \exists Y \in U (\emptyset \in Y \wedge \forall y \in U (y \in Y \Rightarrow y \cup_U \{y\}_U \in Y))$.

Consider the \mathbf{a} -set π postulated by the axiom of infra-infinity **A13** (LTS). By this axiom, $\emptyset \in \pi$, and if $y \in U$ and $y \in \pi$, then $y \cup_{\mathbf{a}} \{y\}_{\mathbf{a}} \in \pi$. Let us verify that $A \equiv y \cup_{\mathbf{a}} \{y\}_{\mathbf{a}} = y \cup_U \{y\}_U \equiv B$. Let $x \in A$. Then $x \in \mathbf{a}$ and $x \in y \vee x = y$. Since by Axiom **A5** (LTS), $\mathbf{a} \subset U$, it follows $x \in U$. Therefore, $x \in B$. Conversely, let $x \in B$, i.e., $x \in U$ and $x \in y \vee x = y$. Since $y \in \pi \in \mathbf{a}$, by axiom **A7** (LTS), we have, $y \in \mathbf{a}$. If $x \in y$, then for the same reason $x \in \mathbf{a}$. If $x = y$, then $x \in \mathbf{a}$ once again. Therefore, in each case, $x \in \mathbf{a}$. Therefore, $x \in A$. From the equality which was proved above, we infer that $y \cup_U \{y\}_U \in \pi$. This means that the translation of Axiom **A8** (ZF) is deduced in the LTS.

On s , the axiom of regularity **A9** is translated into the formula $M \models \mathbf{A9}[s] \equiv \forall X \in U (X \neq \emptyset \Rightarrow \exists x \in U (x \in X \wedge x \cap_U X = \emptyset))$. This formula is obviously deduced from the axiom of transitivity **A7** (LTS) and the axiom of regularity **A12** (LTS).

Finally, on s , the axiom of choice **A10** is translated into the formula $M \models \mathbf{A10}[s] \equiv \forall X \in U (X \neq \emptyset \Rightarrow \exists z \in U ((z \equiv \mathcal{P}_U(X) \setminus \{\emptyset\}_U \rightarrow_U X) \wedge \forall Y \in U (Y \in \mathcal{P}_U(X) \setminus \{\emptyset\}_U \Rightarrow z(Y) \in Y)))$.

If $\emptyset \neq X \in U$, then by axiom of choice **A14** (LTS), there exists the class z such that $(z \equiv \mathcal{P}_U(X) \setminus \{\emptyset\}_U \rightarrow_U X) \wedge \forall Y (Y \in \mathcal{P}_U(X) \setminus \{\emptyset\}_U \Rightarrow z(Y) \in Y)$. By Axiom **A9**, $\mathcal{P}_U(X) \in U$, and, by Axiom **A8**, $A \equiv \mathcal{P}_U(X) \setminus \{\emptyset\}_U \in U$. Therefore, by Lemma 3 of Sec. 2, $B \equiv A *_U X \in U$. From $z \subset B$, according to Axiom **A8**, we infer that $z \in U$, and this gives us the required deducibility. \square

According to Statement 1, in every universal class U , we can use all assertions for U -classes and U -sets which can be proved in ZF for the classes and sets.

5.2. The globalization of local constructions. In the same manner as was done in ZF for classes, in the LTS, for assemblies **A** and **B** and classes A and B , we define the following assemblies:

- (1) $\mathcal{P}(\mathbf{A}) \equiv \{x | x \subset \mathbf{A}\}$;
- (2) $\mathbf{A} \cup \mathbf{B} \equiv \{x | x \in \mathbf{A} \vee x \in \mathbf{B}\}$;
- (3) $\mathbf{A} \cap \mathbf{B} \equiv \{x | x \in \mathbf{A} \wedge x \in \mathbf{B}\}$;
- (4) $\{A\} \equiv \{x | x = A\}$;
- (5) $\{A, B\} \equiv \{A\} \cup \{B\} = \{x | x = A \vee x = B\}$;
- (6) $\langle A, B \rangle \equiv \{A, \{A, B\}\}$;
- (7) $\mathbf{A} * \mathbf{B} \equiv \{x | \exists y \in \mathbf{A} \exists z \in \mathbf{B} (x = \langle y, z \rangle)\}$;
- (8) $\cup \mathbf{A} \equiv \{x | \exists y \in \mathbf{A} (x \in y)\}$.

In the same manner as was done in ZF, changing the word “class” to the word “assembly” and the word “set” to the word “class”, in the LTS, we define a *correspondence* \mathbf{C} with the domain $\text{dom } \mathbf{C}$ and the class of values $\text{rng } \mathbf{C}$, a *function* (\equiv a mapping) \mathbf{F} , a *correspondence* $\mathbf{C} : \mathbf{A} \prec \mathbf{B}$, a *function* $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$, a (*multivalued*) *collection* $\langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$ with the union $\cup \langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$ and the intersection $\cap \langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$, a *simple collection* $(b_a \in \mathbf{B} | a \in \mathbf{A})$ with the assembly of members $\{b_a \in \mathbf{B} | a \in \mathbf{A}\}$, the (*multivalued*) *sequential pair* $\langle \mathbf{A}, \mathbf{A}' \rangle$, *triplet* $\langle \mathbf{A}, \mathbf{A}', \mathbf{A}'' \rangle$, ... of assemblies $\mathbf{A}, \mathbf{A}', \mathbf{A}'', \dots$, the *simple sequential pair* (a, a') , the *triplet* (a, a', a'') , ... of classes a, a', a'', \dots , the *product* $\prod \langle \mathbf{A}_i \subset \mathbf{A} | i \in \mathbf{I} \rangle$ of a *collection* $\langle \mathbf{A}_i \subset \mathbf{A} | i \in \mathbf{I} \rangle$, the *product* $\mathbf{A} \times \mathbf{A}'$, $\mathbf{A} \times \mathbf{A}' \times \mathbf{A}''$, ... of the *pair* $\langle \mathbf{A}, \mathbf{A}' \rangle$, the *triplet* $\langle \mathbf{A}, \mathbf{A}', \mathbf{A}'' \rangle$, ... of assemblies $\mathbf{A}, \mathbf{A}', \mathbf{A}'', \dots$, an *n-placed relation* $\mathbf{R} \subset \mathbf{A}^n \equiv \text{Map}(n, \mathbf{A})$ on an assembly \mathbf{A} , an *n-placed operation* $\mathbf{O} : \mathbf{A}^n \rightarrow \mathbf{A}$ on an assembly \mathbf{A} , and so on.

One can verify that the pairs $\langle a, b \rangle$ and (a, b) possesses the usual property: $\langle a, b \rangle = \langle a', b' \rangle \Leftrightarrow a = a' \wedge b = b'$ and $(a, b) = (a', b') \Leftrightarrow a = a' \wedge b = b'$ for every classes a and b .

With every assembly \mathbf{A} , in a canonical way the *collection* $\langle a \in \overline{\mathbf{V}} | a \in \mathbf{A} \rangle$ of element classes of the assembly \mathbf{A} and the *simple collection* $(a \in \mathbf{A} | a \in \mathbf{A})$ of elements of the assembly \mathbf{A} are associated. The equalities $\cup \mathbf{A} = \cup \langle a \in \overline{\mathbf{V}} | a \in \mathbf{A} \rangle$ and $\mathbf{A} = \{a \in \mathbf{A} | a \in \mathbf{A}\}$ are valid for them.

Now we will state the connection between local notions and constructions and corresponding global ones.

Lemma 1. *Let α and β be universal classes, let $A \in \alpha$, and let $A \in \beta$. Then $\mathcal{P}_\alpha(A) = \mathcal{P}_\beta(A) = \mathcal{P}(A)$.*

Proof. Let $x \in \mathcal{P}_\alpha(A)$, i. e., $x \in \alpha$ and $x \subset A$. Since $A \in \beta$, by Axiom **A8**, $x \in \beta$. Therefore, $x \in \mathcal{P}_\beta(A)$. Hence $\mathcal{P}_\alpha(A) \subset \mathcal{P}_\beta(A)$. The converse implication is verified analogously.

It is clear that $\mathcal{P}_\alpha(A) \subset \mathcal{P}(A)$. The inclusion $\mathcal{P}(A) \subset \mathcal{P}_\alpha(A)$ can be verified as above. \square

Corollary. *For every class A , the assembly $\mathcal{P}(A)$ is a class.*

Proof. By the axiom of universality **A6**, for A , there exists a universal class α such that $A \in \alpha$. Then, by the proved lemma, $\mathcal{P}(A) = \mathcal{P}_\alpha(A)$. But by the axiom scheme **AS2**, $\mathcal{P}_\alpha(A)$ is a class. \square

Lemma 2. *Let α and β be universal classes, $A, B \subset \alpha$, and let $A, B \subset \beta$. Then $A \cup_\alpha B = A \cup_\beta B = A \cup B$ and $A \cap_\alpha B = A \cap_\beta B = A \cap B$.*

Proof. Let $x \in A \cup_\alpha B$, i. e. $x \in \alpha$ and $x \in A \vee x \in B$. Then $x \in \beta$, and, therefore, $x \in A \cup_\beta B$. Thus, $A \cup_\alpha B \subset A \cup_\beta B$. The converse inclusion can be verified in the same way.

It is clear that $A \cup_\alpha B \subset A \cup B$. The inclusion $A \cup B \subset A \cup_\alpha B$ is verified as above. \square

Lemma 3. *Let α and β be universal classes, $A, B \in \alpha$, and let $A, B \in \beta$. Then $\{A\}_\alpha = \{A\}_\beta = \{A\}$, $\{A, B\}_\alpha = \{A, B\}_\beta = \{A, B\}$, and $\langle A, B \rangle_\alpha = \langle A, B \rangle_\beta = \langle A, B \rangle$.*

Proof. If $y \in \{A\}_\alpha \equiv \{x \in \alpha \mid x = A\}$, then $y = A \in \beta$, and, therefore, $y \in \{x \in \beta \mid x = A\} \equiv \{A\}_\beta$. Thus, $\{A\}_\alpha \subset \{A\}_\beta$. The converse inclusion is verified in the same way. It is clear that $\{A\}_\alpha \subset \{a\}$. The inclusion $\{A\} \subset \{A\}_\alpha$ is verified as was done above.

Now according to the proved assertions and Lemma 2, $\{A, B\}_\alpha \equiv \{A\}_\alpha \cup_\alpha \{B\}_\alpha = \{A\}_\beta \cup_\alpha \{B\}_\beta = \{A\}_\beta \cup_\beta \{B\}_\beta \equiv \{A, B\}_\beta$. Similarly, $\{A, B\}_\alpha = \{A, B\}$.

Finally, by Lemma 2 of Sec. 2, $\{A\}_\alpha \in \alpha$, $\{A\}_\beta \in \beta$, $\{A, B\}_\alpha \in \alpha$, and $\{A, B\}_\beta \in \beta$. Therefore, by the properties proved above, $\{A\}_\beta \in \alpha$ and $\{A, B\}_\beta \in \alpha$.

Consequently, applying the equality proved above, we obtain

$$\langle A, B \rangle_\alpha \equiv \{\{A\}_\alpha, \{A, B\}_\alpha\}_\alpha = \{\{A\}_\beta, \{A, B\}_\beta\}_\alpha = \{\{A\}_\beta, \{A, B\}_\beta\}_\beta \equiv \langle A, B \rangle_\beta.$$

Similarly, $\langle A, B \rangle_\alpha = \langle A, B \rangle$. \square

Corollary. *For every class A the assembly $\{A\}$ is a class.*

Proof. By the axiom of universality **A6**, for A , there exists a universal class α such that $A \in \alpha$. Then, by the proved lemma, $\{A\} = \{A\}_\alpha$. But, by the axiom scheme **AS3**, $\{A\}_\alpha$ is a class. \square

Lemma 4. *Let α and β be universal classes, $A, B \subset \alpha$, and let $A, B \subset \beta$. Then $A *_\alpha B = A *_\beta B = A * B$.*

Proof. Let $x \in A *_\alpha B$, i. e., $x \in \alpha$ and $\exists y \exists z (y \in A \wedge z \in B \wedge x = \langle y, z \rangle_\alpha)$. Since $y \in A \subset \beta$, we have $y \in \beta$. Similarly, $z \in \beta$. By Lemmas 2 of Sec. 2 and 3 $x = \langle y, z \rangle_\alpha = \langle y, z \rangle_\beta \in \beta$. Therefore, $x \in \beta$ and $\exists y \exists z (y \in A \wedge z \in B \wedge x = \langle y, z \rangle_\beta)$, i. e., $x \in A *_\beta B$. Therefore, $A *_\alpha B \subset A *_\beta B$. The converse inclusion is verified in the same way.

It is clear that $A *_\alpha B \subset A * B$. The inclusion $A * B \subset A *_\alpha B$ can be verified as above. \square

Lemma 5. *Let α and β be universal classes, $A, B \subset \alpha$, and let $A, B \subset \beta$. Then for every class u , the following assertions are equivalent:*

- (1) $u \rightleftharpoons A \prec_\alpha B$ [respectively $u \rightleftharpoons A \rightarrow_\alpha B$];
- (2) $u \rightleftharpoons A \prec_\beta B$ [respectively $u \rightleftharpoons A \rightarrow_\beta B$];
- (3) $u \rightleftharpoons A \prec B$ [respectively $u \rightleftharpoons A \rightarrow B$].

Moreover, $\text{dom}_\alpha u = \text{dom}_\beta u = \text{dom } u$ and $\text{rng}_\alpha u = \text{rng}_\beta u = \text{rng } u$.

Proof. (1) \vdash (2). By Lemma 4, $u \subset A *_\alpha B = A *_\beta B$. Therefore, $dom_\beta u \subset A$. If $x \in A$, then, by (1), $x \in A = dom_\alpha u$. Therefore, $x \in \alpha$ and $\langle x, y \rangle_\alpha \in u$ for some $y \in B$. By Lemma 3, $\langle x, y \rangle_\beta \in u$. Hence $x \in dom_\beta u$. So $A \subset dom_\beta u$. As a result, $dom_\beta u = A$.

Let $\langle x, y \rangle \in u$ and $\langle x, y' \rangle_\beta \in u$ for some $x \in A$. Then $y, y' \in rng_\beta u \subset B$. Since $x, y, y' \in \alpha$ and $x, y, y' \in \beta$, by Lemma 3, we infer that $\langle x, y \rangle_\alpha = \langle x, y \rangle_\beta \in u$ and $\langle x, y' \rangle_\alpha = \langle x, y' \rangle_\beta \in u$. From (1), we now infer that $y = y'$. This means that $u \simeq A \rightarrow_\beta B$.

All other deducibilities are proved in the same manner.

The equalities $dom_\alpha u = dom_\beta u = dom u$ and $rng_\alpha u = rng_\beta u = rng u$ are verified by using Lemma 3 in an obvious way. \square

Lemma 6. *Let α and β be universal classes, $A \subset \alpha$, and let $A \subset \beta$. Then $\cup_\alpha A = \cup_\beta A = \cup A$.*

Proof. By definition, $\cup_\alpha A \equiv \cup_\alpha \{a \in \alpha \mid a \in A\} \equiv \{z \in \alpha \mid \exists y \in A (z \in y)\}$. Therefore, if $x \in \cup_\alpha A$, then by Axiom **A7**, $x \in y \in A \subset \beta$ implies $x \in \beta$. Therefore, $x \in \cup_\beta A$. Thus, $\cup_\alpha A \subset \cup_\beta A$. The converse inclusion is verified similarly.

It is clear that $\cup_\alpha A \subset \cup A$. The inclusion $\cup A \subset \cup_\alpha A$ is verified in the same way as above. \square

Corollary. *For every class A the assembly $\cup A$, is a class.*

Proof. By the axiom of universality **A6**, for A , there exists a universal class α such that $A \in \alpha$. Then, by the previous lemma, $\cup A = \cup_\alpha A$. But by the axiom scheme **AS2**, $\cup_\alpha A$, is a class. \square

Unfortunately, for classes A and B we cannot yet prove that the assemblies $A \cup B$, $A \cap B$, $\{A, B\}$, $\langle A, B \rangle$, and $A * B$ are classes. This will be done in Sec. 5.4.

5.3. Ordinals and cardinals in the local theory of sets. In the same manner as it was done in ZF, changing the word “class” to the word “assembly” and the word “set” to the word “class,” in the LTS, we can define *ordered* and *well-ordered assemblies*, *ordinals*, and *ordinal numbers*.

In the same manner as it was done in ZF, changing the word “class” to the word “ U -class”, the word “set” to the word “ U -set,” and the word “relation” to the word “ U -relation,” in the LTS, for every universal class U we can define *U -ordered* and *well- U -ordered U -classes*, *U -ordinals*, and *U -ordinal numbers* with the following change of the definition of a well- U -ordered U -class. Precisely, an U -ordered U -class P is said to be *well- U -ordered*, if $\forall Q(Q \subset U \wedge \emptyset \neq Q \subset P \Rightarrow \exists x \in U(x \in Q \wedge \forall y \in U(y \in Q \Rightarrow x \leq y)))$, which means that every nonempty U -subclass of the U -class P has the smallest element. From $x, y \in Q \subset P \subset U$, we infer that this formula is equivalent to the formula $\forall Q(\emptyset \neq Q \subset P \Rightarrow \exists x \in Q(\forall y \in Q(x \leq y)))$ cited in condition (5) from the definition of a well-ordered class in ZF. But in the LTS, this formula has a wider sense; precisely, this means that every nonempty subclass of the U -class P has the smallest element (compare with Lemma 2, Sec. 4 in ZF).

This implies that the following lemma is fulfilled.

Lemma 7. *Let U be a universal class and $\alpha \in U$. Then the following assertions are equivalent:*

- (1) α is an ordinal number;
- (2) α is a U -ordinal number.

Now we will infer from this lemma that the assembly $\overline{\mathbf{On}} \equiv \{x \mid \text{On}(x)\}$ of all ordinal numbers in the LTS is well-ordered by the relation $\in \cup =$.

Lemma 8. *The assembly $\overline{\mathbf{On}}$ is well-ordered by the relation $\in \cup =$.*

Proof. Let α and β be ordinal classes, and let $\alpha \neq \beta$. By the axiom of universality **A6**, $\alpha \in U$ and $\beta \in V$ for some universal classes U and V . Then either $\alpha \not\subset \beta$ or $\beta \not\subset \alpha$. For certainty, let $\beta \not\subset \alpha$. In this case, by the axiom of subset **A8**, the nonempty V -set $\beta \setminus \alpha = \{\eta \in V \mid \eta \in \beta \wedge \eta \notin \alpha\} \in V$ has the smallest element $\gamma \in V$. We have $\gamma \notin \alpha$ by the definition of the V -set $\beta \setminus \alpha$. Since every element γ is an element of α by the minimality of γ , we have $\gamma \subseteq \alpha$. By the axiom of subset, $\gamma \in U$. From $\gamma \notin \alpha$, and $\gamma \subseteq \alpha$ it follows that $\gamma = \alpha$, i.e., $\alpha \in \beta$.

We have proved that $\in \cup =$ is a linear order on the assembly $\overline{\mathbf{On}}$. Let us show that this assembly is well-ordered with respect to the given order. Suppose that we have some nonempty class S of ordinal numbers. Consider a universe U , containing S . Then, by Axiom **A7**, $S \subset U$. By Lemma 7, S is an U -class of U -ordinal numbers. Since, the universe U is a model of the theory ZF by Statement 1, we infer that the U -class S of U -ordinal numbers has the smallest element. \square

The next lemma is similar to Lemma 2, Sec. 4, but it has another proof.

Lemma 9. *Let \mathbf{A} be a nonempty subassembly of the assembly $\overline{\mathbf{On}}$. Then \mathbf{A} has the smallest element.*

Proof. By the condition, there exists some ordinal number $\alpha \in \mathbf{A}$. By the axiom of universality **A6** (LTS), there exists a universal class U such that $\alpha \in U$. Consider the assembly $\mathbf{B} \equiv \{x \in U | x \in \mathbf{A} \wedge x \in \alpha \cup_U \{\alpha\}_U\}$. By the full comprehension axiom scheme **AS2** (LTS), this assembly is an U -class. Since $\alpha \in \mathbf{B} \subset \overline{\mathbf{On}}$ and the assembly $\overline{\mathbf{On}}$ is well-ordered, we infer that the class \mathbf{B} has the smallest element β . Take an arbitrary element $\gamma \in \mathbf{A}$. If $\gamma < \alpha$, then $\gamma \in \alpha \in U$. Whence, by the axiom of transitivity **A7** (LTS), $\gamma \in U$. Therefore, $\gamma \in \mathbf{B}$ in this case, and hence $\gamma \geq \beta$. If $\gamma = \alpha$, then $\gamma \in \mathbf{B}$ once again, and, therefore $\gamma \geq \beta$. Finally, if $\gamma > \alpha$, then $\gamma > \beta$. Therefore, β is the smallest element of the assembly \mathbf{A} . \square

Lemma 10. *Let α be an ordinal number. Then:*

- (1) *the assembly $\alpha^+ \equiv \alpha + 1 \equiv \alpha \cup \{\alpha\}$ is an ordinal number;*
- (2) *α^+ is the smallest of all ordinal numbers which are greater than the number α .*

Proof. (1) By Axiom **A6**, $\alpha \in U$ for some $U \bowtie$. Let $x \in \alpha^+_U \equiv (\alpha + 1)_U \equiv \alpha \cup_U \{\alpha\}_U$. Then $x \in U$ and either $x \in \alpha$ or $x = \alpha$. Therefore, $x \in \alpha^+$. Let $y \in \alpha^+$. In this case either $y \in \alpha$ or $y \in \alpha$. In both cases, $y \in U$. This means that $y \in \alpha^+_U$. Thus, $\alpha^+ = \alpha^+_U$ and α^+ is a U -ordinal number and, therefore, an ordinal number. It is clear that $\alpha^+ > \alpha$.

(2) Let β be an ordinal number such that $\beta > \alpha$. Suppose that $\alpha^+ > \beta$. Then $\beta \in \alpha^+$, i.e., either $\beta \in \alpha$ or $\beta = \alpha$, but this contradicts the condition $\beta > \alpha$. From this contradiction, we infer that $\beta \geq \alpha^+$. \square

An ordinal number α^+ will be called the *successor of the ordinal number α* .

Lemma 11. *If A is a nonempty class of ordinal numbers, then:*

- (1) *$\cup A$ is an ordinal number;*
- (2) *$\cup A = \sup A$ in the ordered assembly $\overline{\mathbf{On}}$.*

Proof. By Axiom **A6**, $A \in U$ for some universal class U . Then, by Axiom **A7**, $A \subset U$. Consider the assembly $X \equiv \cup A$ and the U -class $Y \equiv \cup_U A$. By Lemma 6, $X = Y$. By Lemma 3, Sec. 4 from ZF and Statement 1, Y is an U -ordinal number, and, therefore, by Lemma 7, it is an ordinal number. Thus, X is also an ordinal number.

Let $a \in A$. If $X < a$, then $X \in a \in A$ implies $X \in \cup A = X$, but this is impossible. Whence $a \leq X$, and, therefore, X is an upper bound of the class A . Let α be an ordinal number, and let $\alpha \geq a$ for every $a \in A$. Suppose that $X > \alpha$. Then $\alpha \in X = \cup A$ implies $\alpha \in a$ for some $a \in A$. Hence $\alpha < a$, but this is impossible. Therefore, $X \leq \alpha$. Thus $X = \sup A$ in $\overline{\mathbf{On}}$. \square

A *limiting ordinal* is an ordinal which is not equal to α^+ for any ordinal number α .

As in Sec. 4, classes A and B are said to be *equivalent* ($A \sim B$) if there exists a one-to-one (\equiv bijective) function $u : A \rightleftharpoons B$.

An ordinal number α will be called *cardinal* if for every ordinal number β , the conditions $\beta \leq \alpha$ and $\beta \sim \alpha$ imply $\beta = \alpha$. The assembly of all cardinal numbers will be denoted by $\overline{\mathbf{Cn}}$. The assembly $\overline{\mathbf{Cn}}$ with the order induced from the assembly $\overline{\mathbf{On}}$ is well-ordered.

Let U be a universal class. Two U -classes A and B are said to be *U -equivalent* ($A \sim_U B$) if there exists a bijective U -function $u : A \rightleftharpoons_U B$.

A U -ordinal number α is said to be *U -cardinal* if for every U -ordinal number β , the conditions $\beta \leq \alpha$ and $\beta \sim_U \alpha$ imply $\beta = \alpha$.

Proposition 1. *Let U be a universal class and $\alpha \in U$. Then the following assertions are equivalent:*

- (1) α is a cardinal number;
- (2) α is an U -cardinal number.

Proof. (1) \vdash (2). By Lemma 7, α is a U -ordinal number. Let β be a U -ordinal number such that $\beta \leq \alpha$ and $\beta \sim_U \alpha$. This means that there exists a bijective U -mapping $f : \beta \xrightarrow{\sim} \alpha$. By the axiom of transitivity **A7**, $\beta \subset \alpha \subset U$. Therefore, by Lemma 5, $f \xrightarrow{\sim} \beta \xrightarrow{\sim} \alpha$. Hence $\beta \sim \alpha$. By condition (1), we obtain $\beta = \alpha$.

(2) \vdash (1). By Lemma 7, α is an ordinal number. Let β be an ordinal number such that $\beta \leq \alpha$ and $\beta \sim \alpha$. This means that there exists a bijective mapping $f : \beta \xrightarrow{\sim} \alpha$. By Axiom **A7**, $\beta \subset \alpha \subset U$. Therefore, by Lemma 5, $f \xrightarrow{\sim} \beta \xrightarrow{\sim} \alpha$, i. e., $\beta \sim_U \alpha$. By condition (2), we infer $\beta = \alpha$. \square

The power $\text{card}_U A$ of a set $A \in U$ in a universe U is an U -cardinal $\alpha \in U$ such that there exists a one-to-one U -mapping $f : A \xrightarrow{\sim} \alpha$. The power $\text{card} A$ of a class A is a cardinal α such that $A \sim \alpha$.

Proposition 2. *Suppose that $A \in U \in V$, $U \bowtie$, and $V \bowtie$. Then*

$$\text{card} A = \text{card}_U A = \text{card}_V A < \text{card}_V U.$$

Proof. Let $\text{card}_U A = \alpha$, $\alpha \in U$. Then $\alpha \in V$. By definition, there exists a one-to-one function $f : A \xrightarrow{\sim} \alpha$. By Lemma 5, $f \xrightarrow{\sim} A \xrightarrow{\sim} \alpha$.

By Proposition 1, α is a U -cardinal number. Therefore, $\alpha = \text{card}_V A$.

Similarly, by Lemma 5, $f \xrightarrow{\sim} A \xrightarrow{\sim} \alpha$, and, by Proposition 1, α is a cardinal number. Thus, $\alpha = \text{card} A$.

Let us show now that

$$\text{card}_V A < \text{card}_V U.$$

According to Statement 1, $\text{card}_V A \leq \text{card}_V U$. Suppose that

$$\text{card}_V A = \text{card}_V U = \alpha, \quad \alpha \in V.$$

From the assertions proved above we infer that $\text{card}_U A = \text{card}_V A = \alpha$ implies $\alpha \in U$. By Axiom **A7**, $\alpha \subset U$, $\alpha \subset V$ and $U \subset V$. By the definition of V -power, there exists a bijective V -function $f : \alpha \xrightarrow{\sim} U$. According to Lemma 5 $f \xrightarrow{\sim} \alpha \xrightarrow{\sim} U$. By the axiom of full union **A11**, $U = \text{rng}_U f \in U$.

We infer that $U \in U$, but this is impossible. \square

An inaccessible cardinal number is defined in the LTS as was done in the theory ZF.

An U -cardinal number \varkappa is said to be U -regular if for every U -ordinal number β for which there exists a U -function $f : \beta \rightarrow_U \varkappa$ such that $\cup_U \text{rng}_U f = \varkappa$, $\varkappa \leq \beta$. An U -cardinal number $\varkappa > \omega_0 \equiv \omega$ is said to be U -inaccessible if \varkappa is U -regular and for every U -ordinal number λ , from $\lambda \in \varkappa$, it follows that $\mathcal{P}_U(\lambda) \in \varkappa$.

Proposition 3. *Let U be a universal class and $\alpha \in U$. Then the following conditions are equivalent:*

- (1) α is an inaccessible cardinal number;
- (2) α is a U -inaccessible U -cardinal number.

Proof. (1) \vdash (2). By Proposition 1, α is an U -cardinal number. Let $\beta \in U$, β be an U -ordinal number, and let there exist a U -mapping $f : \beta \rightarrow_U \alpha$ such that $\cup_U \text{rng}_U f = \alpha$. By Lemma 7, β is an ordinal number. Since $\beta, \alpha \in U$, $\beta, \alpha \subset U$ by the axiom of transitivity. Therefore, by Lemma 5, $f \xrightarrow{\sim} \beta \rightarrow \alpha$ and $\text{rng}_U f = \text{rng} f$. Whence $\alpha = \cup_U \text{rng} f$. By Lemma 6, $\alpha = \cup \text{rng} f$. Since α is a regular cardinal number, we infer that $\alpha \leq \beta$. Hence α is a U -regular U -cardinal number.

Let $\beta \in U$, β be an U -ordinal number, and let $\beta \in \alpha$. By Lemma 7, β is an ordinal number. Since α is inaccessible, $\mathcal{P}(\beta) \in \alpha$. By Lemma 1, $\mathcal{P}_U(\beta) = \mathcal{P}(\beta) \in \alpha$. Therefore α is a U -inaccessible U -cardinal number.

(2) \vdash (1). By Proposition 1, α is a cardinal number. Let β be an ordinal number, and let $f : \beta \rightarrow \alpha$ be a mapping such that $\cup \text{rng} f = \alpha$. Suppose that $\beta \in \alpha \in U$. By the axiom of transitivity **A7**, $\beta \in U$. By Lemma 7, β is an U -ordinal number. By **A8**, $\beta, \alpha \subset U$. Therefore, by Lemma 7, $f \xrightarrow{\sim} \beta \rightarrow_U \alpha$ and $\text{rng} f = \text{rng}_U f$. By Lemma 6, $\alpha = \cup \text{rng}_U f = \cup_U \text{rng}_U f$. Since α is a U -regular U -cardinal number, we

infer that $\alpha \leq \beta < \alpha$, but this is impossible. From this contradiction, we infer that the case $\beta \in \alpha$ is impossible. Thus, $\alpha \leq \beta$. Hence α is a regular cardinal number.

Let β be an ordinal number, and let $\beta \in \alpha$. By Axiom **A7**, $\beta \in U$. By Lemma 7, β is a U -ordinal number. Since α is U -inaccessible, $\mathcal{P}_U(\beta) \in \alpha$. By Lemma 1 and Proposition 2, $\text{card } \mathcal{P}(\beta) = \text{card } \mathcal{P}_U(\beta) = \text{card}_U \mathcal{P}_U(\beta) \in \alpha$. Therefore, α is an inaccessible cardinal number. \square

In the LTS, we can use the principle of transfinite induction because of the well-ordering of the assembly of all ordinals. Let us show this.

Theorem 1 (principle of transfinite induction in the LTS). *Let \mathbf{C} be an assembly of ordinal numbers such that:*

- (1) $\emptyset \in \mathbf{C}$;
- (2) $\alpha \in \mathbf{C} \Rightarrow \alpha + 1 \in \mathbf{C}$;
- (3) (α is a limiting ordinal number $\wedge \alpha \in \mathbf{C}$) $\Rightarrow \alpha \in \mathbf{C}$.

Then $\mathbf{C} = \overline{\mathbf{On}}$.

Proof. Suppose that this is not true. Consider the subassembly $\mathbf{D} \equiv \overline{\mathbf{On}} \setminus \mathbf{C}$.

Since the assembly \mathbf{D} is not empty, by Lemma 9, we infer that it has the smallest element.

Then $\gamma \neq \emptyset$, because $\emptyset \in \mathbf{C}$. Therefore γ is either β^+ for some ordinal number β or a limiting ordinal number. Suppose that $\gamma = \beta + 1$. Since $\beta \in \gamma$, it follows that $\beta \notin \mathbf{D}$, and, therefore, $\gamma \in \mathbf{C}$. By condition (2) of the theorem, $\gamma = \beta + 1 \in \mathbf{C}$. From this contradiction, we infer that $\gamma \neq \beta + 1$. Now the case remains where γ is a limiting ordinal number. In this case, $\gamma = \sup \gamma$ and $\gamma \subset \mathbf{C}$; therefore, by the condition (3) of the theorem, $\gamma = \sup \gamma \in \mathbf{C}$. Therefore, the assembly \mathbf{D} is empty, and, therefore, $\mathbf{C} = \overline{\mathbf{On}}$. \square

Theorem 2 (principle of natural induction in the LTS). *Let \mathbf{C} be some assembly in LTS such that:*

- (1) $\emptyset \in \mathbf{C}$;
- (2) for all $n \in \omega$ the condition $n \in \mathbf{C}$ implies $n + 1 \in \mathbf{C}$. Then $\omega \subseteq \mathbf{C}$.

Proof. Consider the assembly $\tilde{\mathbf{C}} = \mathbf{C} \cap \omega$. This assembly is nonempty, because $\emptyset \in \mathbf{C} \wedge \emptyset \in \omega \Rightarrow \emptyset \in \tilde{\mathbf{C}}$, and, moreover, it contains only ordinal numbers. Suppose that the assertion of the theorem is not fulfilled. In this case, the assembly $\overline{\mathbf{C}} = \{x \mid x \in \omega \wedge x \notin \tilde{\mathbf{C}}\}$ is nonempty, because it is a subassembly of $\overline{\mathbf{On}}$ and, therefore, contains the smallest element $\alpha \in \omega$. We know that $\alpha \neq \emptyset$, because $\emptyset \in \tilde{\mathbf{C}}$. Since $\alpha \in \omega$ and $\alpha \neq \emptyset$, there exists β such that $\alpha = \beta + 1$. In this case, $\beta \in \tilde{\mathbf{C}}$, because α is the smallest ordinal number in $\overline{\mathbf{C}}$. By condition (2) of the theorem, $\beta + 1 \in \tilde{\mathbf{C}}$ in this case, i.e., $\alpha \in \tilde{\mathbf{C}}$, and we obtain a contradiction with our assumption. \square

5.4. Mirimanov–von Neumann classes in the local theory of sets and their connection with universal classes. Using all previous material, we will construct Mirimanov–von Neumann classes in the LTS.

Consider an arbitrary universe U . Since it is a model of ZF, using the construction by transfinite induction, we can, define Mirimanov–von Neumann U -sets V_α^U for every U -ordinal number $\alpha \in U$. By the axiom of universality **A6** and Lemma 12, for every ordinal number α , we define some Mirimanov–von Neumann U -set V_α^U for every universe U such that $\alpha \in U$.

Lemma 12. *Let U and W be universal classes, and let α be an ordinal number such that $\alpha \in U$ and $\alpha \in W$. Then $V_\alpha^U = V_\alpha^W$.*

Proof. We will prove this by the transfinite induction. Consider the subassembly \mathbf{C} of the assembly $\overline{\mathbf{On}}$ consisting of all ordinal numbers α such that either $\alpha \in U$, $\alpha \in W$, and $V_\alpha^U = V_\alpha^W$, or $\alpha \notin U$, or $\alpha \notin W$.

Since $\emptyset \in U$, $\emptyset \in W$, $V_\emptyset^U = \emptyset$, and $V_\emptyset^W = \emptyset$, it follows that $V_\emptyset^U = V_\emptyset^W$. Therefore, $\emptyset \in \mathbf{C}$.

Let $\alpha \in \mathbf{C}$. If $\alpha \notin U$ or $\alpha \notin W$, then $\alpha + 1 \notin U$ or $\alpha + 1 \notin W$ respectively, i.e., $\alpha + 1 \in \mathbf{C}$. Let $\alpha \in U$, $\alpha \in W$. By the facts proved in Sec. 4, $V_{\alpha+1}^U = V_\alpha^U \cup_U \mathcal{P}_U(V_\alpha^U)$ and $V_{\alpha+1}^W = V_\alpha^W \cup_W \mathcal{P}_W(V_\alpha^W)$. By

assumption, $V_\alpha^U = V_\alpha^W \equiv V_\alpha$. Since $V_\alpha \in U$ and $V_\alpha \in W$, it follows that $\mathcal{P}_U(V_\alpha) = \mathcal{P}_W(V_\alpha)$ by Lemma 1. Hence $V_{\alpha+1}^U = V_\alpha \cup_U \mathcal{P}_U(V_\alpha) \equiv P$ and $V_{\alpha+1}^W = V_\alpha \cup_W \mathcal{P}_U(V_\alpha) \equiv Q$. By Axiom **A9** (LTS), $\mathcal{P}_U(V_\alpha) \in U$ and $\mathcal{P}_U(V_\alpha) \in W$. Using Axiom **A7**, we can easily verify that $P = Q$. Thus, $V_{\alpha+1}^U = V_{\alpha+1}^W$. Therefore, $\alpha + 1 \in \mathbf{C}$.

Let α be a limiting ordinal number such that $\alpha \subset \mathbf{C}$. If $\alpha \notin U$ or $\alpha \notin W$, then $\alpha \in \mathbf{C}$. Consider the case $\alpha \in U$ and $\alpha \in W$. By Axiom **A7**, $\alpha \subset U$ and $\alpha \subset W$. Since $\alpha \subset \mathbf{C}$, we have $V_\beta^U = V_\beta^W \equiv V_\beta$ for all $\beta \in \alpha$. By the assertions proved in Sec. 4,

$$V_\alpha^U = \cup_U \langle V_\beta^U | \beta \in \alpha \rangle_U \quad \text{and} \quad V_\alpha^W = \cup_W \langle V_\beta^W | \beta \in \alpha \rangle_W.$$

Since $V_\beta^U = V_\beta^W = V_\beta$ and $V_\beta \in U, V_\beta \in W$ for all $\beta \in \alpha$, we can write $V_\alpha^U = \cup_U \langle V_\beta | \beta \in \alpha \rangle_U$ and $V_\alpha^W = \cup_W \langle V_\beta | \beta \in \alpha \rangle_W$. Let us show that $R \equiv \cup_U \langle V_\beta | \beta \in \alpha \rangle_U = \cup_W \langle V_\beta | \beta \in \alpha \rangle_W \equiv S$.

Let $x \in R$. Then $x \in U$ and $x \in V_\beta$ for some $\beta \in \alpha$. Since $x \in V_\beta \in W$ by Axiom **A7**, we have $x \in W$. Therefore, $x \in S$. Thus, $R \subset S$. The innverse implication is verified similarly. Therefore, we infer that $V_\alpha^U = R = S = V_\alpha^W$. This means that $\alpha \in \mathbf{C}$.

By Theorem 3, $\mathbf{C} = \overline{\mathbf{On}}$. □

Using this lemma, for every $\alpha \in \overline{\mathbf{On}}$, we can define the Mirimanov–von Neumann class \overline{V}_α in the LTS (we draw a line over V_α to differentiate these classes from the corresponding classes in ZF) as the U -class V_α^U for every universe U satisfying the condition $\alpha \in U$. As a result, we obtain the *Mirimanov–von Neumann collection* $\langle \overline{V}_\alpha \subset \overline{\mathbf{V}} | \alpha \in \overline{\mathbf{On}} \rangle$ in the LTS. It satisfies properties (1)–(3) of the Mirimanov–von Neumann collection in ZF listed in paragraph 4.

Lemma 13. *The collection $\langle \overline{V}_\alpha \subset \overline{\mathbf{V}} | \alpha \in \overline{\mathbf{On}} \rangle$ possesses the following properties:*

- (1) $\alpha = \beta \iff \overline{V}_\alpha = \overline{V}_\beta$;
- (2) $\alpha < \beta \iff \overline{V}_\alpha \in \overline{V}_\beta$.

Proof. First, we show that $\alpha < \beta$ implies $\overline{V}_\alpha \in \overline{V}_\beta$. Suppose that $\beta \in U$ for some universe U . Then $\alpha \in U$ implies $\overline{V}_\alpha = V_\alpha^U$ and $\overline{V}_\beta = V_\beta^U$. By Statement 1, our assertion follows from the fact that in ZF, by Lemma 7, Sec. 4, $\alpha < \beta \Rightarrow V_\alpha \in V_\beta$. Now we will prove all the assertions of the lemma. The assertion $\alpha = \beta \Rightarrow \overline{V}_\alpha = \overline{V}_\beta$ is proved above. If $\overline{V}_\alpha = \overline{V}_\beta$, then either $\alpha < \beta$ or $\alpha = \beta$, or $\beta < \alpha$. If $\alpha < \beta$, then $\overline{V}_\alpha \in \overline{V}_\beta$; if $\beta < \alpha$, then $\overline{V}_\beta \in \overline{V}_\alpha$; therefore, $\beta = \alpha$.

The assertion $\alpha < \beta \Rightarrow \overline{V}_\alpha \in \overline{V}_\beta$ is already proved. If $\overline{V}_\alpha \in \overline{V}_\beta$, then $\alpha < \beta$, because for $\alpha = \beta$, we have $\overline{V}_\alpha = \overline{V}_\beta$, and for $\beta < \alpha$, we have $\overline{V}_\beta \in \overline{V}_\alpha$. □

The following theorem shows that all universal classes in the LTS are Mirimanov–von Neumann sets for inaccessible cardinal indices.

Theorem 3. *Let U be an arbitrary universal class. Then:*

- (1) $\varkappa \equiv \sup(\overline{\mathbf{On}} \cap U) = \cup(\overline{\mathbf{On}} \cap U) \subset U$ is an inaccessible cardinal number;
- (2) $U = \overline{V}_\varkappa$;
- (3) the correspondence $q : U \mapsto \varkappa$ such that $U = \overline{V}_\varkappa$ is an injective isotone mapping from the assembly \mathbf{U} of all universal classes into the assembly $\overline{\mathbf{In}}$ of all inaccessible cardinal numbers.

Proof. (1) Since $A \equiv \overline{\mathbf{On}} \cap U$ is a nonempty class, because it contains the element $0 \equiv \emptyset$, by Lemma 11 we infer that \varkappa is an ordinal number.

Suppose that \varkappa is not a cardinal number. In this case, there exist an ordinal number $\alpha < \varkappa$ and a bijective function $f : \alpha \rightarrow \varkappa$. Since $\varkappa \subset U$ and $\alpha \subset U$, by Lemma 4, we have $\alpha * \varkappa = \alpha * U \varkappa$. Therefore, f is an U -function $f : \alpha \rightarrow_U \varkappa$. Since $\alpha \in \varkappa \subset U$ and $f(x) \in \varkappa \subset U$ for every $x \in \beta$, by the axiom of full union, for the universal class U , we infer that $\varkappa = \text{rng}_U f \in U \cap \overline{\mathbf{On}}$ and, therefore, by the axiom of binary union, $\varkappa_U^+ \equiv \varkappa \cup_U \{\varkappa\}_U \in U$. By Lemma 10, $\varkappa_U^+ = \varkappa^+ \in \overline{\mathbf{On}}$. Thus, $\varkappa_U^+ \leq \varkappa < \varkappa_U^+$, but this is impossible. From this contradiction, we infer that \varkappa is a cardinal number.

Suppose that the cardinal \varkappa is not regular. Then $\alpha \equiv cf(\varkappa) < \varkappa$. By definition, there exists a function $f : \alpha \rightarrow \varkappa$ such that $\sup f[\alpha] = \varkappa$. As above, f is an U -function $f : \alpha \rightarrow_U \varkappa$ and $rng_U f \in U$. It is clear that $rng_U f \subset f[\alpha]$. Conversely, if $y \in f[\alpha]$, then $y = f(x)$ for some $x \in \alpha$. Since $f(x) \in \varkappa \subset U$, we have $y \in U$. Consequently, $y \in rng_U f$. As a result, $f[\alpha] = rng_U f \in U$. By the axiom of full union,

$$\varkappa = \sup f[\alpha] = \cup f[\alpha] = \cup_U [y \subset U | y \in f[\alpha]]_U \in U.$$

Similarly, as was done before, the property $\varkappa \in U$ leads to a contradiction. Therefore, \varkappa is a regular cardinal.

Let λ be a cardinal number such that $\lambda < \varkappa$. Since $\lambda \in \varkappa \subset U$, by the axiom of full anseble and by Lemma 1, we have $\mathcal{P}(\lambda) = \mathcal{P}_U(\lambda) \in U$. Consequently, $\alpha \equiv card \mathcal{P}(\lambda) = card \mathcal{P}_U(\lambda)$. According to Proposition 2, this last number is equal to the number $card_U \mathcal{P}_U(\lambda) \in U$. Therefore, $\alpha \in U \cap \overline{\mathbf{On}}$. Therefore, $\alpha \leq \varkappa$. Suppose that $\varkappa = \alpha$. Then $\varkappa \in U$. But, as above, this property leads to a contradiction. As a result, we infer that $\alpha < \varkappa$.

Now it remains only to show that $\varkappa > \omega$. Actually, since $\omega \in \mathbf{a}$ (see the end of Sec. 2.1), we have $\omega \in U$, and, therefore, $\omega + 1 = \omega \cup \{\omega\} \in U$. Therefore, $\omega \in \omega + 1 \in A$ implies $\omega \in \cup A = \varkappa$.

The assertion (1) is proved.

(2) From (1) it follows that \varkappa is a limiting ordinal number.

Therefore, $\overline{V_\varkappa} = \cup(\overline{V_\beta} | \beta \in \varkappa)$. Since $\beta \in \varkappa \subset U$, by the definition, we have $\overline{V_\beta} = V_\beta^U \subset U$. Consequently, $\overline{V_\varkappa} \subset U$. Conversely, let $x \in U$. By Lemma 18, Sec. 4, $\mathbf{\Pi} = \mathbf{V}$ in ZF. Similarly $\cup_U(\overline{V_\alpha^U} | \alpha \in \overline{\mathbf{On}} \cap U) = U$ in the LTS. Therefore, $x \in V_\alpha^U$ for some $\alpha \in A \subset U$. Since $V_\alpha^U = \overline{V_\alpha}$, we have $x \in \overline{V_\alpha} \subset \overline{V_\varkappa}$. Therefore, $U \subset \overline{V_\varkappa}$. As a result, we infer that $U = \overline{V_\varkappa}$.

(3) From Lemma 13, we infer that \varkappa is unique. Therefore, we can define a mapping $q : \mathbf{U} \rightarrow \overline{\mathbf{In}}$ such that $q(U) = \varkappa$, where $U = \overline{V_\varkappa}$. From Lemma 13 we also infer that q is isotone. \square

Corollary 1. *If U is a universal class, then $\varkappa \equiv card U$ is an inaccessible cardinal number and $U = \overline{V_\varkappa}$.*

Proof. According to Theorem 3, we need to only show that for any inaccessible cardinal number \varkappa , $card \overline{V_\varkappa} = \varkappa$.

Consider some universal class W such that $\varkappa \in W$. In this case, $\overline{V_\varkappa} = V_\varkappa^W \in W$. By Proposition 2, $card \overline{V_\varkappa} = card_W \overline{V_\varkappa}$. Since the universe W is a model of ZF, by Lemma 15, Sec. 4, the property $\varkappa = card_W V_\varkappa^W = card \overline{V_\varkappa}$ is fulfilled in it.

Therefore, if U is a universal class, then, by Theorem 3, $U = \overline{V_\varkappa}$, where \varkappa is an inaccessible cardinal number. Now our assertion follows directly from the property $card U = card \overline{V_\varkappa} = \varkappa$. \square

Corollary 2. *In the LTS, the equality $\cup(\overline{V_\alpha} | \alpha \in \overline{\mathbf{On}}) = \overline{\mathbf{V}}$ is valid.*

Proof. We need to show that for an arbitrary class x in the LTS, the assertion $x \in \cup(\overline{V_\alpha} | \alpha \in \overline{\mathbf{On}})$ is valid, i.e., there exists $\alpha \in \overline{\mathbf{On}}$ such that $x \in \overline{V_\alpha}$.

By the axiom of universality, there exists a universe U such that $x \in U$, and, by Theorem 3, $U = \overline{V_\varkappa}$ for some $\varkappa \in \overline{\mathbf{On}}$. Therefore, $x \in \overline{V_\varkappa}$, i.e., our assertion is true. \square

Theorem 3 allows to make the following conclusions about the structure of the assembly $\mathbf{U} \equiv \{U | U \bowtie\}$ of all universal classes.

The relation $\in \cup =$ is an order relation on the assembly \mathbf{U} . We will denote it by \leq , i.e., $U \leq V$ if $U \in V$ or $U = V$. By Axiom **A7**, the assembly \mathbf{U} is transitive. Therefore, $U \in V$ implies $U \subset V$. Thus, $U \leq V$ implies $U \subset V$. Now we will prove that these relations are equivalent.

Proposition 4. *Let U and V be universal classes. Then the relation $U \leq V$ is equivalent to the relation $U \subset V$.*

Proof. We need only to verify that $U \subset V$ implies $U \leq V$. By Theorem 3, $U = \overline{V_\pi}$ and $V = \overline{V_\varkappa}$ for some inaccessible cardinals π and \varkappa . If $\pi = \varkappa$, then $U = \overline{V_\pi} = \overline{V_\varkappa} = V$. If $\pi < \varkappa$, then, by Lemma 13, $U = \overline{V_\pi} \in \overline{V_\varkappa} = V$. Finally if $\pi > \varkappa$, then, by the same lemma, $V = \overline{V_\varkappa} \in \overline{V_\pi} = U \subset V$, but this is impossible. Therefore, $U \leq V$. \square

Corollary 1. *The infra-universe \mathbf{a} is the smallest element in the assembly \mathbf{U} of all universal classes.*

Corollary 2. *If U is a universal class then either $U = \mathbf{a}$ or $\mathbf{a} \in U$.*

Corollary 3. *With the universal class \mathbf{a} is associated a unique inaccessible cardinal number \varkappa^* such that $\mathbf{a} = \overline{V_{\varkappa^*}}$. This number is the smallest in the assembly $\overline{\mathbf{In}}$ of all inaccessible cardinal numbers.*

Thus, in the LTS, there exists at least one inaccessible cardinal number.

Now let us prove now that in the LTS, there exists more than one inaccessible cardinal number.

5.5. The structure of the assemblies of all universal classes and all inaccessible cardinals in the local theory of sets.

Proposition 5. *In the LTS for every ordinal number α , there exists an inaccessible cardinal number \varkappa such that $\alpha < \varkappa$.*

Proof. By the axiom of universality, $\alpha \in U$ for some universal class U . By Theorem 3, $U = \overline{V_{\varkappa}}$ for some inaccessible cardinal \varkappa . By definition, $\overline{V_{\alpha}} = V_{\alpha}^U \subset U = \overline{V_{\varkappa}}$. By Lemma 13, $\alpha \leq \varkappa$. Suppose that $\varkappa = \alpha$. Then $\varkappa \in U$. But this property leads to a contradiction, as was shown in the proof of Theorem 3. Therefore, $\alpha < \varkappa$. \square

This property is similar to the axiom of universality, which postulates that every class in the LTS is an element of some universal class.

The parallelism between properties of universal classes and inaccessible cardinals in the LTS is also confirmed by the following assertions.

Theorem 4. *The assembly \mathbf{U} of all universal classes with respect to the order \subset is well-ordered. Furthermore, every subassembly of the assembly \mathbf{U} has the smallest element.*

Proof. Let $\emptyset \neq \mathbf{A} \subset \mathbf{U}$. Using the injective and strictly monotone mapping $q : U \mapsto \varkappa$ from the assembly \mathbf{U} into the assembly $\overline{\mathbf{On}}$ of the form $U = \overline{V_{\varkappa}}$ from Theorem 3, we can consider for the assembly \mathbf{A} the subassembly $\mathbf{B} \equiv q[\mathbf{A}] \equiv \{x \mid x \in \overline{\mathbf{On}} \wedge \exists U \in \mathbf{A}(z = q(U))\}$ of the assembly $\overline{\mathbf{On}}$. By Lemma 9, it has the smallest element π , which is an inaccessible cardinal. Since $\pi \in \mathbf{B}$, we have $\pi = q(U)$ for some $U \in \mathbf{A}$, i.e., $U = \overline{V_{\pi}}$. Since the mapping q is injective and strictly monotone, it follows that U is the smallest element in the assembly \mathbf{A} . \square

Corollary 1. *For every class A there exists a universe $U(A)$, which is the smallest in the assembly of all universes U such that $A \in U$.*

Corollary 2. *The intersection $\cap \mathbf{A} \equiv \{x \mid \forall U \in \mathbf{A}(x \in U)\}$ of any nonempty assembly \mathbf{A} of universal classes is a universal class.*

Proof. By Theorem 4, the subassembly \mathbf{A} of the assembly \mathbf{U} has the smallest element \mathbf{U} . It is clear that $\cap \mathbf{A} \subset U$. If $V \in \mathbf{A}$, then $U \leq V$ implies $U \subset V$. Therefore, $U \subset \cap \mathbf{A}$. So $\cap \mathbf{A} = U$. \square

Theorem 4 allows us to complete the globalization of local constructions, which was started in Sec. 5.2.

Corollary 3. *For every classes A and B , the assemblies $A \cup B$, $A \cap B$, $\{A, B\}$, $\langle A, B \rangle$, and $A * B$ are classes.*

Proof. By the axiom of universality **A6**, for A and B , there exist universal classes β and γ such that $A \in \beta$ and $B \in \gamma$. By Theorem 4, either $\beta \subset \gamma$ or $\gamma \subset \beta$. Therefore, there exists a universal class α ($\alpha = \beta$ or $\alpha = \gamma$) such that $\beta, \gamma \subset \alpha$. Therefore, $A, B \in \alpha$. By the axiom of transitivity **A7**, $A, B \subset \alpha$. Therefore, by Lemmas 2–5, $A \cup B = A \cup_{\alpha} B$, $A \cap B = A \cap_{\alpha} B$, $\{A, B\} = \{A, B\}_{\alpha}$, $\langle A, B \rangle = \langle A, B \rangle_{\alpha}$, and $A * B = A *_{\alpha} B$. By the axiom scheme **AS2**, the right-hand sides of all these equalities are classes, because they are defined by α -predicative formulas. \square

Corollary 4. *Let $n \in \omega \setminus 1$, and let \mathbf{F} be an assembly such that \mathbf{F} is a mapping from the class n into the assembly $\overline{\mathbf{V}}$. Then the assembly \mathbf{F} is a class.*

Proof. Consider the assembly $\mathbf{K}' \subset n$ consisting of all natural numbers $k \in n$ such that the assembly $\mathbf{F}|(k+1)$ is a class. Consider the assembly $\mathbf{K}'' \equiv \omega \setminus n$ and the assembly $\mathbf{K} \equiv \mathbf{K}' \cup \mathbf{K}''$.

Since $\mathbf{F}|1 = \langle 0, \mathbf{F}(0) \rangle$, by Corollary 3 to Theorem 4, we infer that $\mathbf{F}|1$ is a class.

Let $k \in \omega$ and $k \in \mathbf{K}$. If $k < n - 1$, then $k + 1 \in n$. Since $k \in \mathbf{K}'$ in this case, the assembly $\mathbf{F}|(k+1)$ is a class. By Corollary 3 to Theorem 4, the assembly $\langle k+1, \mathbf{F}(k+1) \rangle$ is also a class. Now from the equality $\mathbf{F}|(k+2) = \mathbf{F}|(k+1) \cup \{ \langle k+1, \mathbf{F}(k+1) \rangle \}$, by the mentioned corollary, we infer that the assembly $\mathbf{F}|(k+2)$ also is a class. This means that $k+1 \in \mathbf{K}' \subset \mathbf{K}$.

If $k \geq n - 1$, then $k + 1 \geq n$ implies $k + 1 \in \mathbf{K}'' \subset \mathbf{K}$.

Applying Theorem 2, we conclude that $\omega \subset \mathbf{K} \subset \omega$. Therefore, $\mathbf{K}' = n$. Consequently $n - 1 \in \mathbf{K}'$ means that the assembly $\mathbf{F} = \mathbf{F}|((n - 1) + 1)$ is a class. \square

Corollary 5. *Let A, A', A'', \dots be classes. Then the assemblies (A, A') , (A, A', A'') , \dots are classes.*

Proof. By definition, the assemblies (A, A') , (A, A', A'') , \dots are particular cases of sequences

$$(A_0, \dots, A_{n-1})$$

when $n \in \omega \setminus 2$. But, by the previous corollary, the sequences (A_0, \dots, A_{n-1}) are classes. \square

Thus, the assembly $\overline{\mathbf{V}}$ of all classes in the LTS allows us to produce almost all set-theoretical constructions which are possible in a universal class or in the NBG-universe, except the construction of full union, which is basic for the construction by the transfinite induction. The fact that the construction of full union is actually impossible in the LTS follows from Statement 1, Sec. 8. This means that the global assembly of all classes in the LTS with respect to its constructive possibilities is much poorer than local universal classes in it.

The next theorem describes the structure of the assembly \mathbf{U} of all universes in the LTS. It is proved by using of Theorem 4.

Theorem 5. *In the LTS, for every $n \in \omega$, there exist a unique universal class U and a unique U -sequence of universal classes $u(n) \equiv (U_k \in U | k \in n+1)_U$ such that $U_0 = \mathbf{a}$, $U_k \in U_l$ for every $k \in l \in n+1$ and if V is a universal class and $U_0 \subset V \in U$, then $V = U_k$ for some $k \in n+1$ (the property of uncompressibility).*

From the property of uniqueness, it follows that $u(n)|m+1 = u(m)$ for all $m \leq n$, i. e., these finite sequences continue each other.

Proof. Consider the set N consisting of all $n \in \omega$ for which there exist a unique universal class U and a unique sequence of universal classes $u \equiv u(n) \equiv (U_k \in U | k \in n+1)_U$ such that $U_0 = \mathbf{a}$, $U_k \in U_l$ for any $k \in l \in n+1$ and if V is a universal class and $U_0 \subset V \in U$, then $V = U_k$ for some $k \in n+1$.

By Corollary 1 to Theorem 4, for the infra-universe \mathbf{a} , there exists a universe U^* which is the smallest from all universes U such that $\mathbf{a} \in U$. Since the universe $U \equiv U^*$ and the sequence $(U_k \in U | k \in 1)$ such that $U_0 \equiv \mathbf{a}$ satisfies all mentioned properties, we have $0 \in N$. Let $n \in N$. Consider the assembly $\mathbf{V} \equiv \{ i | i \bowtie \wedge u \neq U \wedge \forall k \in n+1 (u \neq U_k) \}$. By Axiom **A6**, for U , there exists a universal class K such that $U \in K$. Therefore, the assembly \mathbf{V} is nonempty, and, consequently, by Theorem 4, it contains the smallest element V . It is clear that $V \neq U$ and $V \geq U^* > U_0$. Suppose that $V \in U$. Then by supposition, $U_0 \leq V \in U$ implies $V = U_k$ for some $k \in n+1$, but this is impossible. Therefore, $U \in V$.

Thus, in the universe V , we can define a V -sequence $v \equiv (V_k \in V | k \in n+2)_V$ setting $V_k \equiv U_k$ for every $k \in n+1$ and $V_{n+1} \equiv U$. It is clear that $V_0 = \mathbf{a}$ and $V_k \in V_l$ for every $k \in l \in n+2$. Let W be a universe, and let $V_0 \leq W \in V$. Then $U_0 \subset W \in V$. If $W = U$, then $W = V_{n+1}$. If $W \in U$, then $U_0 \subset W \in U$ implies $W = U_k = V_k$ for some $k \in n+1$. Finally, if $U \in W$, then $U_0 \in U_1 \in \dots \in U_n \in U \in W$ implies $W \in \mathbf{V}$. Therefore, $V \subset W$. But this case is impossible. From the two previous cases, we infer that $W = V_k$ for some $k \in n+2$. This means that the pair V and v possesses all the necessary properties. Let us verify its uniqueness. Suppose that there exist a universe W and a W -sequence of universes $w \equiv (W_k \in W | k \in n+2)_W$ such that $W_0 = \mathbf{a}$, $W_k \in W_l$ for all $k \in l \in n+2$, and if K is a universe and $W_0 \subset K \in W$, then $K = W_k$ for some $k \in n+2$.

Consider the universe $U' \equiv W_{n+1}$ and the U' -sequence $u' \equiv (U'_k \in U' | k \in n+1)_{U'}$ such that $U'_k \equiv W_k$ for every $k \in n+1$. It is clear that $U'_0 = W_0 = \mathbf{a}$ and $U'_k \equiv W_k \in W_k \equiv U'_l$ for every $k \in l \in n+1$. If K is a universe and $U'_0 \leq K \in U'$, then $U'_0 \leq K \in U'$ and $W_0 \leq K \in W_{n+1} \in W$. By the axiom of transitivity **A7**, $W_0 \subset K \in W$. Therefore, by assumption, $K = W_k$ for some $k \in n+2$. Since $K \in W_{n+1}$, we have $K = W_k = U'_k$ for some $k \in n+1$. Thus, the pair U' and u' possesses all the properties for n mentioned above. Therefore, because of the uniqueness of this pair, we infer that $U = U' = W_{n+1}$ and $u = u' = w | n+1$, i. e., $W_{n+1} = U \equiv V_{n+1}$ and $W_k = U_k \equiv V_k$ for all $k \in n+1$. Therefore, $w = v$. If $W \in V$, then by the above $V_0 = W_0 \subset W \in V$ implies $W = V_k = W_k$ for some $k \in n+2$, but this is impossible. If $V \in W$, then $W_0 = V_0 \subset V \in W$ in a similar manner implies $V = W_k = V_k$ for some $k \in n+2$, but this is impossible. Therefore, $W = V$ and the uniqueness of the universe V and the sequence v is proved. Therefore, $n+1 \in N$. By the principle of natural induction, $N = \omega$. Therefore, for any $n \in \omega$, there exists the mentioned unique pair V and u . \square

Unfortunately, in the LTS, in contrast to ZF, where there is the axiom scheme of replacement, we have no means to combine all these finite sequences into one infinite sequence of universal classes.

By using Proposition 5 we can prove the following assertion

Theorem 6. *In the LTS, for every $n \in \omega$, there exists a unique sequence $c(n) \equiv (\varkappa_k \in \mathbf{I} | k \in n+1)$ of inaccessible cardinals such that $\varkappa_0 = \varkappa^*$, $\varkappa_k \in \varkappa_l$ for every $k \in l \in n+1$, and if π is an inaccessible cardinal and $\varkappa_0 \leq \pi < \varkappa_n$, then $\pi = \varkappa_k$ for some $k \in n$ (the property of incompressibility).*

From the property of uniqueness, it follows that $c(n) | m+1 = c(m)$ for all $m \leq n$, i. e., these finite sequences continue each other.

The proof is similar to the proof of the previous theorem.

The remark made after Theorem 5 is also valid in this case: in the LTS there are no means to construct from these finite sequences continuing each other one infinite sequence of inaccessible cardinals. In the next section, we show how to do this in the theory ZF.

6. Relative Consistency between LTS and the ZF Theory

6.1. Additional axioms on inaccessible cardinals in the ZF theory. To prove the relative consistency, we need to write the axioms on inaccessible cardinals in a more formal way. Therefore, we are forced to adduce the following formal definitions in the ZF theory:

- α is an ordinal number $\equiv On(\alpha) \equiv (\forall x(x \in \alpha \Rightarrow \forall y(y \in x \Rightarrow y \in \alpha))) \wedge (\forall x, y, z(x, y, z \in \alpha \wedge x \in y \wedge y \in z \Rightarrow x \in z)) \wedge (\forall x, y(x, y \in \alpha \Rightarrow x \in y \vee x = y \vee y \in x)) \wedge \forall z(\emptyset \neq z \subset \alpha \Rightarrow \exists x(x \in z \wedge \forall y(y \in z \Rightarrow x \in y)))$;
- f is a function $\equiv func(f) \equiv \forall x \forall y \forall y'(\langle x, y \rangle \in f \wedge \langle x, y' \rangle \in f \Rightarrow y = y')$;
- $f \rightleftharpoons A \rightarrow B \equiv func(f) \wedge dom f = A \wedge rng f \subset B$;
- \varkappa is a cardinal number $\equiv Cn(\varkappa) \equiv On(\varkappa) \wedge \forall x(On(x) \wedge (x = \varkappa \vee x \in \varkappa) \wedge \exists u(u \rightleftharpoons x \rightleftharpoons \varkappa) \Rightarrow x = \varkappa)$;
- \varkappa is a regular cardinal number $\equiv Rcn(\varkappa) \equiv Cn(\varkappa) \wedge \forall x(On(x) \wedge \exists u(u \rightleftharpoons x \rightarrow \varkappa \wedge \cup rng u = \varkappa) \Rightarrow (\varkappa = x \vee x \in \varkappa))$;
- \varkappa is an inaccessible cardinal number $\equiv Icn(\varkappa) \equiv Rcn(\varkappa) \wedge \forall x(On(x) \wedge x \in \varkappa \Rightarrow card \mathcal{P}(x) \in \varkappa)$;
- IC (inaccessible cardinal) $\equiv \exists x(Icn(x))$;
- ISIC (infinite set of inaccessible cardinals) $\equiv \exists X(\forall x \in X(Icn(x)) \wedge X \neq \emptyset \wedge \forall x \in X \exists x' \in X(x \in x'))$;
- ISIC₂ (infinite set of inaccessible cardinals of type $\omega + \omega$) $\equiv \exists X \exists Y(\forall x \in X \forall y \in Y(Icn(x) \wedge Icn(y) \wedge x \in y) \wedge X \neq \emptyset \wedge Y \neq \emptyset \wedge \forall x \in X \exists x' \in X(x \in x') \wedge \forall y \in Y \exists y' \in Y(y \in y'))$;
- I (inaccessibility) $\equiv \forall x(On(x) \Rightarrow \exists x'(Icn(x') \wedge x \in x'))$.

Consider the class (possibly empty) $\mathbf{In} \equiv \{x | Icn(x)\}$ of all inaccessible cardinal numbers in the ZF theory.

The adduced list will allow us later in thwe process of investigation of corresponding interpretations to look accurately at what values some derivative terms such as $rng u$, $\cup rng u$, $\mathcal{P}(x)$, $\{x\}$, $x \cup \{x\}$, $\langle x, y \rangle$,

$dom f$, $rng f$, and others take under the interpretation, and also what formulas such formulas as $z \subset \alpha$, $u \Leftrightarrow x \rightarrow \varkappa$, and others are translated into.

Theorem 1. *In the ZF theory, the following assertions are equivalent:*

- (1) *ISIC;*
- (2) *for every $n \in \omega$ there exists a finite set of inaccessible cardinals of the power $n + 1$;*
- (3) *for every $n \in \omega$ there exists a finite sequence $u \equiv (\iota_k | k \in n + 1)$ of inaccessible cardinals such that $\iota_k < \iota_l$ for all $k < l \in n + 1$, i. e., the sequence u is strictly increasing;*
- (4) *there exists an inaccessible cardinal \varkappa^* and for every $n \in \omega$, there exists a unique finite strictly increasing sequence $u(n) \equiv (\iota_k^n | k \in n + 1)$ of inaccessible cardinals such that $\iota_0^n = \varkappa^*$, and from the fact that \varkappa is an inaccessible cardinal and $\iota_0^n \leq \varkappa \leq \iota_n^n$, it follows that $\varkappa = \iota_k^n$ for some $k \in n + 1$ (the property of uncompressibility);*
- (5) *there exists a denumerable set of inaccessible cardinals;*
- (6) *there exists an infinite sequence $u \equiv (\iota_n | n \in \omega)$ of inaccessible cardinals such that $\iota_k < \iota_l$ for every $k < l \in \omega$, i. e., the sequence u is strictly increasing;*
- (7) *there exists an infinite strictly increasing sequence $u \equiv (\iota_n | n \in \omega)$ of inaccessible cardinals such that from $n \in \omega$, \varkappa is an inaccessible cardinal, and $\iota_0 \leq \varkappa \leq \iota_n$, it follows that $\varkappa = \iota_k$ for some $k \in n + 1$ (the property of incompressibility);*
- (8) *there exists an infinite set of inaccessible cardinals.*

Proof. (1) \vdash (4). Let I be a nonempty set, existence of which is postulated by Axiom ISIC. Consider the nonempty class $\mathbf{I} \equiv \{x | Icn(x) \wedge \exists y \in I(x \leq y)\}$. If $x \in \mathbf{I}$, then $x \leq y$ for some $y \in I$. By ISIC, for $y \in I$, there exists $z \in I$ such that $y < z$. Therefore, $x < z \in \mathbf{I}$. Therefore, the class \mathbf{I} possesses all the properties listed in formula ISIC.

Since $\emptyset \neq \mathbf{I} \subset \mathbf{In}$, by Lemma 2, Sec. 4, in \mathbf{I} , there exists the smallest element \varkappa^* . From $\varkappa^* \leq y$ for every $y \in I$, we infer that $\varkappa^* \in \mathbf{I}$. The class \mathbf{I} possesses the following property: if $z \in \mathbf{In}$ and $z \leq y$ for some $y \in \mathbf{I}$, then $z \in \mathbf{I}$.

Consider the set N consisting of all $n \in \omega$ for which there exists a unique sequence $u \equiv u(n) \equiv (\iota_k \in \mathbf{I} | k \in n + 1)$ such that $\iota_0 = \varkappa^*$, $\iota_k < \iota_l$ for every $k < l \in n + 1$ and $\varkappa \in \mathbf{In}$ and $\iota_0 \leq \varkappa < \iota_n$ imply $\varkappa = \iota_k$ for some $k \in n$.

Since the sequence $(\iota_k \in \mathbf{I} | k \in 1)$ such that $\iota_0 \equiv \varkappa^*$ possesses all the mentioned properties, we infer that $0 \in N$. Let $n \in N$. By the property of the class \mathbf{I} , for $\iota_n \in \mathbf{I}$, there exists $z \in \mathbf{I}$ such that $\iota_n < z$. Therefore, the class $\mathbf{J} \equiv \{x \in \mathbf{I} | \iota_n < x\}$ is nonempty. Hence, by Lemma 2, Sec. 4 it contains the smallest element α .

Therefore, we can define a sequence $v \equiv (\pi_k \in \mathbf{I} | k \in n + 2)$ setting $\pi_k \equiv \iota_k$ for every $k \in n + 1$ and $\pi_{n+1} \equiv \alpha$, i. e., $v = u \cup \{(n + 1, \alpha)\}$. It is clear that $\pi_0 = \varkappa^*$ and $\pi_k < \pi_l$ for all $k < l \in n + 2$. Let $\varkappa \in \mathbf{In}$ and $\pi_0 \leq \varkappa < \pi_{n+1}$. Then $\varkappa \in \mathbf{I}$ and $\iota_0 \leq \varkappa < \alpha$. If $\varkappa = \iota_n$, then $\varkappa = \pi_n$. If $\varkappa < \iota_n$, then $\iota_0 \leq \varkappa < \iota_n$ implies $\varkappa = \iota_k = \pi_k$ for some $k \in n$. Finally, if $\varkappa > \iota_n$, then $\varkappa \in \mathbf{J}$. Therefore, $\alpha \leq \varkappa$, but this contradicts the property $\varkappa < \alpha$. Therefore, this case is impossible. It follows from the two previous cases that $\varkappa = \pi_k$ for some $k \in n + 1$. This means that the sequence v possesses all the necessary properties. Let us verify its uniqueness. Suppose that there exists a sequence $w \equiv (\varkappa_k \in \mathbf{I} | k \in n + 2)$ such that $\varkappa_0 = \varkappa^*$, $\varkappa_k < \varkappa_l$ for all $k < l \in n + 2$, and $\varkappa \in \mathbf{In}$ and $\varkappa_0 \leq \varkappa < \varkappa_{n+1}$ imply $\varkappa = \varkappa_k$ for some $k \in n + 1$. Since the sequence $w|n + 1 \equiv (\varkappa_k \in \mathbf{I} | k \in n + 1)$ possesses all the mentioned properties for n , by the uniqueness of the sequence u , we infer that $u = w|(n + 1)$, i. e., $\varkappa_k = \iota_k \equiv \pi_k$ for all $k \in n + 1$. If $\varkappa_{n+1} < \pi_{n+1}$, then by the above $\pi_0 = \varkappa_0 \leq \varkappa_{n+1} < \pi_{n+1}$ implies $\varkappa_{n+1} = \pi_k = \varkappa_k$ for some $k \in n + 1$, but this is impossible. If $\pi_{n+1} < \varkappa_{n+1}$, then in a similar way, $\varkappa_0 = \pi_0 \leq \pi_{n+1} < \varkappa_{n+1}$ implies $\pi_{n+1} = \varkappa_k = \pi_k$ for some $k \in n + 1$, but this is also impossible. Therefore, $\varkappa_{n+1} = \pi_{n+1}$. Hence the uniqueness of the sequence v is proved. Consequently, $n + 1 \in N$. By the principle of natural induction, $N = \omega$. Thus, for every $n \in \omega$, there exists a unique mentioned sequence $u(n)$. By its uniqueness, we can denote it by $(\iota_k^n | k \in n + 1)$.

(4) \vdash (7). Consider the following formula of the ZF theory: $\varphi(x, y) \equiv (x \in \omega \Rightarrow y = \iota_x^x) \wedge (x \notin \omega \Rightarrow y = \emptyset)$. By the axiom scheme of replacement **AS6** (ZF), for ω , there exists a set Y such that

$\forall x \in \omega (\forall y (\varphi(x, y) \Rightarrow y \in Y))$. If $n \in \omega$, then $\varphi(n, \iota_n^n)$ implies $\iota_n^n \in Y$. Therefore, in the set $\omega \times Y$, we can define an infinite sequence $u \equiv (\iota_n \in Y | n \in \omega)$ setting $u \equiv \{z \in \omega \times Y | \exists x \in \omega (z = \langle x, \iota_x^x \rangle)\}$. It follows from the above-mentioned property of uniqueness that $u(m) = u(n) | m + 1$ for all $m \leq n$. Hence $u | n + 1 = u(n)$. It is clear that the sequence u possesses all the necessary properties.

(6) \vdash (1). Consider the following formula of the ZF theory: $\varphi(x, y) \equiv (x \in \omega \Rightarrow y = \iota_x) \wedge (x \notin \omega \Rightarrow y = \emptyset)$. By the axiom scheme of replacement **AS6** (ZF), for ω , there exists a set Y such that $\forall x \in \omega (\forall y (\varphi(x, y) \Rightarrow y \in Y))$. If $n \in \omega$, then $\varphi(n, \iota_n)$ implies $\iota_n \in Y$. By the axiom scheme of separation **AS3** (ZF), a class $X \equiv \{\iota_n | n \in \omega\} \equiv \{y | \exists x \in \omega (y = \iota_x)\} = \{y | y \in Y \wedge \exists x \in \omega (y = \iota_x)\}$ is a set. Since the sequence u is strictly increasing, then the set X satisfies the axiom ISIC.

Deducibilities (7) \vdash (6) \vdash (5) \vdash (2) are obvious.

Deducibilities (4) \vdash (3) \vdash (2) are also obvious.

(2) \vdash (3) and (2) \vdash (6). Consider the nonempty class **A** of all finite sets of inaccessible cardinals. Then the class **I** $\equiv \cup \mathbf{A}$ is also nonempty, and, therefore, by Lemma 2, Sec. 4, in **I** there exists the smallest element \varkappa^* .

Consider the set N consisting all $n \in \omega$ for which there exists a unique sequence $u \equiv u(n) \equiv (\iota_k \in \mathbf{I} | k \in n + 1)$ such that $\iota_0 = \varkappa^*$, $\iota_k < \iota_l$ for all $k \in l \in n + 1$ and $\varkappa \in \mathbf{I}$ and $\iota_0 \leq \varkappa < \iota_n$ imply $\varkappa = \iota_k$ for some $k \in n$ (*the property of I-uncompressibility*). Since the sequence $(\iota_k \in \mathbf{I} | k \in 1)$ such that $\iota_0 \equiv \varkappa^*$ possesses all the listed properties, we have $0 \in N$. Let $n \in N$, i.e., for n , the sequence $u \equiv (\iota_k \in \mathbf{I} | k \in n + 1)$ is constructed. Consider the finite set $A \equiv \{\iota_k \in \mathbf{I} | k \in n + 1\}$ of power $n + 1$. By condition (2), for $n + 2$, there exists a finite set $B \in \mathbf{A}$ of power $n + 2$. In B take, the smallest element a and the greatest element b . By definition, $a \geq \varkappa^*$. Suppose that $b \leq \iota_n$. Then for every $c \in B$, the inequality $\iota_0 = \varkappa^* \leq a \leq c \leq b \leq \iota_n$ is valid. If $c < \iota_n$, then from $c \in \mathbf{I}$, by the property of **I**-uncompressibility, we infer that $c = \iota_k$ for some $k \in n$, i.e., $c \in A$. If $c = \iota_n$, then $c \in A$ once again. As a result we come to the inclusion $B \subset A$, but this is impossible. From this contradiction, we infer that $\iota_n < b$. Since $b \in \mathbf{I}$, the class **J** $\equiv \{x \in \mathbf{I} | \iota_n < x\}$ is nonempty. Therefore, by Lemma 2, Sec. 4 it contains the smallest element α .

Consequently, we can define the sequence $v \equiv (\pi_k \in \mathbf{I} | k \in n + 2)$ setting $\pi_k \equiv \iota_k$ for every $k \in n + 1$ and $\pi_{n+1} \equiv \alpha$, i.e. $v = u \cup \{ \langle n + 1, \alpha \rangle \}$. Then in almost the same manner as in the inference (1) \vdash (4) changing **In** by **I**, we make sure that the sequence v possesses all the necessary properties and that it is unique. Therefore, $n + 1 \in N$. By the principle of natural induction, $N = \omega$. Therefore, for every $n \in \omega$, there exists the unique mentioned sequence $u(n)$. By its uniqueness, we can denote it by $(\iota_k^n | k \in n + 1)$. Therefore, inference (2) \vdash (3) is finished.

Now as in deducibility (4) \vdash (7) using the sequences $(\iota_k^n | k \in n + 1)$, we construct an infinite strictly increasing sequence $u \equiv (\iota_n | n \in \omega)$ of inaccessible cardinals. It gives the inference (2) \vdash (6).

Thus, the following deducibilities and equivalences are proved: (1) \vdash (4) \vdash (7) \vdash (6) \vdash (1) and (6) \vdash (5) \vdash (2) \vdash (6) and (2) \sim (3). This immediately implies the equivalence of all assertions (1)–(7).

(8) \vdash (6). Let I be an infinite set of inaccessible cardinals. By Lemma 2, Sec. 4 in I there exists the smallest \varkappa^* .

Consider the set N consisting of all $n \in \omega$ for which there exists a unique sequence $u \equiv u(n) \equiv (\iota_k \in I | k \in n + 1)$ such that $\iota_0 = \varkappa^*$, $\iota_k < \iota_l$ for every $k \in l \in n + 1$, and $\varkappa \in I$ and $\iota_0 \leq \varkappa < \iota_n$ imply $\varkappa = \iota_k$ for some $k \in n$ (*the property of I-incompressibility*).

Since a sequence $(\iota_k \in I | k \in 1)$ such that $\iota_0 = \varkappa^*$ possesses all the listed properties, we infer that $0 \in N$. Let $n \in N$. Consider the set $J \equiv I \setminus \{\iota_k | k \in n + 1\}$. It is nonempty because in the opposite case, the set I must be finite; therefore, it contains the smallest element α . It is clear that $\alpha \neq \iota_n$ and $\alpha \geq \varkappa^* = \iota_0$. Suppose that $\alpha < \iota_n$. Then by the condition $n \in N$, $\iota_0 \leq \alpha < \iota_n$ implies $\alpha = \iota_k$ for some $k \in n$, but this is impossible. Therefore, $\iota_n < \alpha$.

Therefore, we can define a sequence $v \equiv (\pi_k \in I | k \in n + 2)$ setting $\pi_k \equiv \iota_k$ for every $k \in n + 1$ and $\pi_{n+1} \equiv \alpha$, i.e., $v = u \cup \{ \langle n + 1, \alpha \rangle \}$. It is clear that $\pi_0 = \varkappa^*$ and $\pi_k < \pi_l$ for all $k \in l \in n + 2$. Let $\varkappa \in I$, and let $\pi_0 \leq \varkappa < \pi_{n+1}$. Then $\iota_0 \leq \varkappa < \alpha$. If $\varkappa = \iota_n$, then $\varkappa = \pi_n$. If $\varkappa < \iota_n$, then $\iota_0 \leq \varkappa < \iota_n$ implies $\varkappa = \iota_k = \pi_k$ for some $k \in n$. Finally, if $\varkappa > \iota_n$, then $\varkappa > \iota_k$ for all $k \in n + 1$. Hence $\varkappa \in J$. Therefore, $\alpha \leq \varkappa$, but this contradicts the property $\varkappa < \alpha$. Therefore, this case is impossible. This follows from the

two previous cases that $\varkappa = \pi_k$ for some $k \in n + 1$. This means that the sequence v possesses all the necessary properties.

Let us verify its uniqueness. Suppose that there exists a sequence $w \equiv (\varkappa_k \in I | k \in n + 2)$ such that $\varkappa_0 = \varkappa^*$, $\varkappa_k \in \varkappa_l$ for all $k \in l \in n + 2$, and $\varkappa \in I$ and $\varkappa_0 \leq \varkappa < \varkappa_{n+1}$ imply $\varkappa = \varkappa_k$ for some $k \in n + 1$. Since the sequence $w|n + 1 \equiv (\varkappa_k \in I | k \in n + 1)$ possesses all properties for n mentioned above, by the uniqueness of the sequence u , we infer that $u = w|n + 1$, i. e., $\varkappa_k = \iota_k \equiv \pi_k$ for all $k \in n + 1$. If $\varkappa_{n+1} < \pi_{n+1}$, then, by the above, $\pi_0 = \varkappa_0 \leq \varkappa_{n+1} < \pi_{n+1}$ implies $\varkappa_{n+1} = \pi_k = \varkappa_k$ for some $k \in n + 1$, but this is impossible. If $\pi_{n+1} < \varkappa_{n+1}$, then in a similar way, $\varkappa_0 = \pi_0 \leq \pi_{n+1} < \varkappa_{n+1}$ implies $\pi_{n+1} = \varkappa_k = \pi_k$ for some $k \in n + 1$, but this is also impossible. Therefore, $\varkappa_{n+1} = \pi_{n+1}$. Therefore, the uniqueness of the sequence v is proved. Consequently, $n + 1 \in N$. By the principle of natural induction $N = \omega$. Hence for every $n \in \omega$, there exists a unique mentioned sequence $u(n)$. By its uniqueness, we can denote it by $(\iota_k^n | k \in n + 1)$. Further, as in the inference (4) \vdash (7), using the sequences $(\iota_k^n | k \in n + 1)$, we construct the infinite strictly increasing sequence $u \equiv (\iota_n | n \in \omega)$ of inaccessible cardinals.

(6) \vdash (8). In the same manner as in the proof of deducibility (6) \vdash (1) for the sequence u , consider the set $X \equiv \{\iota_n | n \in \omega\}$ of its members. Suppose that the set X is finite. Then X contains the greatest element \varkappa , but this contradicts the fact that the sequence U is strictly increasing. \square

It becomes clear from assertion (6) of this theorem why in the mathematical literature, the axiom ISIC is also denoted by $I(\omega)$ and is called the *axiom of ω -inaccessibility*.

Proposition 1. *In the ZF theory, the following assertions are equivalent:*

- (1) ISIC₂;
- (2) *there exist infinite sequences $u \equiv (\iota_m | m \in \omega)$ and $v \equiv (\varkappa_n | n \in \omega)$ of inaccessible cardinals such that $\iota_k < \iota_m < \varkappa_l < \varkappa_n$ for every $k \in m \in \omega$ and $l \in n \in \omega$, i. e., the sequences u and v are strictly increasing and continue each other.*

Proof. (1) \vdash (2). By ISIC₂, there exist sets X and Y satisfying Axiom ISIC. Therefore, by Theorem 1, there exist strictly increasing infinite sequences $u \equiv (\iota_m \in X | m \in \omega)$ and $v \equiv (\varkappa_n \in Y | n \in \omega)$. By ISIC₂, $\iota_k < \varkappa_l$ for all $k, l \in \omega$.

(2) \vdash (1). Similarly as in the proof of the deducibility (6) \vdash (1) from Theorem 1, we verify that the classes $X \equiv \{\iota_m | m \in \omega\}$ and $Y \equiv \{\varkappa_n | n \in \omega\}$ are sets. These sets satisfy Axiom ISIC₂. \square

It becomes clear from assertion (2) of this proposition why in the mathematical literature the axiom ISIC₂ is denoted also by $I(\omega + \omega)$. Roughly speaking, ZF+ISIC₂ ensures the existence of $\omega + \omega$ different inaccessible cardinals.

Now we will clear up the correlation of the axioms on inaccessible cardinals:

- IC (*inaccessible cardinal*) \equiv *there exists at least one inaccessible cardinal*;
- ISIC \sim *there exists an infinite set of inaccessible cardinals*;
- ISIC₂ \sim *there exist two infinite sets of inaccessible cardinals following one after another*;
- I (*inaccessibility*) \equiv *for every ordinal α there exists an inaccessible cardinal which is greater than α .*

Proposition 2. *In the ZF theory, the deducibilities LIC \vdash ISIC₂ \vdash ISIC \vdash IC are valid.*

Proof. Let us prove that from I , we can infer property (2) from Proposition 1. The arguments completely repeats the proof of deducibility (1) \vdash (2) from Proposition 1 and deducibility (1) \vdash (2) from Theorem 1.

Using now the equivalence of (2) and ISIC₂, we obtain the proof of deducibility $I \vdash$ ISIC₂. All other deducibilities are obvious. \square

In the [8], it is proved that in the ZF theory, the axiom of inaccessibility I is equivalent to the axiom of universality U. Therefore, the set theories ZF+I and ZF+U are equivalent.

It can be also proved that in the ZF theory, the axiom of ω -inaccessibility $I(\omega) \equiv$ ISIC is equivalent to the *axiom of ω -universality*

$$U(\omega) \equiv \exists X (\forall U \in X (U \bowtie) \wedge X \neq \emptyset \wedge \forall U \in X \exists V \in X (U \in V))$$

postulating the existence of an infinite set of universal sets where $U \bowtie$ denotes the property of a set U to be universal. Therefore, the set theories ZF+ISIC and ZF+U(ω) are also equivalent.

6.2. “Forks” of relative consistency. Using the globalization of local constructions in the LTS, which was made above, we can prove the following statement.

Statement 1. *All axioms of the ZF theory, except the axiom scheme of separation **AS3** and the axiom scheme of replacement **AS6**, are deducible in the LTS as formulas of the LTS.*

Proof. For every formula φ in the first-order theory, the formula scheme $\varphi \Rightarrow \varphi$ is deduced. By the definition of equality in the LTS, this gives the formula $\forall u(u \in X \Leftrightarrow u \in Y) \Rightarrow X = Y$. Formula **A1** is inferred from it by the rule Gen.

For classes u and v , by Corollary 3 to Theorem 4, Sec. 5, there exists the class $\{u, v\}$. By definition, $z \in \{u, v\} \equiv z = u \vee z = v$. Using the scheme $\varphi \Rightarrow \varphi$, we obtain the formula $(z \in \{u, v\}) \Leftrightarrow z = u \vee z = v$. By the rule Gen, the logical axiom scheme **LAS12**, and the rule MP, we can infer from it the formula $\exists x \forall z(z \in x \Leftrightarrow z = u \vee z = v)$. Formula **A2** (ZF) is inferred from it by the rule Gen.

For a class X , by Corollary to Lemma 6, Sec. 5, there exists a class $\cup X$. By definition, $u \in \cup X \equiv \exists z(u \in z \wedge z \in X)$. Using the formula scheme $\varphi \Rightarrow \varphi$, we obtain the formula $(u \in \cup X) \Leftrightarrow \exists z(u \in z \wedge z \in X)$. By the rule Gen, the logical axiom scheme **LAS12**, and the rule MP, the formula $\exists Y \forall u(u \in Y \Leftrightarrow \exists z(u \in z \wedge z \in X))$ is inferred from it. Formula **A4** (ZF) is inferred from it by the rule Gen.

For a class X , by Corollary to Lemma 1, Sec. 5, there exists the class $\mathcal{P}(X)$. By definition, $u \in \mathcal{P}(X) \equiv u \subset X$. As was done above, we consecutively infer the formulas $u \in \mathcal{P}(X) \Leftrightarrow u \subset X$, $\forall u(u \in \mathcal{P}(X) \Leftrightarrow u \subset X)$, $\exists Y \forall u(u \in Y \Leftrightarrow u \subset X)$, and **A5** (ZF).

Consider the class π , which exists by the axiom of infra-infinity **A13** (LTS). Let $u \in \pi$. Then $u \in \pi \in \mathfrak{a}$ by the axiom of transitivity **A7** implies $u \in \mathfrak{a}$. Therefore, by Lemma 3, Sec. 5, $\{u\}_{\mathfrak{a}} = \{u\}$. By Lemma 2, Sec. 2, $\{u\} \in \mathfrak{a}$. By **A7**, $u, \{u\} \subset \mathfrak{a}$. Hence, by Lemma 2, Sec. 5, $u \cup \{u\} = u \cup_{\mathfrak{a}} \{u\} = u \cup_{\mathfrak{a}} \{u\}_{\mathfrak{a}}$. By **A13**, we infer $u \cup \{u\} \in \pi$. Since we did not apply the rule of generalization, then by the theorem of deduction (see [28], Ch. 2, § 4) and by the rule Gen, the formula $\forall u(u \in \pi \Rightarrow u \cup \{u\} \in \pi)$ is deduced. Since, by **A13**, $\pi \neq \emptyset$, we infer that by the derivative rule of conjunction, the formula $\pi \neq \emptyset \wedge \forall u(u \in \pi \Rightarrow u \cup \{u\} \in \pi)$ is deduced. Using logical axiom scheme **LAS12** and the rule MP, we infer from it formula **A7** (ZF).

For every nonempty class X , by the axiom of universality **A6**, there exists a universal class α such that $X \in \alpha$. By the axiom of transitivity **A7**, $X \subset \alpha$. From the axiom of regularity **A12** (LTS) the formula $\exists x(x \in X \wedge x \cap_{\alpha} X = \emptyset)$ is deduced. Since $x \in X \subset \alpha$, by **A7**, we have, $x \subset \alpha$. Hence, by Lemma 2, Sec. 5, $x \cap_{\alpha} X = x \cap X$. Thus, the formula $\exists x(x \in X \wedge x \cap X = \emptyset)$ is deduced. By the theorem of deduction, the formula $X \neq \emptyset \Rightarrow \exists x(x \in X \wedge x \cap X = \emptyset)$ is deduced. By the rule Gen we infer from it formula **A8** (ZF).

In the LTS, consider the empty class \emptyset . From **A3** (LTS), the formula $\forall z(z \notin \emptyset)$ is deduced. Using **LAS12** and MP, we infer formula **A9** (ZF) from it.

For every nonempty class X , by **A6**, there exists a universal class α such that $X \in \alpha$. From the axiom of choice **A14** (LTS), we infer the formula $\exists z((z \Leftarrow \mathcal{P}_{\alpha}(X) \setminus \{\emptyset\}_{\alpha} \rightarrow_{\alpha} X) \wedge \forall Y(Y \in \mathcal{P}_{\alpha}(X) \setminus \{\emptyset\}_{\alpha} \Rightarrow z(Y) \in Y)$. By the Lemmas 1, Sec. 5, and 3, Sec. 5, $\mathcal{P}_{\alpha}(X) = \mathcal{P}(X)$ and $\{\emptyset\}_{\alpha} = \{\emptyset\}$. Therefore, $z \Leftarrow \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow_{\alpha} X$. By the axiom of transitivity **A7**, $X \subset \alpha$ and $\emptyset \subset \alpha$. Consequently by, Lemma 5, Sec. 5, $z \Leftarrow \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$. Thus, the formula $\exists z(z \Leftarrow \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X) \wedge \forall Y(Y \in \mathcal{P}(X) \setminus \{\emptyset\} \Rightarrow z(Y) \in Y)$ is deduced. By the theorem of deduction and the rule Gen, formula **A10** (ZF) is deduced from it. \square

The undeducibility of the axiom scheme of replacement in the LTS will be proved later.

The existence of inaccessible cardinal numbers in the LTS proved in the previous section allows to prove the following statement.

Statement 2. *If the LTS is consistent, then the ZF+IC theory is consistent, where IC denotes the axiom postulated the existence of at least one inaccessible cardinal number.*

Proof. By the axiom of universality, for a fixed universe U_0 , there exists a universe U such that $U_0 \in U$. Consider the interpretation $M \equiv (U, I)$ of the ZF theory in the LTS, described in the proof of Statement 1, Sec. 5. Let us prove that this interpretation is a model of the ZF+IC theory. According to the proof of Statement 1, Sec. 5, we need to only consider the translation of Axiom IC and prove its deducibility in the LTS.

We will use the notation of the proof of Statement 1, Sec. 5.

On the sequence s Axiom IC is translated into the formula

- $\varphi_0 \equiv M \vDash IC[s] = IC^U \equiv \exists x' \in U (x' \text{ is an inaccessible cardinal number})^U$, where
- $\varphi_1(x') \equiv (Icn(x'))^U \equiv (Rcn(x'))^U \wedge \forall x'' \in U ((On(x''))^U \wedge x'' \in x' \Rightarrow card_U \mathcal{P}_U(x'') \in x')$, where
- $\varphi_2(x') \equiv (Rcn(x'))^U \equiv (Cn(x'))^U \wedge \forall x''' \in U ((On(x'''))^U \wedge \exists u \in U (u \rightleftharpoons x''' \rightarrow_U x' \wedge \cup_U rng_U u = x') \Rightarrow (x' = x''' \vee x' \in x'''))$; where
- $\varphi_3(x') \equiv (Cn(x'))^U \equiv (On(x'))^U \wedge \forall x''' \in U ((On(x'''))^U \wedge (x''' = x' \vee x''' \in x') \wedge \exists u \in U (u \rightleftharpoons x''' \rightleftharpoons_U x') \Rightarrow x''' = x')$, where
- $\varphi_4(x'') \equiv (On(x''))^U \equiv \forall x \in U (x \in x'' \Rightarrow \forall y \in U (y \in x \Rightarrow y \in x'')) \wedge (\forall x, y, z \in U (x, y, z \in x'' \wedge x \in y \wedge y \in z \Rightarrow x \in z)) \wedge (\forall x, y \in U (x, y \in x'' \Rightarrow x \in y \vee x = y \vee y \in x)) \wedge \forall z \in U (z \neq \emptyset \wedge \forall x \in U (x \in z \Rightarrow x \in x'') \Rightarrow \exists x \in U (x \in z \wedge \forall y \in U (y \in z \Rightarrow x \in y)))$,
- $(On(x'))^U = \varphi_4(x'' \| x') \equiv \varphi_4(x')$,
- $(On(x'''))^U = \varphi_4(x'' \| x''') \equiv \varphi_4(x''')$.

Comparison of the formula $\varphi_4(x'')$ with the definition shows that the subformula $\psi \equiv \forall x \in U (x \in z \Rightarrow x \in x'')$ is unusual in it. But in the place where it is staying, it is equivalent to the formula $\psi' \equiv z \subset x''$. Actually, if $x \in z$, then from $z \in U$, by the axiom of transitivity **A7** (LTS), it follows that $x \in U$, and, therefore, $x \in x''$. Therefore, we can substitute ψ' for ψ . Under this substitution, we see that the formula $\varphi_4(x'')$ means that x'' is an U -ordinal number in the universal class U . Consequently, $\varphi_4(x')$ and $\varphi_4(x''')$ mean that x' and x''' are U -ordinal numbers.

This leads to the following form of the formula $\varphi_3(x')$:

$$\varphi_3(x') = (x' \text{ is an } U\text{-ordinal number}) \wedge \forall x''' \in U ((x''' \text{ is an } U\text{-ordinal number}) \wedge (x''' \leq x') \wedge \exists u \in U (u \rightleftharpoons x''' \rightleftharpoons_U x') \Rightarrow x''' = x').$$

In this formula, the subformula $\chi \equiv \exists u \in U (u \rightleftharpoons x''' \rightleftharpoons_U x')$ is unusual. Since $x''', x' \in U$, we infer that, by the axiom of subset **A8**, $u \subset x''' *_U x' \in U$ implies $u \in U$. Therefore, χ is equivalent to the formula $\chi' \equiv \exists u (u \rightleftharpoons x''' \rightarrow_U x')$. Substituting χ' for χ , we see that $\varphi_3(x')$ means that x' is an U -cardinal number.

This leads to the following form of the formula $\varphi_2(x')$:

$$\varphi_2(x') = (x' \text{ is an } U\text{-cardinal number}) \wedge \forall x''' \in U ((x''' \text{ is an } U\text{-ordinal number}) \wedge \exists u \in U (u \rightleftharpoons x''' \rightarrow_U x' \wedge \cup_U rng_U u = x') \Rightarrow (x' \leq x''')).$$

For the same reasons as above, in φ_2 , the quantifier prefix $\exists u \in U$ can be changed by $\exists u$. But then the formula $\varphi_2(x')$ means that x' is an U -regular U -cardinal number.

This leads to the following form of the formula $\varphi_1(x')$:

$$\varphi_1(x') = (x' \text{ is an } U\text{-regular } U\text{-cardinal number}) \wedge \forall x'' \in U ((x'' \text{ is an } U\text{-ordinal number}) \wedge x'' \in x' \Rightarrow card_U \mathcal{P}_U(x'') \in x').$$

This means that x' is an U -inaccessible U -cardinal number in the universe U . Thus, Axiom IC was translated into the formula $\varphi_0 \equiv M \vDash IC[s] = \exists x' \in U (x' \text{ is a } U\text{-inaccessible } U\text{-cardinal number})$.

Let us infer this formula in the LTS. By Corollary 3 to Proposition 4, Sec. 5, there exists an inaccessible cardinal number \varkappa such that $\mathfrak{a} = \overline{V_\varkappa}$. Since $\varkappa \subset \mathfrak{a} \subset U_0 \in U$, by the axiom of subset **A8**, we have, $\varkappa \in U$. By Proposition 3, Sec. 5, \varkappa is an U -inaccessible U -cardinal number. As a result, we deduced the desired formula. \square

Statement 2 was proved by constructing a model of the ZF+IC theory in the LTS. It follows from Theorem 6, Sec. 5 that to construct a model of the LTS in the ZF theory, it is necessary to have the “metasequence” $c(0), c(1), \dots, c(n), \dots$ of finite incompressible sequences $c(n) \equiv (\varkappa_k^n | k \in n + 1)$ of inaccessible cardinals in ZF at least the same as in Theorem 6, Sec. 5. But in ZF, this metasequence can be

globalized by the axiom scheme of replacement into the usual unfinite sequence $c \equiv (\varkappa_n \equiv \varkappa_n^n | n \in \omega)$. As Theorem 1 shows, the existence of such an infinite sequence of inaccessible cardinals is equivalent to Axiom ISIC.

Using Theorem 1, we can prove the following statement.

Statement 3. *If the ZF+ISIC theory is consistent, then the LTS is consistent.*

Proof. Consider the sequence $(\iota_n | n \in \omega)$ from Theorem 1 and the set $A \equiv \{\iota_n | n \in \omega\}$. By Lemma 3, Sec. 4, $\alpha \equiv \cup A = \sup A$ is an ordinal number. Further, instead of V_{ι_n} we will write W_n . Since $\iota_n \leq \alpha$, we have $W_n \subset V_\alpha$. Therefore, $D \equiv \cup \{W_n | n \in \omega\} \subset V_\alpha$. By **AS3**, D is a set. Since $0 < \iota_n < \iota_n + 1 \leq \iota_{n+1}$, by Lemma 7, Sec. 4, we obtain, $\emptyset = V_0 \in W_n \in W_{n+1} \subset D$ for every $n \in \omega$.

The set D is transitive. Actually, if $y \in D$, then by Lemma 9, Sec. 4, $y \in W_m$ for some m implies $y \subset W_m \subset D$. Similarly, using Lemma 8, Sec. 4, we can verify that if $x \subset y \in D$, then $x \in D$. Later, we will often use these two properties.

Choose the set D in the capacity of the domain of interpretation of the LTS in the set ZF+ISIC theory. Consider the subset $R \equiv \{x \in D | \exists n \in \omega (x = W_n)\}$ in D . Define a correspondence I assigning to the predicate symbol \in in the LTS the two-placed relation $B \equiv \{z \in D \times D | \exists x, y \in D (z = (x, y) \wedge x \in y)\}$, to the symbol \ni in LTS the one-placed relation R , and to the constant symbols \emptyset and \mathfrak{a} in the LTS the elements \emptyset and W_0 of the set D , respectively.

Let s be some sequence x_0, \dots, x_q, \dots of elements of D . We will consider translations $M \models \varphi[s]$ of axioms and axiom schemes of the LTS on the sequence s under the interpretation M and will prove their deducibility in the ZF+ISIC set theory. Instead of $\theta_M[s]$ and $M \models \varphi[s]$, we will write θ^t and φ^t for terms θ and formulas φ , respectively.

To make our further account more simple, consider, first, the translations of some basic formulas. Let u and v be some classes in the LTS.

The formula $u \in v$ is translated into the formula $(u \in v)^t = ((u^t, v^t) \in B)$. Denote this last formula by γ . By definition, this formula is equivalent to the formula $(\exists x \exists y (x \in D \wedge y \in D \wedge (u^t, v^t) = (x, y) \wedge (x \in y)))$. Using the property of a sequential pair, we conclude that $u^t = x$ and $v^t = y$. Consequently, the formula $\delta \equiv (u^t \in v^t)$ is deduced from γ . By the theorem of deduction, $\gamma \Rightarrow \delta$. Conversely, consider the formula δ . In the ZF theory, one can prove that for sets u^t and v^t , there exists a set z such that $z = (u^t, v^t)$. By **LAS3**, the formula $(z = (u^t, v^t) \Rightarrow u^t \in D \wedge v^t \in D \wedge z = (u^t, v^t) \wedge u^t \in v^t)$ is deduced from δ . Since the formula $z = (u^t, v^t)$ is deduced from axioms, we infer that the formula $(u^t \in D \wedge v^t \in D \wedge z = (u^t, v^t) \wedge u^t \in v^t)$ is also deduced. By **LAS12**, the formula $\exists x \exists y (x \in D \wedge y \in D \wedge z = (x, y) \wedge x \in y)$ is deduced, and it is equivalent to the formula $z \in B$ and, therefore, to the formula γ . By the theorem of deduction, $\delta \Rightarrow \gamma$. Thus, the first equivalence $(u \in v)^t \Leftrightarrow u^t \in v^t$ is valid.

The formula $v \subset w$ is translated into the formula $(v \subset w)^t$. Denote this last formula by ε . By the equivalence proved above it is equivalent to the formula $\varepsilon' \equiv \forall u \in D (u \in v^t \Rightarrow u \in w^t)$. According to **LAS11**, the formula $\varepsilon'' \equiv (x \in D \Rightarrow (x \in v^t \Rightarrow x \in w^t))$ is deduced from the formula ε' . If $x \in v^t$, then $v^t \in D$, and the transitivity of the set D implies $x \in D$. Then the formula ε'' implies $x \in v^t \Rightarrow x \in w^t$. Consequently, by the theorem of deduction, the formula $(\varepsilon \Rightarrow (x \in v^t \Rightarrow x \in w^t))$ is deduced. By the rule of generalization, the formula $\forall x (\varepsilon \Rightarrow (x \in v^t \Rightarrow x \in w^t))$ is deduced. By **LAS13**, we infer the formula $(\varepsilon \Rightarrow \forall x (x \in v^t \Rightarrow x \in w^t))$, i.e., the formula $(\varepsilon \Rightarrow v^t \subset w^t)$.

Conversely, let the formula $v^t \subset w^t$ be given. Using the logical axioms, we can consecutively infer from it the formulas $(u \in v^t \Rightarrow u \in w^t)$ and $(u \in D \Rightarrow (u \in v^t \Rightarrow u \in w^t))$. By the rule Gen, the formula ε' is deduced. Therefore, by the theorem of deduction, the formula $(v^t \subset w^t \Rightarrow \varepsilon)$ is deduced. Thus, the second equivalence $(v \subset w)^t \Leftrightarrow v^t \subset w^t$ is valid.

It follows from this equivalence that we obtain the equivalence $(v =_{LTS} w)^t \Leftrightarrow (v^t \subset w^t) \wedge (w^t \subset v^t)$. By the axiom of extensionality **A1** (ZF), the last formula implies $v^t =_{ZF} w^t$. By the theorem of deduction, in the ZF theory, the formula $((v^t \subset w^t) \wedge (w^t \subset v^t) \Rightarrow v^t = w^t)$ is deduced. Conversely, if $v^t =_{ZF} w^t$, then, by the principle of changing equals (see Sec.4.1), the formula $(u \in v^t \Rightarrow u \in w^t)$ is deduced. By the rule Gen, the formula $v^t \subset w^t$ is deduced. Similarly the formula $w^t \subset v^t$ is deduced. Therefore,

the formula $(v^t \subset w^t) \wedge (w^t \subset v^t)$ is also deduced. By the theorem of deduction, in the ZF theory, the formula $(v^t =_{ZF} w^t \Rightarrow (v^t \subset w^t) \wedge (w^t \subset v^t))$ is deduced. Thus, we obtain the third equivalence $(v =_{LTS} w)^t \Leftrightarrow v^t =_{ZF} w^t$.

Further, we will write not literal translations of axioms and axiom schemes but their equivalent variants, which are obtained by using the proved equivalences $(u \in_{LTS} v)^t \Leftrightarrow u^t \in_{ZF} v^t$, $(v \subset_{LTS} w)^t \Leftrightarrow v^t \subset_{ZF} w^t$, and $(v =_{LTS} w)^t \Leftrightarrow v^t =_{ZF} w^t$. We will denote these equivalent variants using the sign “ $\widetilde{\sim}$ ” over them.

$\widetilde{\mathbf{A}^t1}$. $\forall y \in D \forall z \in D ((y = z) \Rightarrow (\forall X \in D (y \in X \Leftrightarrow z \in X)))$.

In ZF, the formula $y = z \Rightarrow z = y$ can be proved in the following way. By the property of changing equals, we have $(y = z \Rightarrow (y = y \Rightarrow z = y))$. Since the formula $y = y$ is valid for any y , we obtain $y = z \Rightarrow z = y$. Moreover, by the property of changing equals, $(y = z) \Rightarrow (y \in X \Rightarrow z \in X)$ and $(z = y) \Rightarrow (z \in X \Rightarrow y \in X)$ are deduced. By using $y = z \Rightarrow z = y$, $(y = z) \Rightarrow (y \in X \Leftrightarrow z \in X)$ and $(y = z) \Rightarrow (y \in X \Leftrightarrow z \in X)$ are deduced. Now, by **LAS1** and the rule of generalization, $\forall X \in D ((y = z) \Rightarrow (y \in X \Leftrightarrow z \in X))$ is deduced. Whence, by **LAS13** and the rule of generalization, the formula **\mathbf{A}^t1** is deduced.

$\widetilde{\mathbf{A}^t2}$. Let $\varphi(x)$ be an X -predicative formula of the LTS such that the substitution $\varphi(x||y)$ is admissible and φ does not contain Z as a free variable. Then $\forall X \in D (\exists Z \in D (\forall y \in D ((y \in Z) \Leftrightarrow (y \in X \wedge \varphi^\tau(y)))))$, where φ^τ denotes the formula $M \models [s\tau]$ in which by s^τ we denote the corresponding change of the sequence s under the translation of the quantifier over formulas $\forall X(\dots)$, $\exists Y(\dots)$ and $\forall y(\dots)$ indicated above.

Since φ^τ is a formula of ZF, we infer that **$\widetilde{\mathbf{A}^t2}$** is deduced from **AS3** (ZF). Actually, by **AS3** (ZF), for $X \in D$, there exists Z such that $\forall y (y \in Z \Leftrightarrow y \in X \wedge \varphi^\tau(y))$. Therefore, $Z \subset X \in D$. By the definition of D , there exists $n \in \omega$ such that $X \in W_n$. By Lemma 8, Sec. 4, $Z \in W_n \subset D$. Thus, for $X \in D$, there exists $Z \in D$ such that $\forall y \in D (y \in Z \Leftrightarrow y \in X \wedge \varphi^\tau(y))$.

$\widetilde{\mathbf{A}^t3}$. $\forall Z \in D ((\forall x \in D (x \notin Z)) \Leftrightarrow Z = \emptyset)$.

Fix the condition $Z \in D$. Consider the formula $\chi \equiv \forall x (x \in D \Rightarrow x \notin Z)$. If $x \in Z$ then, by the condition, $x \in D$ and then χ implies $x \notin Z$. If $x \notin Z$, then, obviously, χ implies $x \notin Z$. Consequently, under our condition, from χ is deduced $x \notin Z$. By the rule of generalization, from χ is deduced $\forall x (x \notin Z)$. By Axioms **A1** and **A7** (ZF), $\forall x (x \notin Z)$ implies $Z = \emptyset$. Thus, from the totality $Z \in D$ and χ , $Z = \emptyset$ is deduced. By the theorem of deduction, $Z \in D$ implies the formula $\chi \Rightarrow Z = \emptyset$. Conversely, by **A1** and **A7**, $Z = \emptyset$ implies $\forall x (x \notin Z)$. Therefore, from the totality $Z \in D$ and $Z = \emptyset$, we can infer χ . By the theorem of deduction, from $Z \in D$, we can infer the formula $Z = \emptyset \Rightarrow \chi$. Therefore, from the condition $Z \in D$, the formula $\chi \Leftrightarrow Z = \emptyset$ is deduced. By the theorem of deduction, the formula $Z \in D \Rightarrow (\chi \Leftrightarrow Z = \emptyset)$ is deduced. Therefore, by the rule of generalization, the formula **\mathbf{A}^t3** is deduced.

$\widetilde{\mathbf{A}^t4}$. $\forall U \in D \forall V \in D ((U = V) \Rightarrow (U \in R \Leftrightarrow V \in R))$.

By the property of changing equals, we have $(U = V) \Rightarrow (U \in R \Rightarrow V \in R)$. By the proved above, $(U = V) \Rightarrow (V = U)$, and, therefore, $(U = V) \Rightarrow (V \in R \Rightarrow U \in R)$. It follows that $(U = V) \Rightarrow (U \in R \Leftrightarrow V \in R)$, and by the rule of generalization, we infer the formula **\mathbf{A}^t4** .

$\widetilde{\mathbf{A}^t5}$. $W_0 \in R \wedge \forall U \in D (U \in R \Rightarrow W_0 \subset U)$.

The formula $W_0 \in R$ is deduced in ZF+ISIC by **AS3**. Therefore, we need to only deduce the formula $\forall U \in D (U \in R \Rightarrow W_0 \subset U)$. We can write this formula in another form as $\forall U \in D (\exists n \in \omega (U = W_n) \Rightarrow W_0 \subset U)$. By Corollary 1 to Lemma 9 from Sec. 4.3, $\iota_0 \leq \iota_n$ implies $W_0 \subset W_n$ for every $n \in \omega$. It follows from this assertion that the mentioned formula is deduced in ZF+ISIC.

$\widetilde{\mathbf{A}^t6}$. $\forall X \in D \exists U \in D (U \in R \wedge X \in U)$.

It follows from $X \in D$ that $X \in W_n$ for some $n \in \omega$. Consider $U = W_n$. Then $X \in D$ implies $U \in D \wedge U \in R \wedge X \in U$, and it gives **\mathbf{A}^t6** as a result.

A^t7. $\forall U \in D(U \in R \Rightarrow \forall X \in D(X \in U \Rightarrow X \subset U))$.

It follows from $U \in R$ that $U = W_n$ for some $n \in \omega$, and $X \in W_n$ implies $X \subset W_n$ by Lemma 9, Sec. 4. This gives the desired formula.

A^t8. $\forall U \in D(U \in R \Rightarrow \forall X \in D \forall Y \in D(X \in U \wedge Y \subset X \Rightarrow Y \in U))$.

Since from $U \in R$ it follows that $U = W_n$ for some $n \in \omega$, we need only to prove that for any X, Y from D ($X \in W_n \wedge Y \subset X \Rightarrow Y \in W_n$) holds. But this directly follows from Lemma 8, Sec. 4.

A^t9. $\forall U \in D(U \in R \Rightarrow \forall X \in D(X \in U \Rightarrow \mathcal{P}_U(X)^\tau \in U))$, where the set $Z \equiv \mathcal{P}_U(X)^\tau$ is defined by the formula $\exists Z \in D(\forall y \in D((y \in Z) \Leftrightarrow (y \in U \wedge y \subset X)))$.

First, let us verify that if $U \in R$, $X \in D$, and $X \in U$, then $Z = \mathcal{P}(X)$. Let $y \in Z$. Since $Z \in D$, it follows from the transitivity property that $Z \subset D$. Consequently, $y \in D$. But, in this case, $y \in D$ and $y \in Z$ imply $y \subset X$, i.e., $y \in \mathcal{P}(X)$. Conversely, let $y \in \mathcal{P}(X)$, i.e., $y \subset X$. From $X \in D$, by the property of the set D proved above, we obtain $y \in D$. It follows from $U \in R$ that $U = W_n$ for some $n \in \omega$. Consequently, by Lemma 8, Sec. 4, $y \subset X \in W_n$ implies $y \in W_n = U$. But then $y \in D$, $y \in U$, and $y \subset X$ imply $y \in Z$, and this proves the desired equality.

By Lemma 12, Sec. 4, $X \in W_n$ implies $Z = \mathcal{P}(X) \in W_n = U$. From here, the formula **A^t9** is deduced by the logical means.

A^t10. $\forall U \in D(U \in R \Rightarrow \forall X \in D \forall Y \in D(X \in U \wedge Y \in U \Rightarrow (X \cup_U Y)^\tau \in U))$, where the set $Z \equiv (X \cup_U Y)^\tau$ is defined from the formula $\exists Z \in D(\forall y \in D((y \in Z) \Leftrightarrow (y \in U \wedge (y \in X \vee y \in Y))))$.

In the same way as in the deduction of the formula **A^t9**, we verify that the conditions $U \in R$, $X \in D$, $Y \in D$, $X \in U$ and $Y \in U$ imply the equality $Z = X \cup Y$, and also $U = W_n$ for some $n \in \omega$. By Lemma 11, Sec. 4, $X, Y \in W_n$ implies $Z = X \cup Y \in W_n = U$. From here, the formula **A^t10** is deduced.

A^t11. $\forall U \in D(U \in R \Rightarrow \forall X \in D \forall Y \in D \forall z \in D((X \in U \wedge Y \subset U \wedge (z \subset (X *_U Y)^\sigma) \wedge (\forall x \in D(x \in X \Rightarrow z \langle x \rangle^\tau \in U))) \Rightarrow ((rng_U z)^\sigma \in U)))$, where:

- the set $Z_1 \equiv (X *_U Y)^\sigma$ is defined from the formula $\exists Z_1 \in D(\forall y \in D((y \in Z_1) \Leftrightarrow (y \in U \wedge (\exists u \in D \exists v \in D(u \in X \wedge v \in Y \wedge y = \langle u, v \rangle_U^*))))$,
- the set $Z_2 \equiv Z_2(x) \equiv z \langle x \rangle^\tau$ is defined from the formula $\exists Z_2 \in D(\forall y \in D((y \in Z_2) \Leftrightarrow (y \in U \wedge y \in Y \wedge \langle x, y \rangle_U^* \in z)))$,
- the set $Z_3 \equiv (rng_U z)^\sigma$ is defined from the formula $\exists Z_3 \in D(\forall y \in D((y \in Z_3) \Leftrightarrow (y \in U \wedge y \in Y \wedge (\exists x \in D(x \in X \wedge \langle x, y \rangle_U^* \in z))))$.

As before, we verify that the conditions $U \in R$, $u \in D$, $v \in D$, $u \in U$, and $v \in U$ consecutively imply the equalities $\{u\}_U^* = \{u\}$, $\{u, v\}_U^* = \{u, v\}$, and $\langle u, v \rangle_U^* = \langle u, v \rangle$, where $U = W_n$ for some $n \in \omega$. By Corollary 1 to Lemma 12, Sec. 4 $u, v \in W_n$ implies $\langle u, v \rangle_U^* = \langle u, v \rangle \in W_n = U$.

Therefore, in turn, we infer that the conditions $U \in R$, $X \in D$, $Y \in D$, $X \in U$, $Y \subset U$, $x \in D$, and $x \in X$ imply the equalities $Z_1 = X *_U Y$, $Z_2 = z \langle x \rangle$ and $Z_3 = rng z$. Let us also have the condition $\forall x \in D(x \in X \Rightarrow z \langle x \rangle^\tau \in U)$. Since z is a correspondence from X into $Y \subset U$, it follows that z is a correspondence from X into W_n . If $x \in X \in D$, then from the transitivity of D follows $x \in D$. Therefore, this additional condition implies $z \langle x \rangle = z \langle x \rangle^\tau \in W_n$. Since $X \in W_n$, by Lemma 17, Sec. 4, it follows that $(rng_U z)^\sigma = rng z \in W_n = U$. From here, by logical means, we infer the formula **A^t11**.

A^t12. $\forall U \in D(U \in R \Rightarrow \forall X \in D(X \subset U \wedge X \neq \emptyset \Rightarrow \exists x \in D(x \in X \wedge (x \cap_U X)^\tau = \emptyset)))$, where the set $Z \equiv (x \cap_U X)^\tau$ is defined from the formula $\exists Z \in D(\forall y \in D((y \in Z) \Leftrightarrow (y \in U \wedge (y \in x \wedge y \in X))))$.

If we fix some $U \in R$, i.e., $U = W_n$ for $n \in \omega$, then we need to prove the formula $\forall X \in D(X \subset W_n \wedge X \neq \emptyset \Rightarrow \exists x \in D(x \in X \wedge (x \cap_{W_n} X)^\tau = \emptyset))$. Since $X \subset W_n$ implies $X \in D$, and $x \in X$ implies $x \in D$, it follows that this formula can be transformed in to the formula $\forall X(X \subset W_n \wedge X \neq \emptyset \Rightarrow \exists x(x \in X \wedge (x \cap_{W_n} X)^\tau = \emptyset))$. For $x, X \in W_n$, we have $((x \cap_{W_n} X)^\tau = x \cap X)$; therefore, we need only to prove

that $\forall X \subset W_n (X \neq \emptyset \Rightarrow \exists x (x \in X \wedge x \cap X = \emptyset))$. But it is a direct consequence of the axiom of regularity in ZF.

A^t13. $\exists X \in D (X \in W_0 \wedge \emptyset \in X \wedge \forall x \in D (x \in X \Rightarrow ((x \cup_{W_0} \{x\}_{W_0})^\tau \in X)))$, where:

- the set $Z_1 \equiv (x \cup_{W_0} \{x\}_{W_0})^\tau$ is defined from the formula $\exists Z_1 \in D (\forall y \in D ((y \in Z_1) \Leftrightarrow (y \in W_0 \wedge (y \in x \vee y \in \{x\}_{W_0}^*))))$,
- the set $Z_2 \equiv \{x\}_{W_0}^*$ is defined from the formula $\exists Z_2 \in D (\forall y \in D ((y \in Z_2) \Leftrightarrow (y \in W_0 \wedge y = x)))$.

From the conditions $X \in D$, $X \in W_0$, $x \in D$, and $x \in X$, it follows that $Z_2 = \{x\}$, and, therefore, $Z_1 = x \cup \{x\}$. In ZF, consider the set $X \equiv \omega$. This set obviously satisfies the condition $\emptyset \in X \wedge \forall x \in X (x \cup \{x\} \in X)$. We need to prove that $\omega \in W_0$. Actually, since $W_0 = V_{\iota_0}$ and $\iota_0 > \omega$, by the definition of an inaccessible cardinal, by Lemma 13, Sec. 4, it follows that $\omega \in W_0$.

A^t14. $\forall U \in D (U \in R \Rightarrow \forall X \in D (X \in U \wedge X \neq \emptyset \Rightarrow \exists z \in D ((z \Leftrightarrow \mathcal{P}_U(X) \setminus \{\emptyset\}_U \rightarrow_U X)^\sigma \wedge \forall Y \in D (Y \in (\mathcal{P}_U(X) \setminus \{\emptyset\}_U)^\tau \Rightarrow z(Y)^\tau \in Y))))$, where:

- the set $Z_1 \equiv (\mathcal{P}_U(X) \setminus \{\emptyset\}_U)^\tau$ is defined from the formula $\exists Z_1 \in D (\forall y \in D ((y \in Z_1) \Leftrightarrow (y \in U \wedge (y \in \mathcal{P}_U(X)^* \wedge y \notin \{\emptyset\}_U^*))))$,
- the set $Z_2 \equiv z(Y)^\tau$ is defined from the formula $Z_2 \in U \wedge \langle Y, Z_2 \rangle_U^\tau \in z$.
- η^τ denotes the formula $M \models \eta[s^\tau]$ in which s^τ denotes the corresponding change of the sequence s under the translation of the quantifier over-formulas $\forall U(\dots)$, $\forall X(\dots)$, and $\exists z(\dots)$ indicated above.

Fix the conditions $U \in D$, $U \in R$, $X \in D$, and $X \in U$. Denote $\mathcal{P}_U(X) \setminus \{\emptyset\}_U$ by S and $\mathcal{P}(X) \setminus \{\emptyset\}$ by T . We have proved above that $X \in U$ and $\emptyset \in U$ imply $\mathcal{P}_U(X)^* = \mathcal{P}(X)$ and $\{\emptyset\}_U^* = \{\emptyset\}$. Since $U \in R$, it follows that $U = W_n$ for some $n \in \omega$. Therefore, by Lemma 12, Sec. 4, $X \in U$ implies $\mathcal{P}(X) \in U$, and, by Lemma 8, Sec. 4, it implies $T \in U$. By Lemma 9, Sec. 4, $y \in T \in U$ implies $y \in U$. All these properties imply $Z_1 = T$.

If $Y \in D$ and $Y \in Z_1$, then $Y \in T \in U$ implies $Y \in U$. As was stated above, $Z_2 \in U$ and $Y \in U$ imply $\langle Y, Z_2 \rangle_U^\tau = \langle Y, Z_2 \rangle$. Then $\langle Y, Z_2 \rangle \in z$ implies $Z_2 \in z \langle Y \rangle$. From here and from the previous conditions, we cannot yet infer that $Z_2 = z \langle Y \rangle$.

Consider the formula $\varphi \equiv (z \Leftrightarrow \mathcal{P}_U(X) \setminus \{\emptyset\}_U \rightarrow_U X)$. It is the conjunction of the three following formulas: $\varphi_1 \equiv (z \subset S * U X)$, $\varphi_2 \equiv (dom_U z = S)$, and $\varphi_3 \equiv (\forall x (x \in S \Rightarrow \forall y (y \in X \Rightarrow \forall y' (y' \in X \Rightarrow (\langle x, y \rangle_U \in z \wedge \langle x, y' \rangle_U \in z \Rightarrow y = y')))))$.

Therefore, $\varphi^\sigma = \varphi_1^\sigma \wedge \varphi_2^\sigma \wedge \varphi_3^\sigma$. Since $\varphi_1 = (\forall u (u \in z \Rightarrow u \in U \wedge \exists x \exists y (x \in S \wedge y \in X \wedge u = \langle x, y \rangle_U))$, it follows that $\varphi_1^\sigma \Leftrightarrow (\forall u \in D (u \in z \Rightarrow u \in U \wedge \exists x \in D \exists y \in D (x \in Z_1 \wedge y \in X \wedge u = \langle x, y \rangle_U^*))$. Similarly, $\varphi_2 = (\forall x (x \in S \Rightarrow x \in U \wedge x \in S \wedge \exists y (y \in X \wedge \langle x, y \rangle_U \in z))$ implies $\varphi_2^\sigma \Leftrightarrow (\forall x \in D (x \in Z_1 \Rightarrow x \in U \wedge x \in Z_1 \wedge \exists y \in D (y \in X \wedge \langle x, y \rangle_U^* \in z))$.

Finally, $\varphi_3^\sigma \Leftrightarrow (\forall x \in D (x \in Z_1 \Rightarrow \forall y \in D (y \in X \Rightarrow \forall y' \in D (y' \in X \Rightarrow (\langle x, y \rangle_U^* \in z \wedge \langle x, y' \rangle_U^* \in z \Rightarrow y = y')))))$.

For X , by the axiom of choice **A10** (ZF), there exists z such that $\chi \equiv (z \Leftrightarrow \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X) \wedge \forall Y (Y \in \mathcal{P}(X) \setminus \{\emptyset\} \Rightarrow z \langle Y \rangle \in Y)$.

Consider the formula $\psi \equiv (z \Leftrightarrow \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X)$. As above, $\psi = \psi_1 \wedge \psi_2 \wedge \psi_3$, where $\psi_1 = (\forall u (u \in z \Rightarrow \exists x \exists y (x \in T \wedge y \in X \wedge u = \langle x, y \rangle))$, $\psi_2 = (\forall x (x \in T \Rightarrow x \in T \wedge \exists y (y \in X \wedge \langle x, y \rangle \in z))$ and $\psi_3 = (\forall x (x \in T \Rightarrow \forall y (y \in X \Rightarrow \forall y' (y' \in X \Rightarrow (\langle x, y \rangle \in z \wedge \langle x, y' \rangle \in z \Rightarrow y = y')))))$.

Since $T \in U$, by Corollary 2 to Lemma 12, Sec. 4, we obtain $T * X \in U$. The formula ψ_1 means that $z \subset T * X$. Therefore, $z \in U$ and $z \in D$.

Let us deduce from these properties the formula φ_1^σ . Let $u \in D$ and $u \in z$. Then, by Lemma 9, Sec. 4, $u \in z \in U$ implies $u \in U$. From the formula ψ_1 , it follows that for u , there exist $x \in T$ and $y \in X$ such that $u = \langle x, y \rangle$. By Lemma 9, Sec. 4, $x, y \in U \subset D$. We have stated above that $\langle x, y \rangle_U^* = \langle x, y \rangle$ in this case. Since $x \in T$ and $T = Z_1$, it follows that $x \in Z_1$. Thus, from $u \in z$, the formula $(u \in U \wedge \exists x \in D \exists y \in D (x \in Z_1 \wedge y \in X \wedge u = \langle x, y \rangle_U^*))$ is deduced. Applying the theorem of deduction and the rules of deduction, we deduce the formula φ_1^σ .

Let $x \in D$, and let $x \in Z_1 = T$. Since $T \in U$, by Lemma 9, Sec. 4, it follows, that $x \in U$. From the formula ψ_2 , we infer that for x , there exists $y \in X$ such that $\langle x, y \rangle \in z$. By the transitivity of D , the condition $y \in X \in D$, implies $y \in D$. Similarly, by Lemma 9, Sec. 4, $y \in X \in U$ implies $y \in U$. But $\langle x, y \rangle = \langle x, y \rangle_U^*$ in this case. Thus from $x \in Z_1$, the formula $(x \in U \wedge x \in Z_1 \wedge \exists y \in D(y \in X \wedge \langle x, y \rangle_U^* \in z))$ is deduced. As above, from here, we deduce the formula φ_2^σ .

Let $x \in Z_1 = T$, $y \in D$, $y \in X$, $y' \in D$, $y' \in X$, $\langle x, y \rangle_U^* \in z$, and let $\langle x, y' \rangle_U^* \in z$. As above, these conditions imply $\langle x, y \rangle \in z$ and $\langle x, y' \rangle \in z$. But then the formula ψ_3 implies $y = y'$. In turn, applying several times the theorem of deduction and the rules of deduction, we deduce the formula φ_3^σ from the formula ψ .

Thus, the formula φ^σ is deduced.

Since $z \rightleftharpoons T \rightarrow X$, it follows that $z\langle Y \rangle = \{z(Y)\}$. Consequently, from $Z_2 \in \{z(Y)\}$, we infer that $Z_2 = z(Y)$. Therefore, for the function z , the conditions $Y \in D$ and $Y \in Z_1 = T$ imply $Z_2 = z(Y) \in Y$.

All this means that from Axiom **A10** (ZF) is deduced the existence of an object z satisfying the formula χ , from which the formula $\xi \equiv \varphi^\sigma \wedge \forall Y \in D(Y \in Z_1 \Rightarrow Z_2 \in Y)$ is deduced. Therefore, in ZF, from the fixed conditions, the formula $\exists z \in D(\xi)$ is deduced. In turn, applying several times the theorem of deduction and the rule of generalization, we deduce as result the formula **A^t14**.

Since all the translations of the axioms of LTS turn out to be deducible formulas of the ZF+ISIC theory, it follows that the LTS is consistent. \square

The absence of an axiom scheme like the axiom scheme of replacement in the ZF theory in the LTS apparently makes impossible an interpretation of the ZF+ISIC theory in the LTS. But this interpretation becomes possible if we strengthen the LTS by the *axiom of an infinite class of universes*

$$ICU \equiv \exists X(\forall U \in X(U \bowtie) \wedge X \neq \emptyset \wedge \forall V \in X \exists W \in X(V \in W)).$$

Lemma 1. *The following assertions are equivalent in the LTS:*

- (1) ICU ;
- (2) $ITCU$ (an infinite transitive class of universes) \equiv there exists a class Y such that:
 - (a) $\forall U \in Y(U \bowtie)$;
 - (b) $Y \neq \emptyset$;
 - (c) $\forall U \forall V(U \bowtie \wedge U \in V \in Y \Rightarrow U \in Y)$ (the property of transitivity with respect to universes);
 - (d) $\forall V \in Y \exists W \in Y(V \in W)$ (the property of unboundedness).

Proof. (1) \vdash (2). Denote by D the class the existence of which is postulated by Axiom ICU. It follows from the axiom of universality **A6** that $D \in \alpha$ for some universal class α . By the axiom of transitivity **A7**, $\forall V \in D(V \in \alpha)$. By Corollary 2 to Proposition 4, $\mathfrak{a} \in \alpha$ or $\mathfrak{a} = \alpha$. It is clear that the second case contradicts ICU. Therefore, $\mathfrak{a} \in \alpha$.

Consider the class $E \equiv \{U \in \alpha \mid U \bowtie \wedge \exists V \in D(U \in V)\}$. If $U \in D$, then, by ICU, $\exists V \in D(U \in V)$. Therefore, $D \subset E$. The class E is universally transitive. Actually, if $U \bowtie$ and $U \in V \in E$, then $U \in V \in W \in D$ for some W . Using Axiom **A7**, we obtain $U \in W \in D \in \alpha$, $U \in W \in \alpha$, and $U \in \alpha$. Therefore $U \in E$.

If $V \in E$, then $V \in W \in D \subset E$ for some W , by definition. Therefore, E has property (d).

The deducibility (2) \vdash (1) is obvious. \square

A similar lemma is valid for inaccessible cardinals in the ZF theory if we replace ICU by ISIC.

Lemma 2. *Let E be a nonempty class of universes with the transitivity property with respect to universes, i.e., E possesses properties (a)–(c) from Lemma 1. Then $\mathfrak{a} \in E$.*

Proof. Let $V \in E$. By Corollary 2 to Proposition 4, Sec. 5, $V = \mathfrak{a}$ or $\mathfrak{a} \in V$. In the first case, $\mathfrak{a} \in E$. In the second case, $\mathfrak{a} \in V \in E$ implies $\mathfrak{a} \in E$. \square

Statement 4. *If the LTS+ICU is consistent, then the ZF+ISIC theory is consistent.*

Proof. Consider the class C the existence of which is postulated by axiom ICU. According to the axiom of universality **A6**, there exists a universal class D such that $C \in D$. Consider the interpretation $M \equiv (D, I)$ of the ZF theory in the LTS described in the proof of Statement 1, Sec. 5. In the proof of Statement 1, Sec. 5, it was established that the interpretation M is a model of ZF in the LTS, and therefore, a model of ZF in the LTS+ICU. Let us verify the deducibility of the translation of Axiom ISIC under the interpretation M on an arbitrary infinite sequence $s \equiv x_0, \dots, x_q, \dots$ of elements of the class D . This translation has the form

$$\varphi \equiv M \models ISIC[s] = \exists X \in D (\forall x \in D (x \in X \Rightarrow Icn(x)^D) \wedge X \neq \emptyset \wedge \forall y \in D (y \in X \Rightarrow \exists z \in D (z \in X \wedge y \in z))).$$

In the proof of Statement 2, it was established that the formula $Icn(x)^D$ means that x is a D -inaccessible D -cardinal number in the LTS.

By Theorem 3, Sec. 5, there exists the injective mapping $q : \mathbf{U} \rightarrow \overline{\mathbf{In}}$ such that $q(\alpha) \subset \alpha$ and $\alpha = \overline{V_{q(\alpha)}}$. If $\alpha \in C$, then $q(\alpha) \subset \alpha \in C \in D$, by Axioms **A7** and **A8**, implies $q(\alpha) \in D$. Therefore, $p \equiv q|C$ is an injective mapping from C into D . Since $C \subset D$, it follows, by Lemma 5, Sec. 5, that $p \simeq C \rightarrow_D D$ and $K \equiv rng p = rng_{Dp} \subset D$. Hence $p \simeq C \simeq_D K$. Since $p(\alpha) \in D$, by the axiom of the full union **A11**, it follows that $K \in D$. From $C \neq \emptyset$, it follows that $K \neq \emptyset$. If $\varkappa \in K$, then $\varkappa \in D$. By Proposition 3, Sec. 5, \varkappa is a D -inaccessible D -cardinal. Consequently K consists of D -inaccessible D -cardinals.

By Axiom ICU, for $\varkappa \in K$ and $\beta \equiv \overline{V_\varkappa} \in C$, there exists $\gamma \in C$ such that $\beta \in \gamma$. Consider the inaccessible cardinal $\lambda \equiv p(\gamma) \in K \subset D$. Since q is strictly monotone, it follows that $\varkappa \in \lambda$. Consequently, the formula $\psi(K) \equiv (\forall x \in D (x \in K \Rightarrow (x \text{ is a } D\text{-inaccessible } D\text{-cardinal})) \wedge K \neq \emptyset \wedge \forall y \in D (y \in K \Rightarrow \exists z \in D (z \in K \wedge y \in z)))$ is deduced in the LTS+ICU. Therefore, the formula $\varphi = \exists X \in D \psi(X)$ is also deduced. \square

Axiom ICU allows us to combine the finite sequences of universal classes continuing each other from Theorem 5, Sec. 5 into one infinite sequence.

Proposition 3. *In the LTS, the following assertions are equivalent:*

- (1) ICU;
- (2) *there exist a universal class X and an infinite strictly increasing X -sequence $u \equiv (U_n \in X | n \in \omega)_X$ of universal classes such that from $n \in \omega$, V is a universal class, and $U_0 \subset V \subset V_n$ it follows that $V = V_k$ for some $k \in n + 1$ (the property of incompressibility);*
- (3) *there exists a universal class X such that $U_k^n \in X$ for any $k \in n + 1$ and any $n \in \omega$, where $u(n) \equiv (U_k^n \in U(n) | k \in n + 1)_{U(n)}$ are the increasing incompressible $U(n)$ -sequences of universal classes continuing each other strictly from Theorem 5, Sec. 5.*

Proof. (1) \vdash (2). Consider the class A the existence of which is postulated by Axiom ICU. By the axiom of universality, there exists a universal class X such that $A \in X$. By Theorem 4, Sec. 5, in the class A , there exists the smallest element α .

In the same way as in the proof of Theorem 5, Sec. 5, it is proved that for every $n \in \omega$, there exists a unique X -sequence of universal classes $u(n) \equiv (U_k \in A | k \in n + 1)_X$ such that $U_0 = \alpha$, $U_k \in U_l$ for every $k \in l \in n + 1$, and if V is a universal class and $U_0 \subset V \subset U_n$, then $V = U_k$ for some $k \in n + 1$ (the property of incompressibility). It follows from the property of uniqueness that $u(n)|(m + 1) = u(m)$ for all $m \leq n$, i.e., these finite sequences continue each other. By this uniqueness, we can denote the sequence $u(n)$ by $(U_k^n | k \in n + 1)$.

Consider the X -class $\omega *_{X} A$ and the X -correspondence $u \equiv \{z \in \omega *_{X} A | \exists x \in \omega (z = \langle x, U_x^n \rangle_X)\}$. Since $u\langle n \rangle = \{U_k^n\}_X \in X$ for every $n \in \omega$, it follows that $u \simeq \omega \rightarrow_X A$. By the axiom of transitivity, $U_n \equiv U_n^n \in A \in X$ implies $U_n \in X$. Therefore, u is an X -sequence $(U_n \in X | n \in \omega)$.

(2) \vdash (3). From the property of uniqueness of the $U(n)$ -sequence $u(n)$ and the class $U(n)$ from Theorem 5, Sec. 5, it follows that $u|n + 1 = u(n)$, i.e., $U_{n+1} = U(n)$ and $U_k = U_k^n$ for any $n \in \omega$ and any $k \in n + 1$. Therefore, $U_k^n \in X$.

(3) \vdash (1). Consider the X -class $Y \equiv \{y \in X \mid \exists x \in \omega(y = U_x^x)\} = \{U_n^n \in X \mid n \in \omega\}_X$. Since the $U(n)$ -sequences $u(n)$ are strictly increasing and continue each other, it follows that the class Y satisfies Axiom ICU. \square

It follows from assumption (2) of this statement that the LTS+ICU resembles the axiomatics of da Costa (see [6] and [7]) with the denumerable set of constants $\mathfrak{U}_1, \dots, \mathfrak{U}_n, \dots$ for universes with axioms like the axiom $X \subset \mathfrak{U}_n \Rightarrow X \in \mathfrak{U}_{n+1}$. However, the theory of da Costa uses the nonconstructive rule of deduction (precisely, the ω -rule of Carnap) and some properties of natural numbers.

Corollary. *In the LTS+ICU there exist a universal class X and an infinite strictly increasing incompressible X -sequence of universal classes $u \equiv (U_m \in X \mid m \in \omega)_X$, and also for every $n \in \omega$, there exist a unique universal class V and a unique finite strictly increasing incompressible V -sequence of universal classes $v(n) \equiv (V_k \in V \mid k \in n+1)_V$ such that $V_0 = X$.*

From the property of uniqueness, it follows that $v(n) \mid (l+1) = v(l)$ for all $l \leq n$, i. e., these finite sequences continue each other.

Proof. By Proposition 3, there exist the corresponding X and $u \equiv (U_m \in X \mid m \in \omega)_X$. Further similarly to the proof of Theorem 5, Sec. 5 is deduced the existence of the corresponding V and $v(n) \equiv (V_k \in V \mid k \in n+1)_V$. \square

Roughly speaking, the LTS+ICU ensures the existence of $\omega + \forall n$ different universal classes.

It follows from this corollary and the remark made before Theorem 1 that to construct a model of the LTS+ICU in the ZF theory, it is necessary to have at least two infinite sequences $u \equiv (\iota_m \mid m \in \omega)$ and $v \equiv (\varkappa_n \mid n \in \omega)$ of inaccessible cardinals in ZF such that $\iota_k < \iota_m < \varkappa_l < \varkappa_n$ for every $k \in m \in \omega$ and $l \in n \in \omega$. But, as Proposition 1 shows, the existence of such infinite sequences is equivalent to Axiom ISIC₂.

With the help of Proposition 1 we can prove the following statement.

Statement 5. *If the ZF+ISIC₂ theory is consistent, then the LTS+ICU is also consistent.*

Proof. In the same way as in the proof of Statement 3, consider the sequences $(\iota_m \mid m \in \omega)$ and $(\varkappa_n \mid n \in \omega)$ from Proposition 1. Consider the sets $A \equiv \{\iota_m \mid m \in \omega\}$ and $B \equiv \{\varkappa_n \mid n \in \omega\}$. Further, instead of V_{ι_m} and V_{\varkappa_n} we will write W_m' and W_n'' , respectively. Consider the ordinal numbers $\alpha \equiv \cup A = \sup A$ and $\beta \equiv \cup B = \sup B$ and the sets $D \equiv \cup \{W_m' \mid m \in \omega\} \subset V_\alpha$ and $E \equiv D \cup \cup \{W_n'' \mid n \in \omega\} \subset V_\beta$. It is clear that $\emptyset = W_0' \in W_k' \in W_m' \subset D \subset V_\alpha \subset W_0'' \in W_l'' \in W_n'' \subset E \subset V_\beta$ for every $0 \in k \in m \in \omega$ and $0 \in l \in n \in \omega$. Therefore, by Lemma 8, Sec. 4, we infer that $W_k', W_l'' \in E$ for all $k, l \in \omega$.

Choose the set E as the domain of interpretation of the LTS+ICU in the set ZF+ISIC₂ theory. In E , consider the subsets $R \equiv \{x \in E \mid \exists m \in \omega(x = W_m')\}$ and $S \equiv \{x \in E \mid \exists n \in \omega(x = W_n' \vee x = W_n'')\}$. Define a correspondence J assigning to the predicate symbol \in in the LTS the two-placed relation $C \equiv \{z \in E \times E \mid \exists x, y \in E(z = (x, y) \wedge x \in y)\}$, to the symbol \ni in the LTS the one-placed relation $S \subset E$, and to the constant symbols \emptyset and \mathfrak{a} in LTS the elements \emptyset and W_0' of the domain E . Consider the interpretation $N \equiv (E, J)$ of the LTS in the ZF+ISIC₂ set theory, which is similar to the interpretation $M \equiv (D, I)$, described in the proof of Statement 3. Let us prove that this interpretation is a model of the LTS+ICU. According to the proof of Statement 3, we need to only consider the translation of Axiom ICU and to prove its deducibility in ZF+ISIC₂.

Let $s \equiv x_0, \dots, x_q, \dots$ be an arbitrary sequence of elements of the set E . The translation of Axiom ICU under the interpretation N on the sequence s has the form

$$\varphi \equiv N \vDash ICU[s] \equiv \exists X \in E(\forall U \in E(U \in X \Rightarrow U \in R) \wedge X \neq \emptyset \wedge \forall V \in E(V \in X \Rightarrow \exists W \in E(W \in X \wedge V \in W))).$$

Since $\iota_m < \varkappa_0 < \varkappa_1$ for every $m \in \omega$, it follows that $W_m' \in W_0'' \in W_1''$. By Lemma 8, Sec. 4, $R \in W_1''$ and, therefore, $R \in E$. From $W_0' \in R$, it follows that $R \neq \emptyset$. If $U \in E$ and $U \in R$, then $U = W_m'$ for some $m \in \omega$. Since $\iota_m < \iota_{m+1} \leq \iota_{m_1}$, by Lemma 7, Sec. 4, we infer that $U = W_m' \in W_{m+1}' \equiv V \in R \subset E$. Consequently, for the set R in ZF+ISIC₂, the formula $\psi(R) \equiv (\forall U \in E(U \in R \Rightarrow U \in R) \wedge R \neq \emptyset \wedge \forall V \in$

$E(V \in R \Rightarrow \exists W \in E(W \in R \wedge V \in W))$ is deduced. Therefore, the formula $\varphi \equiv \exists X \in E\psi(X)$ is also deduced. \square

Thus, we have proved the following chain of interpretations:

$$ZF + IC \prec LTS \prec ZF + ISIC \prec LTS + ICU \prec ZF + ISIC_2 \prec ZF + I,$$

where $T \prec S$ denotes the interpretation of the theory T in the set S theory. Denoting by $cons(T)$ the consistency of the theory T , we obtain the inverse chain of relative consistency:

$$\begin{aligned} cons(ZF + I) \Rightarrow cons(ZF + ISIC_2) \Rightarrow cons(LTS + ICU) \Rightarrow \\ \Rightarrow cons(ZF + ISIC) \Rightarrow cons(LTS) \Rightarrow cons(ZF + IC). \end{aligned}$$

The similar chains remain valid, apparently, if we replace the ZF theory by the NBG theory.

It follows from Proposition 2 that inside the ZF+I theory, which is the strongest of the mentioned ones, the chain of mutual interpretations can be continued further if we take as the next steps the $LTS + ICU_2$, $ZF + ISIC_3$ theories, and so on.

Since, according to Sec. 6.1, $ZF + U \sim ZF + I$, it follows from Proposition 2 that the $ZF + ISIC$ theory is weaker than the $ZF + U$ theory. From Statement 3 of this section, it follows that the LTS is weaker than $ZF + ISIC$. Therefore, the LTS is weaker than the $ZF + U$ theory. Thus, the LTS satisfies condition (1) formulated in the Introduction.

It follows from Theorem 5, Sec. 5, that the LTS, as well as the theory $ZF + U(\omega) \sim ZF + ISIC$ (see Sec. 6.1), has a countable totality of different universes and, therefore, satisfies all needs of category theory to such an extent as the theory $ZF + U(\omega)$ does. Therefore, the LTS satisfies condition (2).

Finally, the axiom of universality **A6** from the Sec. 2.1 asserts that in the LTS, there are no objects which are not elements of this countable totality of universes. For the same reasons, the LTS satisfies condition (3).

It follows from here that the LTS is a more adequate foundation for category theory than the $ZF + U$ and $ZF + U(\omega)$ theories. Moreover, the consistency of the $ZF + U(\omega)$ theory implies the consistency of the LTS. Moreover, the LTS is a more adequate foundation for category theory than the set theory of da Costa, because the existence of a countable totality of universes in the LTS is deduced but is not postulated beforehand as in the axiomatics of da Costa. This saves us from the necessity of attracting externally the natural numbers and their properties.

Note also that in Sec. 8.2 we will show that in the LTS we cannot prove that the assembly sequence of finite $U(n)$ -sequences $u(n) \equiv (U_k^n \in U(n) | k \in n+1)_{U(n)}$ of universal classes U_k^n constructed in Theorem 5, Sec. 5 can be continued as was done in the assertion (3) of Proposition 3. This means that in the LTS we have only a countable assembly of universes from Theorem 5, Sec. 5.

7. Proof of Relative Consistency by the Method of Abstract Interpretation

The method of abstract interpretation going back to Gödel (see [19], 10) is a direct generalization of the method of interpretation stated in Sec. 1.3.

7.1. Abstracts of a set theory. Let S be some set theory (see Sec. 1.3). We will consider that either S is a theory with the equality or in S the equality is introduced by the formula $(A \subset B) \wedge (B \subset A)$.

In exactly the same way as in the ZF set theory, we introduced classes (see 4.1), and in the LTS, we introduced assemblies (see 2.1), and in an arbitrary set theory S , we introduce the *abstracts* $\mathbf{C} \equiv \{x | \varphi(x)\}$ and $\mathbf{C}(\vec{u}) \equiv \{x | \varphi(x, \vec{u})\}$. For the abstracts $\mathbf{C} \equiv \{x | \varphi(x)\}$ and $\mathbf{D} \equiv \{x | \psi(x)\}$, as in Sec. 2.1, we define the formulas $\mathbf{C} \subset \mathbf{D}$ and $\mathbf{C} = \mathbf{D}$. Define the formula $\mathbf{C} \in y$ as the notation for the formula $\exists z(z \in y \wedge z = \mathbf{C})$.

As in the Sec. 5.2, for abstracts \mathbf{A} and \mathbf{B} and objects A and B of the S set theory, we introduce the abstracts $\mathcal{P}(\mathbf{A})$, $\mathbf{A} \cup \mathbf{B}$, $\mathbf{A} \cap \mathbf{B}$, $\{A\}$, $\{A, B\}$, $\langle A, B \rangle$, $\mathbf{A} * \mathbf{B}$, and $\cup \mathbf{A}$.

Similarly, in S , we define a *correspondence* \mathbf{C} with the domain of definition $dom \mathbf{C}$ and the range of values $rng \mathbf{C}$, a *function* (\equiv a mapping) \mathbf{F} , a *correspondence* $\mathbf{C} : \mathbf{A} \prec \mathbf{B}$, a *function* $\mathbf{F} : \mathbf{A} \rightarrow \mathbf{B}$, a (*multivalued*) *collection* $\langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$ with the union $\cup \langle \mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A} \rangle$ and the intersection

$\cap(\mathbf{B}_a \subset \mathbf{B} | a \in \mathbf{A})$, a simple collection $(b_a \in \mathbf{B} | a \in \mathbf{A})$ with the abstract of members $\{b_a \in \mathbf{B} | a \in \mathbf{A}\}$, the (multivalued) sequential pair $(\mathbf{A}, \mathbf{A}')$, triplet $(\mathbf{A}, \mathbf{A}', \mathbf{A}'')$, ... of abstracts $\mathbf{A}, \mathbf{A}', \mathbf{A}'', \dots$, the simple sequential pair (a, a') , triplet (a, a', a'') , ... of objects a, a', a'', \dots , the product $\prod(\mathbf{A}_i \subset \mathbf{A} | i \in \mathbf{I})$ of a collection $(\mathbf{A}_i \subset \mathbf{A} | i \in \mathbf{I})$, the product $\mathbf{A} \times \mathbf{A}', \mathbf{A} \times \mathbf{A}' \times \mathbf{A}'', \dots$ of the pair $(\mathbf{A}, \mathbf{A}')$, triplet $(\mathbf{A}, \mathbf{A}', \mathbf{A}'')$, ... of abstracts $\mathbf{A}, \mathbf{A}', \mathbf{A}'', \dots$, an n -placed relation $\mathbf{R} \subset \mathbf{A}^n \equiv \text{Map}(n, \mathbf{A})$ on an abstract \mathbf{A} , an n -placed operation $\mathbf{O} : \mathbf{A}^n \rightarrow \mathbf{A}$, and so on.

The abstract $\mathfrak{U} \equiv \{x | x = x\}$ consisting of all objects of the theory S is said to be *universal*.

Note that operating with abstracts, we always stay within the framework of formulas of the S set theory.

7.2. The abstract interpretation of a first-order theory in a set theory. A set theory S is said to be *finitely-closed* (\equiv closed up to finite collections) if in the theory S , some formula $on(x)$ defines natural numbers as the objects of this theory, and for every natural number $n \geq 1$ every abstract \mathbf{F} that is a mapping from the object n into the universal abstract \mathfrak{U} is an object of the S theory.

In a finitely-closed S set theory, we define the abstract $\omega \equiv \{x | on(x)\}$ consisting of all natural numbers. Therefore, in such a theory, abstracts $\mathbf{s} : \omega \rightarrow \mathbf{A}$ are defined. They can be called *abstract infinite sequences of elements of the abstract \mathbf{A}* .

Suppose that in a S set theory, by means of this theory, we have selected some abstract $\mathbf{D} \equiv \{x | \varphi(x)\}$.

Let S be some fixed finitely-closed set theory with some selected abstract \mathbf{D} .

An *abstract interpretation of a first-order theory T in the finitely-closed set theory S with the selected abstract \mathbf{D}* is a pair \mathbf{M} consisting of the abstract \mathbf{D} and some correspondence I assigning to every predicate letter P_i^n some n -placed relation $I(P_i^n)$ in \mathbf{D} , every functional letter F_i^n some n -placed operation $I(F_i^n)$ in \mathbf{D} , and every constant symbol a_i some element $I(a_i)$ of \mathbf{D} .

Let \mathbf{s} be some abstract infinite sequence x_0, \dots, x_q, \dots of elements of the abstract $\mathbf{D} \equiv \{x | \varphi(x)\}$.

Define the *value of a term t of the theory T on the sequence \mathbf{s} under the abstract interpretation \mathbf{M} of the theory T in the S set theory* (in the notation $t_{\mathbf{M}}[\mathbf{s}]$) by induction in the following way:

- if $t \equiv v_i$, then $t_{\mathbf{M}}[\mathbf{s}] \equiv x_i$;
- if $t \equiv a_i$, then $t_{\mathbf{M}}[\mathbf{s}] \equiv I(a_i)$;
- if $t \equiv F(t_0, \dots, t_{n-1})$, where F is an n -placed functional symbol and t_0, \dots, t_{n-1} are terms, then $t_{\mathbf{M}}[\mathbf{s}] \equiv I(F)(t_{0\mathbf{M}}[\mathbf{s}], \dots, t_{n-1\mathbf{M}}[\mathbf{s}])$.

Since $I(F)$ is an operation from \mathbf{D}^n into \mathbf{D} , it follows that for the term $t \equiv F(t_0, \dots, t_{n-1})$, we have $t_{\mathbf{M}}[\mathbf{s}] \in \mathbf{D}$. Consequently, the value of a term is always an element of the abstract \mathbf{D} , i. e., it is some object of the theory S .

Define the *translation of the formula φ on the sequence \mathbf{s} under the abstract interpretation \mathbf{M} of the theory T in the finitely-closed set theory S* (in the notation $\mathbf{M} \models \varphi[\mathbf{s}]$) by induction in the following way:

- if $\varphi \equiv (P(t_0, \dots, t_{n-1}))$, where P is a n -placed predicate symbol and t_0, \dots, t_{n-1} are terms, then $\mathbf{M} \models \varphi[\mathbf{s}] \equiv ((t_{0\mathbf{M}}[\mathbf{s}], \dots, t_{n-1\mathbf{M}}[\mathbf{s}]) \in I(P))$;
- if $\varphi \equiv (\neg\theta)$, then $\mathbf{M} \models \varphi[\mathbf{s}] \equiv (\neg\mathbf{M} \models \theta[\mathbf{s}])$;
- if $\varphi \equiv (\theta_1 \Rightarrow \theta_2)$, then $\mathbf{M} \models \varphi[\mathbf{s}] \equiv (\mathbf{M} \models \theta_1[\mathbf{s}] \Rightarrow \mathbf{M} \models \theta_2[\mathbf{s}])$;
- if $\varphi \equiv (\forall v_i \theta)$, then $\mathbf{M} \models \varphi[\mathbf{s}] \equiv (\forall x(x \in \mathbf{D} \Rightarrow \mathbf{M} \models \theta[x_0, \dots, x_{i-1}, x, x_{i+1}, \dots, x_q, \dots]))$.

On other formulas the translation is continued similarly to Sec. 1.3.

This definition needs some explanation.

For the formula $\varphi \equiv (P(t_0, \dots, t_{n-1}))$, the symbol-string $((t_{0\mathbf{M}}[\mathbf{s}], \dots, t_{n-1\mathbf{M}}[\mathbf{s}]) \in I(P))$ is a formula of the theory S . Actually, we have mentioned above that the values of the terms $t_{i\mathbf{M}}[\mathbf{s}]$ are objects of the theory S . Since the theory S is finitely-closed, it follows that the abstract $v \equiv (t_{0\mathbf{M}}[\mathbf{s}], \dots, t_{n-1\mathbf{M}}[\mathbf{s}])$ is an object of the theory S . By definition, $I(P)$ is a subabstract of the abstract \mathbf{D}^n . Therefore, the symbol-string $(v \in I(P))$ is, in fact, a formula of the theory S . It follows from the other items of this definition that as a result of the translation, we always obtain formulas of the theory S . Thus $\mathbf{M} \models \varphi[\mathbf{s}]$ is a formula of the theory S .

An abstract interpretation \mathbf{M} is called an *abstract model of the axiomatic theory* (T, Φ_a) in the *axiomatic finitely-closed set theory* (S, Ξ_a) with the selected abstract \mathbf{D} if for every abstract sequence \mathbf{s} of elements of \mathbf{D} , the translation $\mathbf{M} \models \varphi[\mathbf{s}]$ of every axiom φ of the theory T is a deducible formula in the theory (S, Ξ_a) .

All other definitions and assertions from Sec. 1.3 are transferred to the case of abstract interpretation under the corresponding insignificant changes.

8. Undeducibility of Some Axioms in LTS

In this section, all proofs are given by the method of abstract interpretation described in the previous section.

8.1. The undeducibility of the axiom scheme of replacement. In Secs. 5.2 and 5.4, we have made the globalization of almost all main constructions used in naive set theory.

Now we need only to know if one can deduce the global axiom scheme of replacement in the LTS:

ASR (*the axiom scheme of replacement*). Let $\varphi(x, y)$ be a formula of the LTS, and let φ not contain Y as a free variable. Then $\forall x \forall y \forall y' (\varphi(x, y) \wedge \varphi(x, y') \Rightarrow y = y') \Rightarrow \forall X \exists Y \forall x \in X \forall y (\varphi(x, y) \Rightarrow y \in Y)$.

Statement 1. (1) *If the LTS is consistent, then $LTS + \neg ASR$ is consistent.*

(2) *If the LTS is consistent, then the axiom scheme ASR is not deducible in the LTS.*

Proof. (1) For $n \in \omega$, consider the class U and the U -sequence $u(n)$ from Theorem 5, Sec. 5. By their uniqueness, we can denote them by $U(n)$ and $u(n) \equiv (U_k^n \in U(n) | k \in n + 1)_{U(n)}$.

Consider the assembly $\mathbf{C} \equiv \{z | \exists x \exists y (x \in \omega \wedge y \in U(x) \wedge z = \langle x, y \rangle)\}$. It is clear that \mathbf{C} is a correspondence from the class ω into the assembly $\overline{\mathbf{V}}$ such that $\mathbf{C}\langle n \rangle = U_n$ for every $n \in \omega$. Therefore, \mathbf{C} can be written as the collection $\mathbf{C} \equiv \langle U(n) \in \overline{\mathbf{V}} | n \in \omega \rangle$. Consider the assemblies $\mathbf{D} \equiv \cup \langle U(n) | n \in \omega \rangle = \{y | \exists x \in \omega (y \in U(x))\}$ and $\mathbf{R} \equiv \{y | \exists x \in \omega (y = U(x))\}$. By Theorem 5, Sec. 5, $U(m) = U_{m+1}^{m+1} = U_{m+1}^{n+1} \in U_{n+1}^{n+1} = U(n) \subset \mathbf{D}$ for any $m \in n \in \omega$. Therefore, $\mathbf{R} \subset \mathbf{D}$. Moreover, $\emptyset \in U_0 \subset \mathbf{D}$ implies $\emptyset \in \mathbf{D}$.

By Corollaries 4 and 5 to Theorem 4, Sec. 5, the LTS is a finitely-closed set theory.

Choose the assembly \mathbf{D} in the capacity of the domain of abstract interpretation of the theory $T \equiv LTS + \neg ASR$ in the finitely-closed set theory $S \equiv LTS$.

Define a correspondence I assigning to the predicate symbol \in in T the two-placed relation $\mathbf{B} \equiv \{z | \exists x \exists y (x \in \mathbf{D} \wedge y \in \mathbf{D} \wedge z = (x, y) \wedge x \in y)\}$ on \mathbf{D} , to the predicate symbol \ni in T the one-placed relation \mathbf{R} on \mathbf{D} , and to the constants \emptyset and \mathbf{a} in T the elements \emptyset and U_0 of the domain \mathbf{D} , respectively. Consider the abstract interpretation $\mathbf{M} \equiv (\mathbf{D}, I)$.

Let \mathbf{s} be an abstract sequence x_0, \dots, x_q, \dots of elements of \mathbf{D} . We will consider the translations $\mathbf{M} \models \varphi[\mathbf{s}]$ of axioms and axiom schemes of the theory T on the sequence \mathbf{s} under the interpretation \mathbf{M} and will prove their deducibility in the S set theory. Instead of $\theta_{\mathbf{M}}[\mathbf{s}]$ and $\mathbf{M} \models \varphi[\mathbf{s}]$, we will write θ^t and φ^t for terms θ and formulas φ , respectively.

The assembly \mathbf{D} is transitive. Actually, if $y \in \mathbf{D}$, then $y \in U(m)$ for some m . By Axiom **A7**, $y \subset U(m) \subset \mathbf{D}$. Moreover, if $x \subset y \in \mathbf{D}$, then $x \in \mathbf{D}$. Actually, by Axiom **A8**, $x \subset y \in U(m)$, implies $x \in U(m) \subset \mathbf{D}$. We will often use these two properties in our proofs further.

For simplifying the further presentation, we first consider the translations of some simple formulas. Let u and v be some classes in the LTS.

The formula $u \in v$ is translated into the formula $(u \in v)^t = ((u^t, v^t) \in \mathbf{B})$. By Corollary 5 to Theorem 4, Sec. 5, the assembly (u^t, v^t) is a class, i.e., there exists a class z such that $z = (u^t, v^t)$. Therefore, similarly to the arguments from the proof of Statement 3, Sec. 6, we establish the first equivalence $(u \in v)^t \Leftrightarrow u^t \in v^t$.

Further, similarly to the proof of Statement 3, Sec. 6, we establish the second equivalence $(v \subset w)^t \Leftrightarrow v^t \subset w^t$.

This equivalence immediately implies the third equivalence $(v = w)^t \Leftrightarrow v^t = w^t$.

We will now write not the literal translations of axioms and axiom schemes but their equivalent variants which are obtained by using the proved equivalences $(u \in v)^t \Leftrightarrow u^t \in v^t$, $(v \subset w)^t \Leftrightarrow v^t \subset w^t$ and $(v = w)^t \Leftrightarrow v^t = w^t$. We will denote these equivalent variants using the sign “ $\widetilde{}$ ” over them.

$\widetilde{\mathbf{A}^t1}$. $\forall y \in \mathbf{D} \forall z \in \mathbf{D} ((y = z) \Rightarrow (\forall X \in \mathbf{D} (y \in X \Leftrightarrow z \in X)))$.

This formula is directly deduced from the axiom of extensionality **A1** in the LTS.

$\widetilde{\mathbf{A}^t2}$. Let $\varphi(x)$ be an X -predicative formula in the LTS such that the substitution $\varphi(x||y)$ is admissible and φ does not contain Y as a free variable. Then $\forall X \in \mathbf{D} (\exists Y \in \mathbf{D} (\forall y \in \mathbf{D} ((y \in Y) \Leftrightarrow (y \in X \wedge \varphi^t(y))))))$, where φ^t denotes the formula $\mathbf{M} \models \varphi[\mathbf{s}^t]$ in which by \mathbf{s}^t we denote the corresponding change of the sequence \mathbf{s} under the translation of the quantifier over-formulas $\forall X(\dots)$, $\exists Y(\dots)$, and $\forall y(\dots)$ indicated above.

Let $X \in \mathbf{D}$. Let us verify that the formula φ^t is also X -predicative.

Suppose that the subformula $\psi \equiv \exists x(x \in X \wedge \dots)$ occurs in the formula φ . Then the subformula ψ^t occurs in the formulas φ^t . By the proved equivalence, taking into account the external quantification by X , the formula ψ^t is equivalent to the formula $\psi' \equiv \exists x(x \in \mathbf{D} \wedge x \in X \wedge \dots)$. If $x \in X$, then it follows from the transitivity of \mathbf{D} that $x \in \mathbf{D}$. Consequently, the formula ψ' is equivalent to the formula $\psi^* \equiv \exists x(x \in X \wedge \dots)$. Substituting the subformula ψ^t in the formula φ^t by the equivalent formula ψ^* , we obtain the formula φ^* , which is equivalent to the formula φ^t and contains the subformula $\psi^* \equiv \exists x(x \in X \wedge \dots)$. Making this substitution with all subformulas of the form ψ in the formula φ , we obtain the formula φ^\times , which is equivalent to the formula φ^t and contains only subformulas of the form ψ^* .

Now suppose that the formula φ contains the subformula $\chi \equiv \forall x(x \in X \Rightarrow \dots)$. Then the subformula χ^t occurs in the formula φ^t . Therefore, it also occurs in the formula φ^\times . As above, the formula χ^t is equivalent to the formula $\chi' \equiv \forall x(x \in \mathbf{D} \Rightarrow (x \in X \Rightarrow \dots))$.

Consider the formula χ' . We deduce from it the formula $\sigma \equiv (x \in \mathbf{D} \Rightarrow (x \in X \Rightarrow \dots))$. If $x \in X$, then, by the transitivity of \mathbf{D} , $x \in X \in \mathbf{D}$ implies $x \in \mathbf{D}$. This means that χ' and $x \in X$ imply the subformula $(x \in X \Rightarrow \dots)$ of the formula σ and, therefore, the formula $\chi^* \equiv \forall x(x \in X \Rightarrow \dots)$. By the theorem of deduction in the LTS, we deduce the formula $\chi' \Rightarrow \chi^*$.

Conversely, consider the formula χ^* . By **LAS12**, it implies the subformula $(x \in X \Rightarrow \dots)$. By **LAS1**, the formula σ is deduced. By the deduction theorem in the LTS, the formula $\chi^* \Rightarrow \sigma$ is deduced. By the generalization rule, the formula $\forall x(\chi^* \Rightarrow \sigma)$ is deduced. In virtue of **LAS13**, the formula $\chi^* \Rightarrow \chi'$ is deduced. Thus, in the LTS, we deduce the equivalence of the formulas χ' and χ^* .

Substituting the subformula χ^t in the formula φ^\times by the equivalent formula χ^* , we obtain the formula $\varphi^{\times*}$, which is equivalent to the formula φ^\times and contains the subformula $\chi^* \equiv \forall x(x \in X \Rightarrow \dots)$. Making this substitution with all subformulas of the form χ in the formula φ , we obtain the formula $\varphi^{\times\times}$, which is equivalent to the formula φ^t and contains only subformulas of the form ψ^* and χ^* . Consequently, the formula $\varphi^{\times\times}$ is X -predicative.

Therefore, by **AS2** from the LTS, $X \in \mathbf{D}$ implies the formula $\pi \equiv (\exists Y (\forall y (y \in Y \Leftrightarrow (y \in X \wedge \varphi^{\times\times}(y))))))$. Consider the formula $\alpha \equiv (\forall y \in Y \Leftrightarrow (y \in X \wedge \varphi^{\times\times}(y)))$. By **LAS11**, the formula α implies the formula $\beta \equiv (y \in Y \Leftrightarrow (y \in X \wedge \varphi^{\times\times}(y)))$. By **LAS1**, from α , we infer the formula $\gamma \equiv (y \in \mathbf{D} \Rightarrow \beta)$. Consequently, from $X \in \mathbf{D}$ and α , we infer the formula $\delta \equiv (y \in \mathbf{D} \Rightarrow (y \in Y \Leftrightarrow (y \in X \wedge \varphi^t(y))))$. By the generalization rule, we infer the formula $\varepsilon \equiv (\forall y \in \mathbf{D} (y \in Y \Leftrightarrow (y \in X \wedge \varphi^t(y))))$.

Moreover, from α , we infer the formula $Y \subset X$. Since $X \in \mathbf{D}$, by the second property of the totality \mathbf{D} proved above, we obtain $Y \in \mathbf{D}$. Therefore, from $X \in \mathbf{D}$ and α , we infer the formula $Y \in \mathbf{D} \wedge \varepsilon$. By **LAS12** is implied the formula $\varkappa \equiv \exists Y \in \mathbf{D} \varepsilon$.

Thus, by the deduction theorem, from $X \in \mathbf{D}$, we infer the formula $\alpha \Rightarrow \varkappa$. By the generalization rule, we infer the formula $\forall Y (\alpha \Rightarrow \varkappa)$. Then, by **LAS14**, from $X \in \mathbf{D}$, we infer the formula $\pi \Rightarrow \varkappa$.

Since we have already deduced above the formula π under the condition $X \in \mathbf{D}$, by the implication rule, we infer the formula \varkappa . By the theorem of deduction in the LTS, the formula $(X \in \mathbf{D} \Rightarrow \varkappa)$ is deduced. Therefore, by the rule of generalization, the formula $\widetilde{\mathbf{A}^{\mathbf{t}2}}$ is deduced.

$\widetilde{\mathbf{A}^{\mathbf{t}3}}$. $\forall Z \in \mathbf{D}((\forall x \in \mathbf{D}(x \notin Z)) \Leftrightarrow z = \emptyset)$.

Fix the condition $Z \in \mathbf{D}$. Consider the formula $\chi \equiv \forall x(x \in \mathbf{D} \Rightarrow x \notin Z)$. If $x \in Z$, then, by the property of transitivity, $x \in \mathbf{D}$ and then χ implies $x \notin Z$. If $x \notin Z$, then obviously χ implies $x \notin Z$. Therefore, under our condition, we infer that χ implies $x \notin Z$. By the rule of generalization, χ implies $\forall x(x \notin Z)$. By Axiom **A3**, χ implies $Z = \emptyset$. By the theorem of deduction $Z \in \mathbf{D}$ implies the formula $\chi \Rightarrow Z = \emptyset$. Conversely, by **A3**, $Z = \emptyset$, implies $\forall x(x \notin Z)$. Therefore, $Z \in \mathbf{D}$ and $Z = \emptyset$ imply the formula $(Z = \emptyset \Rightarrow \chi)$. Thus, the condition $Z \in \mathbf{D}$ implies the formula $(\chi \Leftrightarrow Z = \emptyset)$. By the theorem of deduction, we infer the formula $Z \in \mathbf{D} \Rightarrow (\chi \Leftrightarrow Z = \emptyset)$. Therefore, by the rule of generalization, we infer the formula $\widetilde{\mathbf{A}^{\mathbf{t}3}}$.

$\widetilde{\mathbf{A}^{\mathbf{t}4}}$. $\forall U \in \mathbf{D} \forall V \in \mathbf{D}((U = V \Rightarrow (U \in \mathbf{D} \Leftrightarrow V \in \mathbf{R}))$.

Let $U = V$. If $U \in \mathbf{R}$, then $U = U_n$ for some $n \in \omega$. Then $V = U_n$ implies $V \in \mathbf{R}$. And vice versa. Therefore, $U \in \mathbf{R} \Leftrightarrow V \in \mathbf{R}$.

$\widetilde{\mathbf{A}^{\mathbf{t}5}}$. $U_0 \in \mathbf{R} \wedge \forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow U_0 \subset U)$.

Consider the formulas $\varphi(x, y) \equiv (x \in \omega \wedge y = U(x))$ and $\psi \equiv \exists x \varphi(x, y)$. Since $0 \in \omega \wedge U(0) = U(0)$, by **LAS12** in the LTS, we infer the formula $\exists x(x \in \omega \wedge U(0) = U(x))$, i. e., the formula $\psi(y|U(0))$. By definition, this means that $U(0) \in \mathbf{R}$.

If $U \in \mathbf{R}$, then $U = U(n)$ for some $n \in \omega$. If $n = 0$, then $U(0) = U$. If $n > 0$, then, as was indicated at the beginning of the proof, $U(0) \in U(n)$. By the axiom of transitivity **A7**, $U(0) \subset U(n) = U$. By the theorem of deduction in the LTS, we infer the formula $\alpha \equiv (U \in \mathbf{R} \Rightarrow U(0) \subset U)$. By **LAS1**, we infer the formula $(U \in \mathbf{D} \Rightarrow \alpha)$ and, by the rule of generalization, we infer the formula $\forall U \in \mathbf{D}\alpha$. Therefore, we infer the formula $\widetilde{\mathbf{A}^{\mathbf{t}5}}$.

$\widetilde{\mathbf{A}^{\mathbf{t}6}}$. $\forall X \in \mathbf{D} \exists U \in \mathbf{D}(U \in \mathbf{R} \wedge X \in U)$.

From $X \in \mathbf{D}$, it follows that $X \in U(n)$ for some $n \in \omega$. In the same way as in the deduction of **A^t5**, we prove the deducibility of the formula $U(n) \in \mathbf{R} \subset \mathbf{D}$. Consequently, from $X \in \mathbf{D}$, we infer the formula $\alpha \equiv (U(n) \in \mathbf{D} \wedge U(n) \in \mathbf{R} \wedge X \in U(n))$. By **LAS12**, we infer the formula $\beta \equiv \exists U \in \mathbf{D}(U \in \mathbf{R} \wedge X \in U)$. By the theorem of deduction in the LTS, we infer the formula $\gamma \equiv (X \in \mathbf{D} \Rightarrow \beta)$. By the rule of generalization, we infer $\widetilde{\mathbf{A}^{\mathbf{t}6}}$.

$\widetilde{\mathbf{A}^{\mathbf{t}7}}$. $\forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow \forall X \in \mathbf{D}(x \in U \Rightarrow x \subset U))$.

This formula is deduced from the axiom of transitivity **A7** in the LTS.

$\widetilde{\mathbf{A}^{\mathbf{t}8}}$. $\forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow \forall X \in \mathbf{D} \forall Y \in \mathbf{D}(X \in U \wedge Y \subset X \Rightarrow Y \in U))$.

This formula is deduced from the axiom of subset **A8** in the LTS.

$\widetilde{\mathbf{A}^{\mathbf{t}9}}$. $\forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow \forall X \in \mathbf{D}(X \in U \Rightarrow \mathcal{P}_U(X)^\tau \in U))$, where the U -class $Z \equiv \mathcal{P}_U(X)^\tau$ is defined from the formula $\exists Z \in \mathbf{D}(\forall y \in \mathbf{D}((y \in Z) \Leftrightarrow (y \in U \wedge y \subset X)))$.

First, verify that if $U \in \mathbf{R}$, $X \in \mathbf{D}$, and $X \in U$, then $Z = \mathcal{P}_U(X) \equiv Y$. Let $y \in Z$. Since $Z \in \mathbf{D}$, by the proved-above transitivity, $y \in \mathbf{D}$. But then $y \in \mathbf{D}$ and $y \in Z$ imply $y \in U \wedge y \subset X$, i. e., $y \in Y$. Conversely, let $y \in Y$, i. e., $y \in U \wedge y \subset X$. Since $X \in \mathbf{D}$, by the proved-above second property of the assembly **D**, we obtain $y \in \mathbf{D}$. From $U \in \mathbf{R}$, it follows that U is a universal class. Therefore, by the axiom of subset **A8**, $y \subset X \in U$ implies $y \in U$. But then $y \in \mathbf{D}$, $y \in U$, and $y \subset X$ imply $y \in Z$, which proves the required equality.

By Axiom **A9**, $X \in U$ implies $Z = Y \in U$. From here, by logical means, we infer the formula $\widetilde{\mathbf{A}^{\mathbf{t}9}}$.

$\widetilde{\mathbf{A}^{\mathbf{t}10}}$. $\forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow \forall X \in \mathbf{D} \forall Y \in \mathbf{D}(X \in U \wedge Y \in U \Rightarrow (X \cup_U Y)^\tau \in U))$, where the U -class $Z \equiv (X \cup_U Y)^\tau$ is defined from the formula $\exists z \in \mathbf{D}(\forall y \in \mathbf{D}((y \in z) \Leftrightarrow (y \in U \wedge (y \in X \vee y \in Y))))$.

In the same way as in the deduction of the formula $\widetilde{\mathbf{A}^t\mathbf{9}}$, we verify that the conditions $U \in \mathbf{R}$, $X \in \mathbf{D}$, $Y \in \mathbf{D}$, $X \in U$, and $Y \in U$ imply the equality $Z = X \cup_U Y$, where U is a universal class. By Axiom $\mathbf{A10}$, $Z = X \cup_U Y \in U$. From here, we infer the formula $\mathbf{A}^t\mathbf{10}$.

$\widetilde{\mathbf{A}^t\mathbf{11}}$. $\forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow \forall X \in \mathbf{D} \forall Y \in \mathbf{D} \forall z \in \mathbf{D}((X \in U \wedge Y \subset U \wedge (z \subset (X *_U Y)^\sigma) \wedge (\forall x \in \mathbf{D}(x \in X \Rightarrow z\langle x \rangle^\tau \in U))) \Rightarrow ((rng_U z)^\sigma \in U)))$, where:

- the U -class $Z_1 \equiv (X *_U Y)^\sigma$ is defined from the formula $\exists Z_1 \in \mathbf{D}((\forall y \in \mathbf{D}((y \in Z_1) \Leftrightarrow (y \in U \wedge (\exists u \in \mathbf{D} \exists v \in \mathbf{D}(u \in X \wedge v \in Y \wedge y = \langle u, v \rangle_U^*))))))$,
- the U -class $Z_2 \equiv Z_2(x) \equiv z\langle x \rangle^\tau$ is defined from the formula $\exists Z_2 \in \mathbf{D}(\forall y \in \mathbf{D}((y \in Z_2) \Leftrightarrow (y \in U \wedge y \in Y \wedge \langle x, y \rangle_U^* \in z)))$,
- the U -class $Z_3 \equiv (rng_U z)^\sigma$ is defined from the formula $\exists Z_3 \in \mathbf{D}(\forall y \in \mathbf{D}((y \in Z_3) \Leftrightarrow (y \in U \wedge y \in Y \wedge (\exists x \in \mathbf{D}(x \in X \wedge \langle x, y \rangle_U^* \in z))))))$.

As above, we verify that the conditions $U \in \mathbf{R}$, $u \in \mathbf{D}$, $v \in \mathbf{D}$, $u \in U$, and $v \in U$ successively imply the equalities $\{u\}_U^* = \{u\}_U$, $\{u, v\}_U^* = \{u, v\}_U$, and $\langle u, v \rangle_U^* = \langle u, v \rangle_U$, where U is a universal class. By Lemma 2, Sec. 2, $u, v \in U$ implies $\langle u, v \rangle_U^* = \langle u, v \rangle_U \in U$.

From here, in turn, we infer that the conditions $U \in \mathbf{R}$, $X \in \mathbf{D}$, $Y \in \mathbf{D}$, $X \in U$, $Y \subset U$, $x \in \mathbf{D}$, and $x \in X$ imply the equalities $Z_1 = X *_U Y$, $Z_2 = z\langle x \rangle$, and $Z_3 = rng_U z$. Let us have one more condition $\forall x \in \mathbf{D}(x \in X \Rightarrow z\langle x \rangle^\tau \in U)$. Since z is an U -correspondence from X into $Y \subset U$, it follows that z is an U -correspondence from X into U . If $x \in X \in \mathbf{D}$, then from the transitivity of \mathbf{D} , we infer that $x \in \mathbf{D}$. Therefore, the additional condition implies $z\langle x \rangle = z\langle x \rangle^\tau \in U$. Since $X \in U$, by the axiom of full union $\mathbf{A11}$, it follows that $Z_3 = rng_U z \in U$. From here, by logical means, we infer the formula $\widetilde{\mathbf{A}^t\mathbf{11}}$.

$\widetilde{\mathbf{A}^t\mathbf{12}}$. $\forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow \forall X \in \mathbf{D}(X \subset U \wedge X \neq \emptyset \Rightarrow \exists x \in \mathbf{D}(x \in X \wedge (x \cap_U X)^\tau = \emptyset)))$, where the U -class $Z \equiv (x \cap_U X)^\tau$ is defined from the formula $\exists Z \in \mathbf{D}(\forall y \in \mathbf{D}((y \in Z) \Leftrightarrow (y \in U \wedge (y \in x \wedge y \in X))))$.

Let us verify that the conditions $U \in \mathbf{R}$ and $X \in \mathbf{D}$ imply the equality $Z = x \cap_U X \equiv Y$. Let $y \in Z$. Since $X \in \mathbf{D}$, it follows that $y \in \mathbf{D}$. But in this case, $y \in \mathbf{D}$ and $y \in Z$ imply $y \in U \wedge y \in x \wedge y \in X$, i.e., $y \in Y$. Conversely, let $y \in Y$, i.e., $y \in U \wedge y \in x \wedge y \in X$. Since $y \in X \in \mathbf{D}$, it follows that $y \in \mathbf{D}$. Consequently, $y \in Z$, which proves the required equality. From $U \in \mathbf{R}$, it follows that U is a universal class.

By the axiom of regularity $\mathbf{A12}$, for $\emptyset \neq X \subset U$, there exists $x \in X$ such that $Z = Y = \emptyset$. Since $x \in X \in \mathbf{D}$, it follows that $x \in \mathbf{D}$. From here, by logical means, we infer the formula $\widetilde{\mathbf{A}^t\mathbf{12}}$.

$\widetilde{\mathbf{A}^t\mathbf{13}}$. $\exists X \in \mathbf{D}(X \in U_0 \wedge \emptyset \in X \wedge \forall x \in \mathbf{D}(x \in X \Rightarrow ((x \cup_{U_0} \{x\}_{U_0})^\tau \in X)))$, where:

- the U_0 -class $Z_1 \equiv Z_1(x) \equiv (x \cup_{U_0} \{x\}_{U_0})^\tau$ is defined from the formula $\exists Z_1 \in \mathbf{D}(\forall y \in \mathbf{D}((y \in Z_1) \Leftrightarrow (y \in U_0 \wedge (y \in x \vee y \in \{x\}_{U_0}^*))))$,
- the U_0 -class $Z_2 \equiv Z_2(x) \equiv \{x\}_{U_0}^*$ is defined from the formula $\exists Z_2 \in \mathbf{D}(\forall y \in \mathbf{D}((y \in Z_2) \Leftrightarrow (y \in U_0 \wedge y = x)))$.

From the conditions $X \in \mathbf{D}$, $X \in U_0 = \mathbf{a}$, $x \in \mathbf{D}$, and $x \in X$ it follows that $Z_2 = \{x\}_{\mathbf{a}}$, and, therefore, $Z_1 = x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}}$.

Consider the \mathbf{a} -set π from the axiom of infra-infiniteness $\mathbf{A13}$. It is clear that $\pi \in \mathbf{a} = U_0 \subset \mathbf{D}$. Since π possesses the property $\pi \in U_0 \wedge \emptyset \in \pi \wedge \forall x \in \pi(Z_1(x) \in \pi)$, it follows that in the LTS the formula $\widetilde{\mathbf{A}^t\mathbf{13}}$ is deduced.

$\widetilde{\mathbf{A}^t\mathbf{14}}$. $\forall U \in \mathbf{D}(U \in \mathbf{R} \Rightarrow \forall X \in \mathbf{D}(X \in U \wedge X \neq \emptyset \Rightarrow \exists z \in \mathbf{D}((z \Leftarrow \mathcal{P}_U(X) \setminus \{\emptyset\}_U \rightarrow_U X)^\sigma \wedge \forall Y \in \mathbf{D}(Y \in \mathcal{P}_U(X) \setminus \{\emptyset\}_U)^\tau \Rightarrow z(Y)^\tau \in Y))))$, where:

- the U -class $Z_1 \equiv Z_1(X) \equiv (\mathcal{P}_U(X) \setminus \{\emptyset\}_U)^\tau$ is defined from the formula $\exists Z_1 \in \mathbf{D}(\forall y \in \mathbf{D}((y \in Z_1) \Leftrightarrow (y \in U \wedge (y \in \mathcal{P}_U(X)^* \wedge y \notin \{\emptyset\}_U^*))))$,
- the U -class $Z_2 \equiv Z_2(Y) \equiv z(Y)^\tau$ is defined from the formula $Z_2 \in U \wedge \langle Y, Z_2 \rangle_U^\tau \in z$,

— η^τ denotes the formula $\mathbf{M} \models \eta[\mathbf{s}^\tau]$ in which \mathbf{s}^τ denotes the corresponding change of the sequence \mathbf{s} under the translation of the quantifier over-subformulas $\forall U(\dots)$, $\forall X(\dots)$, $\exists z(\dots)$, and $\forall Y(\dots)$, indicated above.

Fix the conditions $U \in \mathbf{U}$, $U \in \mathbf{R}$, $X \in \mathbf{D}$, and $X \in U$. We established above that under these conditions, $\mathcal{P}_U(X)^* = \mathcal{P}_U(X)$ and $\{\emptyset\}_U^* = \{\emptyset\}_{U^*}$. From here, by the transitivity of \mathbf{D} , as above, we infer $Z_1 = \mathcal{P}_U(X) \setminus \{\emptyset\}_U \equiv T$. From $U \in \mathbf{R}$, it follows that U is a universal class. Consequently, by **A9**, $\mathcal{P}_U(X) \in U$. Therefore, $T \in U$ by **A8**.

If $Y \in \mathbf{D}$ and $Y \in Z_1$, then $Y \in T \in U$ implies $Y \in U$. As was established above, $Z_2 \in U$ and $Y \in U$ imply $\langle Y, Z_2 \rangle_U^\tau = \langle Y, Z_2 \rangle_U$. Then $\langle Y, Z_2 \rangle_U \in z$ implies $Z_2 \in z\langle Y \rangle$. From here and from the previous conditions, we cannot yet infer that $Z_2 = z\langle Y \rangle$.

Consider the formula $\varphi \equiv (z \rightleftharpoons T \rightarrow_U X)$. It is the conjunction of the three following formulas: $\varphi_1 \equiv (z \subset T *_U X)$, $\varphi_2 \equiv (\text{dom}_U z = T)$, and $\varphi_3 \equiv (\forall x(x \in T \Rightarrow \forall y(y \in X \Rightarrow \forall y'(y' \in X \Rightarrow (\langle x, y \rangle_u \in z \wedge \langle x, y' \rangle_U \in z \Rightarrow y = y')))))$.

Therefore, $\varphi^\sigma = \varphi_1^\sigma \wedge \varphi_2^\sigma \wedge \varphi_3^\sigma$. Since $\varphi_1 = (\forall u(u \in z \Rightarrow u \in U \wedge \exists x \exists y(x \in T \wedge y \in X \wedge u = \langle x, y \rangle_U))$, it follows that $\varphi_1^\sigma \Leftrightarrow (\forall u \in \mathbf{D}(u \in z \Rightarrow u \in U \wedge \exists x \in \mathbf{D} \exists y \in \mathbf{D}(x \in Z_1 \wedge y \in X \wedge u = \langle x, y \rangle_U^*))$. Similarly, $\varphi_2 = (\forall x(x \in T \Rightarrow x \in U \wedge x \in T \wedge \exists y(y \in X \wedge \langle x, y \rangle_U \in z))$ implies $\varphi_2^\sigma \Leftrightarrow (\forall x \in \mathbf{D}(x \in Z_1 \Rightarrow x \in U \wedge x \in Z_1 \wedge \exists y \in \mathbf{D}(y \in X \wedge \langle x, y \rangle_U^* \in z))$.

Finally, $\varphi_3^\sigma \Leftrightarrow (\forall x \in \mathbf{D}(x \in Z_1 \Rightarrow \forall y \in \mathbf{D}(y \in X \Rightarrow \forall y' \in \mathbf{D}(y' \in X \Rightarrow (\langle x, y \rangle_U^* \in z \wedge \langle x, y' \rangle_U^* \in z \Rightarrow y = y')))))$.

By the properties of transitivity for x, y , and y' in the formulas φ_1^σ , φ_2^σ , and φ_3^σ , we have $x, y, y' \in U$. Therefore, by the proved above (see the proof of deducibility of **A^t11**), in this formulas, we have the equalities $Z_1 = T$, $\langle x, y \rangle_U^* = \langle x, y \rangle_U$, and $\langle x, y' \rangle_U^* = \langle x, y' \rangle_U$. It follows from here that the formulas $\varphi_1^\sigma, \varphi_2^\sigma$, and φ_3^σ differ from the formulas φ_1, φ_2 , and φ_3 , respectively, only by the bounded quantifier prefixes $\forall \dots \in \mathbf{D}$ and $\exists \dots \in \mathbf{D}$.

For X , by the axiom of choice **A14**, there exists z such that $\chi \equiv (z \rightleftharpoons \mathcal{P}_U(X) \setminus \{\emptyset\}_U \rightarrow_U X) \wedge \forall Y(Y \in \mathcal{P}_U(X) \setminus \{\emptyset\}_U \Rightarrow z\langle Y \rangle \in Y)$.

Therefore, in the LTS, we infer the formula $\varphi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ and, consequently, the formulas φ_1, φ_2 , and φ_3 .

Let $u \in \mathbf{D}$ and $u \in z$. Then from the formula φ_1 , we infer that there exist $x \in T$ and $y \in X$ such that $u = \langle x, y \rangle_U$. Since $x \in T \in U$ and $y \in X \in U$, by the transitivity property, it follows that $x, y \in U \subset \mathbf{D}$. This means that under the given conditions $u \in \mathbf{D}$ and $u \in z$, in the LTS we infer the formula $(u \in U \wedge \exists x \in \mathbf{D} \exists y \in \mathbf{D}(x \in T \wedge y \in X \wedge u = \langle x, y \rangle_U))$. Applying the deduction theorem and the deduction rules two times, we infer the formula φ_1^σ .

Let $x \in \mathbf{D}$ and $x \in Z_1 = T$. Then from the formula φ_2 , we infer that for x , there exists $y \in X$ such that $\langle x, y \rangle_U \in z$. From $y \in X \in \mathbf{D}$, by the transitivity of \mathbf{D} , it follows that $y \in \mathbf{D}$. This means that under the given conditions $x \in \mathbf{D}$ and $x \in T$, in the LTS, we infer the formula $(x \in U \wedge x \in T \wedge \exists y \in \mathbf{D}(y \in X \wedge \langle x, y \rangle_U \in z))$. From here, as above, we infer the formula φ_2^σ .

Let $x \in \mathbf{D}$, $x \in Z_1 = T$, $y \in \mathbf{D}$, $y \in X$, $y' \in \mathbf{D}$, $y' \in X$, $\langle x, y \rangle_U \in z$, and let $\langle x, y' \rangle_U \in z$. Then from the formula φ_3 , we infer that $y = y'$. In turn applying several times the deduction theorem and the deduction rule, we infer the formula φ_3^σ .

Thus, the formula φ^σ is deduced.

Since $z \rightleftharpoons T \rightarrow_U X$, it follows that $z\langle Y \rangle = \{z\langle Y \rangle\}_U$.

Consequently, from $Z_2 \in U\{z\langle Y \rangle\}_U$, we conclude that $Z_2 = z\langle Y \rangle$. Therefore, for the U -mapping z , the conditions $Y \in \mathbf{D}$ and $Y \in Z_1 = T$ imply $Z_2 = z\langle Y \rangle \in Y$.

Since $T \in U$ and $X \in U$, by Lemma 3, Sec. 2, it follows that, $T *_U X \in U$. By Axiom **A8**, from $z \subset T *_U X$, it follows that $z \in U \subset \mathbf{D}$.

All this means that from axiom **A14**, we deduce the existence of an object z satisfying the formula χ from which we infer the formula $\xi \equiv \varphi^\sigma \wedge \forall Y \in \mathbf{D}(Y \in Z_1 \Rightarrow Z_2 \in Y)$. For the same reason, in the LTS,

from the fixed conditions, we infer the formula $\exists z \in \mathbf{D}\xi$. In turn, applying several times the deduction theorem and the generalization rule, as a result, we infer the formula **A^t14**.

Consider now the translation of the axiom scheme of replacement ASR.

ASR^t. Let $\varphi(x, y)$ be a formula of the theory T that does not contain Y as a free variable. Then $\forall x \in \mathbf{D} \forall y \in \mathbf{D} \forall y' \in \mathbf{D} (\varphi^\vee(x, y) \wedge \varphi^\vee(x, y') \Rightarrow y = y') \Rightarrow \forall X \in \mathbf{D} \exists Y \in \mathbf{D} \forall x \in \mathbf{D} (x \in X \Rightarrow \forall y \in \mathbf{D} (\varphi^\wedge(c, y) \Rightarrow y \in Y))$, where φ^\vee and φ^\wedge denote the formulas $\mathbf{M} \models \varphi[\mathbf{s}^\vee]$ and $\mathbf{M} \models \varphi[\mathbf{s}^\wedge]$, respectively, in which by \mathbf{s}^\vee we denote the corresponding change of the sequence \mathbf{s} under the translation of the quantifier over-formulas $\forall x(\dots)$, $\forall y(\dots)$, and $\forall y'(\dots)$ indicated above, and by \mathbf{s}^\wedge indicated above we denote the corresponding change of the sequence \mathbf{s} under the translation of the quantifier over-formulas $\forall X(\dots)$, $\exists Y(\dots)$, $\forall x(\dots)$, and $\forall y(\dots)$.

Denote the first part of this scheme by α and the second part by β . Then $\widetilde{\text{ASR}}^t = (\alpha \Rightarrow \beta)$. Therefore, the equivalence $(\neg(\alpha \Rightarrow \beta)) \Leftrightarrow (\alpha \wedge \neg\beta)$ implies

$(\neg\widetilde{\text{ASR}})^t = (\forall y \in \mathbf{D} \forall y' \in \mathbf{D} \forall y'' \in \mathbf{D} (\varphi^\vee(x, y) \wedge \varphi^\wedge(x, y') \Rightarrow y = y')) \wedge (\exists X \in \mathbf{D} \forall Y \in \mathbf{D} \exists x \in \mathbf{D} (x \in X \wedge \exists y \in \mathbf{D} (\varphi^\wedge(x, y) \wedge y \notin Y)))$.

Further, the first part of this scheme we will denote by α' , and the second part by β' . For the adduced below concrete formula φ the symbol-strings α' and β' will be formulas of the LTS.

Consider the following formulas of the theory T :

- $\varphi(x, y, z) \equiv (x \in \omega \wedge \mathbf{a} \in y \wedge y \bowtie \wedge (z \Leftarrow x + 1 \rightarrow_y y) \wedge \forall k \in x + 1 (z(k) \bowtie) \wedge z(0) = \mathbf{a} \wedge \forall k \in x + 1 (\forall l \in x + 1 (k \in l \Rightarrow z(k) \in z(l))) \wedge \forall V ((V \bowtie \wedge z(0) \subset V \wedge V \in y) \Rightarrow \exists k \in x + 1 (V = z(k))))$,
- $\varphi(x, y) \equiv \exists z \psi(x, y, z)$,

where $x + 1$ denotes the class $x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}}$.

For the formula φ we have the following translations:

- $\varphi^\vee(x, y) \Leftrightarrow \exists z \in \mathbf{D} (x \in \omega^\sigma \wedge \mathbf{a} \in y \wedge y \in \mathbf{R} \wedge (z \Leftarrow x + 1 \rightarrow_y y)^\sigma \wedge \forall k \in \mathbf{D} (k \in (x + 1)^\sigma \Rightarrow z(k) (k \in (x + 1)^\sigma \Rightarrow \forall l \in \mathbf{D} (l \in (x + 1)^\sigma \Rightarrow (k \in l \Rightarrow z(k)^\sigma \in z(l)^\sigma))) \wedge \forall V \in \mathbf{D} ((V \in \mathbf{R} \wedge (z(0)^\sigma \subset V) \wedge V \in y) \Rightarrow \exists k \in \mathbf{D} (k \in (x + 1)^\sigma \wedge (V = z(k)^\sigma))))$,
- $\varphi^\wedge(x, y) \Leftrightarrow \exists z \in \mathbf{D} (x \in \omega^\tau \wedge \mathbf{a} \in y \wedge y \in \mathbf{R} \wedge (z \Leftarrow x + 1 \rightarrow_y y)^\tau \wedge \forall k \in \mathbf{D} (k \in (x + 1)^\tau \Rightarrow z(k)^\tau \in \mathbf{R}) \wedge z(0)^\tau = \mathbf{a} \wedge \forall k \in \mathbf{D} (k \in (x + 1)^\tau \Rightarrow \forall l \in \mathbf{D} (l \in (x + 1)^\tau \Rightarrow (k \in l \Rightarrow z(k)^\tau \in z(l)^\tau))) \wedge \forall V \in \mathbf{D} ((V \in \mathbf{R} \wedge (z(0)^\tau \subset V) \wedge V \in y) \Rightarrow \exists k \in \mathbf{D} (k \in (x + 1)^\tau \wedge (V = z(k)^\tau))))$,

where θ^σ , θ^τ , η^σ , and η^τ denote the terms $\theta_{\mathbf{M}}[\mathbf{s}^\sigma]$ and $\theta_{\mathbf{M}}[\mathbf{s}^\tau]$ and the formulas $\mathbf{M} \models \eta[\mathbf{s}^\sigma]$ and $\mathbf{M} \models \eta[\mathbf{s}^\tau]$ in which \mathbf{s}^σ and \mathbf{s}^τ denote the corresponding changes of the sequences \mathbf{s}^\vee and \mathbf{s}^\wedge under the translation of the quantifier over-formula $\exists z \in \mathbf{D}(\dots)$.

Let us verify that $\omega^\sigma = \omega$ and $\omega^\tau = \omega$. By the formula $On(x)$ defined in Sec. 6.1, the class ω is assigned by the formula $\nu \equiv \exists! z (On(z) \wedge z \neq \emptyset \wedge \forall x (On(x) \Rightarrow z \neq x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}}) \wedge \forall y ((On(y) \wedge y \neq \emptyset \wedge \forall x (On(x) \Rightarrow y \neq x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}})) \Rightarrow z \subset y))$, which is deduced in the theory T . Therefore, by what was the three equivalences proved above, the value ω^σ is defined from the formula $\nu^\sigma \Leftrightarrow \exists! z \in \mathbf{D} (On^{\mathbf{D}}(z) \wedge z \neq \emptyset \wedge z \in \mathbf{a} \wedge \forall x \in \mathbf{D} (On^{\mathbf{D}}(x) \wedge x \in \mathbf{a} \Rightarrow z \neq (x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}})^* \wedge \forall y \in \mathbf{D} ((On^{\mathbf{D}}(y) \wedge y \neq \emptyset \wedge y \in \mathbf{a} \wedge \forall x \in \mathbf{D} (On^{\mathbf{D}}(x) \wedge x \in \mathbf{a} \Rightarrow y \neq (x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}})^*)) \Rightarrow z \subset y))$, because \mathbf{a} is translated into \mathbf{a} .

Since the assembly \mathbf{D} is transitive, by a direct verification, using the definition of the formula $On(z)$ from 6.1, we can prove that for $z \in \mathbf{D}$, the equivalence $\widetilde{on}^{\mathbf{D}}(z) \Leftrightarrow On(z)$ holds.

When we verified the deducibility of the formula **A^t13**, we established that the conditions $x \in \mathbf{D}$ and $x \in \mathbf{a}$ imply the equality $(x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}})^* = x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}}$. Therefore, the formula ν^σ is equivalent to the formula ν' , which differs from the formula ν by only bounded quantifier prefixes $\exists \dots \in \mathbf{D}$ and $\forall \dots \in \mathbf{D}$. But since the formula ν and the formula ν' contain the subformulas $z \in \mathbf{a}$, $x \in \mathbf{a}$, and $y \in \mathbf{a}$, which immediately imply the restrictions $z \in \mathbf{D}$, $x \in \mathbf{D}$, and $y \in \mathbf{D}$, it follows that in the LTS, we infer the equivalence $\nu \Leftrightarrow \nu' \Leftrightarrow \nu^\sigma$. From the equivalence $\nu \Leftrightarrow \nu^\sigma$, it follows that $\omega^\sigma = \omega$. In just the same way, it is verified that $\omega^\tau = \omega$.

We have verified above that under the conditions $x \in \omega$, $\omega \in \mathbf{a}$, and $\mathbf{a} \in \mathbf{D}$, by the transitivity \mathbf{D} , the equality $(x + 1)^\sigma = x \cup_{\mathbf{a}} \{x\}_{\mathbf{a}} = x + 1$ holds. We also verified that under the condition $y \in \mathbf{R}$, the equivalence $(z \rightleftharpoons x + 1 \rightarrow_y y)^\sigma \Leftrightarrow (z \rightleftharpoons x + 1 \rightarrow_y y)$ holds. Therefore, $z(i)^\sigma = z(i)$ for every $i \in x + 1$. The same holds for the variant with the sign τ .

Therefore, $\varphi^\vee(x, y)$ and $\varphi^\wedge(x, y)$ are equivalent to the same formula $\varphi^*(x, y) \equiv \exists z \in \mathbf{D}(x \in \omega \wedge \mathbf{a} \in y \wedge y \in \mathbf{R} \wedge (z \rightleftharpoons x + 1 \rightarrow_y y) \wedge \forall k \in \mathbf{D}(k \in x + 1 \Rightarrow z(k) \in \mathbf{R}) \wedge z(0) = \mathbf{a} \wedge \forall k \in \mathbf{D}(k \in x + 1 \Rightarrow \forall l \in \mathbf{D}(l \in x + 1 \Rightarrow (k \in l \Rightarrow z(k) \in z(l)))) \wedge \forall V \in \mathbf{D}((V \in \mathbf{D} \wedge (z(0) \subset V) \wedge V \in y) \Rightarrow \exists k \in \mathbf{D}(k \in x + 1 \wedge V = z(k))))$.

In the same way as in the proof of Theorem 4, Sec. 6, in the LTS, we infer the formula

$$\forall x \forall y \forall y' \forall z \forall z' (\psi(x, y, z) \wedge \psi(x, y', z') \Rightarrow y = y' \wedge z = z'),$$

which means that y and z are uniquely defined by x . We also infer the formula

$$\forall x \forall x' \forall y \forall y' \forall z \forall z' ((\psi(x, y, z) \wedge \psi(x', y', z') \wedge x \in x') \Rightarrow \forall k \in k + 1 (z'(k) = z(k)) \wedge (z'(x + 1) = y')),$$

which means that for $m \in n$, the sequence $u(n)$ continues the sequence $u(m)$ and $U_{m+1}^n = U(m)$.

Now, let us deduce the formula

$$\forall x \forall y \forall z (\psi(x, y, z) \Rightarrow \forall k \in ((x + 1) \setminus 1) \exists x' \exists y' \exists z' (x' \in x \wedge \varphi(x', y', z') \wedge z(k) = y')),$$

which means that all members of the sequence $u(n)$ starting from the first one are constructed from the previous classes $U(m)$ for $m \in n$.

Denote the formula $\forall k \in ((x + 1) \setminus 1) \exists x' \exists y' \exists z' (x' \in x \wedge \psi(x', y', z') \wedge z(k) = y')$ by $\eta(x, y, z)$. We will infer this formula under the condition $\psi(x, y, z)$. The formula $\psi(x, y, z)$ postulates that x is a natural number, y is a universal class greater than \mathbf{a} and there exists a finite y -sequence of universal classes $u(x) \equiv (y_k \in y | k \in x + 1)_y$ such that $y_0 = \mathbf{a}$, $y_k \in y_l$ for all $k \in l \in x + 1$, and if V is a universal class and $y_0 \subset V \in y$, then $V = y_k$ for some $k \in x + 1$. By Theorem 5, Sec. 5, for every $x \in \omega$ such a sequence exists and is unique, and, moreover, $u(x) | m + 1 = u(m)$ for all $m \leq x$.

The formula $\eta(x, y, z)$ has the form $\forall k (k \in (x + 1) \setminus 1 \Rightarrow \exists x' \exists y' \exists z' (x' \in x \wedge \psi(x', y', z') \wedge z(k) = y'))$. Let us show that from our conditions and the condition $k \in (x + 1) \setminus 1$, the formula $\eta'(x, y, z, k) \equiv \exists x' \exists y' \exists z' (x' \in x \wedge \psi(x', y', z') \wedge z(k) = y')$ is deduced.

Consider x' such that $x' + 1 = k$. This is possible, because $1 \subseteq k$. By Theorem 5, Sec. 5, for the given x' there exist y' and z' such that $\psi(x', y', z')$. Since $x' \in k \subseteq x$, it follows that $x' \in x$. It remains to show that $z(k) = y'$, i. e., $y' = y_k$ in the sequence $u(x)$. Since $\psi(x', y', z')$, it follows that $\mathbf{a} = y_0 \in y'$, i. e., $y_0 \in y'$. Since $y' \bowtie$, there are only three possibilities: (1) $y \in y'$; (2) $y = y'$; (3) $y' \in y$. In the first case, $y = z'(l)$ for some $l \leq x' < x$; but this is impossible; in the second case, the sequences $u(x)$ and $u(x')$ must coincide, but this is impossible, because $x' < x$. As a result, only case (3) $y' \in y$ is possible. Thus, $y_0 \subset y' \in y$, and, consequently, by the condition, $y' = y_m$ for some $m \in x + 1$. Let us show that $m = k$. Actually, if $m < k$, then $y_m = z'(m)$, but this is impossible, because $z'(m) \in y'$ for all $m < k$. If $m > k$, then, taking $V \equiv y_k$, we obtain the condition $y_0' \subset V \in y'$, which implies $V = y_l'$ for $l < k$, but this is also impossible. Therefore, $y' = y_k$, i. e., $z(k) = y'$. Thus, under the conditions $\psi(x, y, z)$ and $k \in (x + 1) \setminus 1$, we infer the formula $\eta'(x, y, z, k)$, and, therefore, by the deduction theorem, under the condition $\psi(x, y, z)$, we infer the formula $k \in (x + 1) \setminus 1 \Rightarrow \eta'(x, y, z, k)$; from here, by the generalization rule, the formula $\eta(x, y, z)$ is deduced. Then, applying once more the deduction theorem and the generalization rule, we obtain the required formula.

Consequently, $z(k) \in \mathbf{R}$ for all $k \in x + 1$. Take in the capacity of value of the variable X the class ω . Since $\omega \in \mathbf{a} = U(0) \subset \mathbf{D}$, it follows that $\omega \in \mathbf{D}$. Take any $Y \in \mathbf{D}$. Then $Y \in U(x_0)$ for some $x_0 \in \omega$. This means that in the LTS, the formula $\exists x \exists y (\exists! z (\psi(x, y, z) \wedge Y \in y))$ is deduced. Denote the unique values of the variables y and z corresponding to the value x_0 by y_0 and z_0 . From all this, it follows that the formula $\psi(x_0, y_0, z_0) \wedge Y \in y_0$ is deduced.

By definition, $y_0 \in \mathbf{R} \subset \mathbf{D}$. From the previous arguments, it follows that $z_0(k) \in \mathbf{R}$ for any $k \in x_0 + 1$. It is clear that $k \in \mathbf{D}$ and $l \in \mathbf{D}$.

If $V \in \mathbf{D}$ and $V \in \mathbf{R}$, then $V \bowtie$, and, therefore, the conditions $z_0(0) \subset V$ and $V \in y_0$ imply $\exists k \in \mathbf{D}(k \in x_0 + 1 \wedge V = z_0(k))$. Thus, in the LTS, the following formula is deduced:

$\delta(x_0, y_0, z_0) \equiv (x_0 \in \omega \wedge \mathbf{a} \in y_0 \wedge y_0 \in \mathbf{R} \wedge (z_0 \rightleftharpoons x_0 + 1 \rightarrow_{y_0} y_0) \wedge \forall k \in \mathbf{D}(k \in x_0 + 1 \Rightarrow z_0(k) \in \mathbf{R}) \wedge z_0(0) = \mathbf{a} \wedge \forall k \in \mathbf{D}(k \in x_0 + 1 \Rightarrow \forall l \in \mathbf{D}(l \in x_0 + 1 \Rightarrow (k \in l \Rightarrow z_0(k) \in z_0(l)))) \wedge \forall V \in \mathbf{D}((V \in \mathbf{R} \wedge z_0(0) \subset V \wedge V \in y_0) \Rightarrow \exists k \in \mathbf{D}(k \in x_0 + 1 \wedge V = z_0(k))))$.

Since $z_0 \rightleftharpoons x_0 + 1 \rightarrow_{y_0} y_0$, by Axiom **A11**, it follows that, $P \equiv \text{rng}_{y_0} z_0 \in y_0$. Therefore, $z_0 \subset ((x_0 + 1)_{y_0}^* P) \equiv Q$. Moreover, $x_0 + 1 \in \omega \in \mathbf{a} \in y_0$ implies $x_0 + 1 \in y_0$. By Lemma 3, Sec. 2, $Q \in y_0$. Consequently, by Axiom **A8**, $z_0 \in y_0 \in \mathbf{D}$. Hence we obtain $z_0 \in \mathbf{D}$. Therefore, in the LTS, the formula $\varphi^*(x_0, y_0) = \exists z \in \mathbf{D}\delta(x_0, y_0, z_0)$ is deduced. Moreover, the formula $Y \in y_0$ was deduced. By the axiom of regularity **A12**, the formula $y_0 \notin Y$ is deduced.

For the same reason, we deduced the formula $\varphi \wedge (x_0, y_0) \wedge y_0 \notin Y$. Since $x_0 \in \mathbf{D}$, $x_0 \in \omega$, and $y_0 \in \mathbf{D}$, by logical means, we infer the formula $\exists x \in \mathbf{D}(x \in \omega \wedge \exists y \in \mathbf{D}(\varphi \wedge (x, y) \wedge y \in Y))$. Since $\omega \in \mathbf{D}$ and $Y \in \mathbf{D}$, then by logical means, we infer the formula β' .

Now let $x, y, y' \in \mathbf{D}$, $x \in \omega$, $\in y$, $\mathbf{a} \in y'$, $y \in \mathbf{R}$, and let $y' \in \mathbf{R}$.

Let us infer the formula $\varphi^*(x, y) \wedge \varphi^*(x, y') \Rightarrow y = y'$. Consider the condition, $\mu(x, y, y') \equiv \varphi^*(x, y) \wedge \varphi^*(x, y')$. According to this condition for a natural number x , there exist universal classes y and y' from \mathbf{R} greater than \mathbf{a} and finite sequences of universal classes $u(x) \equiv (y_k \in y | k \in x + 1)_y$ and $u'(x) \equiv (y'_k \in y' | k \in x + 1)_{y'}$ such that $y_0 = y'_0 = \mathbf{a}$, $y_k \in y_l$ and $y'_k \in y'_l$ for any $k \in l \in x + 1$, and if V and W are universal classes from \mathbf{R} and $y_0 \subset V \in y$, $y'_0 \subset W \in y'$, then $V = y_m$, $W = y'_l$ for some $m, l \in x + 1$. Suppose that $y \neq y'$. Since y and y' are universal classes, it follows that in this case either $y \in y'$ or $y' \in y$. Suppose, for example, that $y \in y'$. Set $W \equiv y$. We obtain $W \in \mathbf{R} \wedge \mathbf{a} \subset W \wedge W \in y'$, $W = y'_l$ for some $l \in x + 1$. Thus $y = y'_l$ for $l \in x + 1$. Similarly, for any $k < x + 1$ there exists $l(k) < x + 1$ such that $y_k = y_{l(k)}$. Since $\forall k \in x + 1(y_k \in y)$, it follows that $\forall k \in x + 1(y_{l(k)} \in y')$, i. e., $l(k) \in l$ for all $k \in x + 1$. Moreover, if $k, m \in x + 1$ and $k \neq m$, then $y_{l(k)} \neq y_{l(m)}$. Consequently, there exists an inclusion of the set $x + 1$ into the set $l \in x + 1$, but this is impossible. The case $y' \in y$ is verified in just the same way. Consequently, $y = y'$. Applying the deduction theorem, we deduce the formula $\mu(x, y, y') \Rightarrow y = y'$.

Therefore, the formula $\varphi^\vee(x, y) \wedge \varphi^\vee(x, y') \Rightarrow y = y'$ is deduced. From here, by logical means, the formula α' is deduced.

As a result, in the theory S , the formula $\alpha' \wedge \beta'$ is deduced, which is equal to the formula $(\widetilde{\neg ASR})^t$ with a given concrete formula φ .

Since all the translations of the axioms of the theory T turned out to be deducible formulas of the theory S , it follows that the theory T is consistent.

(2) We will argue in the naive propositional logic with the symbol of implication \supset .

Denote by Φ_a and Ξ_a the totalities of axioms of the theories $T \equiv LTS + \neg ASR$ and $S \equiv LTS$, respectively.

Consider the propositions $A \equiv \text{cons}(S) \supset \neg(\Xi_a \vdash ASR)$ and $B \equiv \text{cons}(S) \wedge (\Xi_a \vdash ASR)$. Then $\neg A = \text{cons}(S) \wedge \neg(\Xi_a \vdash ASR)$. Using **LAS10**, we obtain $\neg A \supset B$.

It is clear that $B \supset (\Phi_a \vdash ASR)$ and $\Phi_a \vdash \neg ASR$. Therefore, the proposition $B \supset (\Phi_a \vdash ASR) \wedge (\Phi_a \vdash \neg ASR)$, i. e., the expression $B \supset \neg \text{cons}(T)$ is valid.

By the deduction rule, $\neg A \supset \neg \text{cons}(T)$.

According to item (1) of our statement, the proposition $\text{cons}(S) \supset \text{cons}(T)$ is valid. Therefore, $B \supset \text{cons}(T)$ is valid. By the deduction rule, we have $\neg A \supset \text{cons}(T)$.

Thus, the proposition $(\neg A \supset \text{cons}(T)) \wedge (\neg A \supset \neg \text{cons}(T))$ is deduced. Applying the tautology $(\neg A \supset C) \wedge (\neg A \supset \neg C) \supset A$ (see [21], chap. 1, § 7), we infer the proposition A . \square

Using a more complicated abstract interpretation, one can also prove that if the LTS is consistent, then the axiom scheme ASR is not deducible in the LTS+ICU.

In Secs. 5.2 and 5.4, we have proved the closurenens of the assembly $\overline{\mathbf{V}}$ of all classes in the LTS with respect to all basic finite set-theoretical operations. Therefore, in the assembly $\overline{\mathbf{V}}$ we can define such basic mathematical systems as groups, topological spaces, automata, and other on classes, and also define morphisms between them. For the same reason in $\overline{\mathbf{V}}$, we can consider abstract categories of all such mathematical systems and morphisms between them.

From the undeducibility in the LTS of the global axiom scheme of replacement, it follows that the assembly $\overline{\mathbf{V}}$ does not possess the fourth of the five properties of the Eresmann–Dedecker–Sonner–Grothendieck universe listed in the Introduction and necessary for developing in this universe valuable category theories. Therefore, the theory of abstract categories in the LTS will be essentially poorer than the theory of (local) categories in the LTS.

In particular, these abstract categories of mathematical systems will not be closed with respect to such infinite operations as the sum $\cup(A_i | i \in I)$ and the product $\prod(A_i | i \in I)$ of the collection $(A_i | i \in I)$ of objects A_i of these categories and consequently will be abstract categories without direct and inverse limits (see [5], ch. 2, § 1; chap. 3, § 2).

On the contrary, in the ZF or NBG set theories with the axiom of universality abstract categories of mathematical systems from the point of view of naive category theory do not differ absolutely from local U -categories of mathematical systems.

8.2. The nondependence of axiom ICU on the axioms of the LTS. In Sec. 6, we have compared the LTS with the ZF theory with some additional axioms. The fact that axiom IC postulating the existence of an inaccessible cardinal is independent of the axioms of the ZF theory is well-known (see [19], 13). Consequently, Axioms *ISIC* and *ISIC*₂ postulating the existence of infinite sets of inaccessible cardinals are also independent of the axioms of the ZF theory.

It remains only to clarify the nondependence of Axiom *ICU* postulating the existence of an infinite class of universal classes on the axioms of local theory of sets.

Statement 2. (1) *If the LTS is consistent, then $LTS + ICU$ is consistent.*
(2) *If the LTS is consistent, then Axiom ICU is not deducible in LTS.*

Proof. (1) By Lemma 1, Sec. 6, Axiom ICU is equivalent to Axiom ITCU. Therefore, we will consider the equivalent theory $T \equiv LTS + \neg ITCU$. Consider the abstract interpretation $\mathbf{M} \equiv (\mathbf{D}, I)$ of the theory T in the finitely-closed set theory $S \equiv LTS$, described in the proof of Statement 1. In the proof of Statement 1, we have established that the interpretation \mathbf{M} is an abstract model of the LTS in the S set theory.

In virtue of Lemma 2, Sec. 6, in the conjunctive kernel of Axiom ITCU, we can insert one more formula $\mathbf{a} \in Y$. Therefore, consider the formula

$$\varphi \equiv ITCU \equiv \exists Y (\forall U (U \in Y \Rightarrow U \in \mathbf{a}) \wedge \mathbf{a} \in Y \wedge \forall U \forall V (U \in V \wedge V \in Y \Rightarrow U \in Y) \wedge \forall V (V \in Y \Rightarrow \exists W (W \in Y \wedge V \in W))).$$

The translation of this formula on some abstract sequence \mathbf{s} of elements of the assembly \mathbf{D} under the interpretation \mathbf{M} has the form of the formula

$$\psi \equiv \mathbf{M} \models \varphi[\mathbf{s}] = \exists Y \in \mathbf{D} (\forall U \in \mathbf{D} (U \in Y \Rightarrow U \in \mathbf{R}) \wedge \mathbf{a} \in Y \wedge \forall U \in \mathbf{D} \forall V \in \mathbf{D} (U \in \mathbf{R} \wedge U \in V \wedge V \in Y \Rightarrow U \in Y) \wedge \forall V \in \mathbf{D} (V \in Y \Rightarrow \exists W \in \mathbf{D} (W \in Y \wedge V \in W))).$$

Suppose that the condition ψ is fixed and consider a class $E \in \mathbf{D}$ the existence of which follows from this condition. Consider the classes $A_n \equiv \{x \in U(n) | \exists k \in n + 1 (x = U_k^n)\}$, consisting of all members of the $U(n)$ -sequences $u(n)$ from Theorem 5, Sec. 6. Further, along with U_n^n , we will write U_n .

Let us prove by the natural induction that $A_n \subset E$ for every $n \in \omega$. Consider the assembly $\mathbf{X} \equiv \{x | x \in \omega \wedge Ax \subset E\}$. If $n = 0$, then $U_0 \in E$ implies $A_0 \equiv \{x \in U(0) | \exists k \in 1 (x = U_k^0)\} \subset E$. Therefore, $0 \in \mathbf{X}$.

Let $n \in \mathbf{X}$, i. e., $U_k^n \in E$ for every $k \in n + 1$. By the property of the class E , there exists $V \in \mathbf{D}$ such that $V \in E$ and $U_n \in V$. Moreover, $U_n \equiv U_n^n = U_n^{n+1} \in U_{n+1}^{n+1} \equiv U_{n+1}$. If $U_{n+1} = V$, then $U_{n+1} \in E$. Let $U_{n+1} \in V$. In the proof of Theorem 5, Sec. 6, we have established that $U_{n+1} = U(n) \in \mathbf{R}$. Since $V \in E$, by the property of the class E , we conclude that $U_{n+1} \in E$. Finally, let $V \in U_{n+1}$. Then $V \in \mathbf{D}$ and $V \in E$ implies $V \in \mathbf{R}$. Consequently, V is a universal class. From $V \in U_{n+1} \in U(n + 1)$, by the axiom of universality, we infer that $V \in U(n + 1)$. Therefore, by the property of incompressibility from Theorem 5, Sec. 6, the condition $U_0^{n+1} = \mathbf{a} \subset V \in U(n + 1)$ implies the equality $U = U_k^{n+1}$ for some $k \in n + 2$. From $V \in U_{n+1}^{n+1}$, it follows that $k < n + 1$. Hence $U_n^{n+1} = U_n^n \equiv U_n \in V = U_k^{n+1}$ implies $n < k < n + 1$, but this is impossible. From the obtained contradiction, we infer that the third case $V \in U_{n+1}$ is impossible. By Proposition 4, Sec. 6, and Theorem 4, Sec. 6, there is no other possibilities, except the considered three

cases. In the first and the second cases, we obtain $U_{n+1} \in E$. Moreover, $U_k^{n+1} = U_k^n \in E$ for all $k \in n+1$. Therefore, $A_{n+1} \subset E$ implies $n+1 \in \mathbf{X}$. By the principle of natural induction in the LTS (Theorem 2, Sec. 5), $\omega \subset \mathbf{X}$.

Thus, from the formula ψ , we deduced the formula $\chi \equiv \forall x \in \omega (U_x \in E)$. Moreover, from ψ , one can infer $E \in \mathbf{D}$. Consequently, $E \in U(m) = U_{m+1}$ for some $m \in \omega$. Therefore, $U_{m+1} \notin E$. This means that from ψ the formula $\neg\chi$ is inferred. By the theorem of deduction, in the theory S , we deduce the formulas $(\psi \Rightarrow \chi)$ and $(\psi \Rightarrow \neg\chi)$.

Applying **LAS9** and the logical explicit axiom $(\psi \Rightarrow \chi) \Rightarrow ((\psi \Rightarrow \neg\chi) \Rightarrow \neg\psi)$, we consecutively deduce the formulas $(\psi \Rightarrow \neg\chi) \Rightarrow \neg\psi$ and $\neg\psi$. The last formula is equal to the formula $\mathbf{M} \models (\neg\varphi)[\mathbf{s}]$. Therefore, \mathbf{M} is an abstract model of T in S .

(2) The proof is similar to the proof of assertion (2) from Statement 1. □

Thus, ICU does not depend on axioms of the LTS, i. e., is actually a new axiom for the LTS.

Corollary. *If the LTS is consistent, then the assertion (3) from Proposition 3, Sec. 6, is not deducible in the LTS.*

It follows from here that in the LTS there is only a countable assembly of universes, constructed in Theorem 5, Sec. 5, and there are no means to continue it further on like what was done in assertion 3) from Proposition 3, Sec. 6.

8.3. Locally-minimal theory of sets. A local theory of sets with an additional axiom, which states that infinite class of universes does not exist, will be called a *locally-minimal theory of sets* (LMTS), i.e. $LMTS \equiv LTS + \neg ICU$. Statement 2 implies that LTS and LMTS are mutually consistent.

In LMTS, the assembly \mathbf{U} of all universal classes has the following complete description.

Theorem 1. *In LMTS, an assembly sequence $(u(n)|n \in \omega)$ of finite $U(n)$ -sequences $u(n) \equiv (U_k^n \in U(n)|k \in n+1)_{U(n)}$ of universal classes from Theorem 5, Sec. 5, includes all universal classes; more strictly, $\forall U(U \bowtie \Leftrightarrow \exists n \in \omega(U = U_n^n))$.*

Proof. Consider an assembly $\mathbf{U}^* \equiv \{U|U \bowtie \wedge \exists n \in \omega(U = U_n^n)\}$. Suppose that $\mathbf{U}^* \neq \mathbf{U}$, i.e., there exists a universal class X such that $X \notin \mathbf{U}^*$.

Fix numbers $n \in \omega$ and $k \in n+1$. Suppose that $X = U_k^n$. Then by Theorem 5, Sec. 5, $X = U_k^n = U_k^k \in \mathbf{U}^*$, but this is not true. Therefore, $X \neq U_k^n$ for all $n \in \omega$ and $k \in n+1$. By Theorem 4, Sec. 5, and Proposition 4, Sec. 5, either $X \in U_k^n$ or $U_k^n \in X$. Suppose that $X \in U_k^n$. By Corollary 2 to Proposition 4, Sec. 5, $U_0^n = \mathbf{a} \subset X \in U_k^n$. By Axiom **A8**, from LTS, we have $U_0^n \in U_k^n$. Since $k \leq n$, we have $n \geq 1$. Thus, $U_0^{n-1} = \mathbf{a} \subset X \in U_k^n \subset U_n^n - U(n-1)$. By the property of incompressibility from Theorem 5, Sec. 5, $X = U_l^n$ for some $k \in n+1$. But we have proved above that this is impossible. This contradiction implies that $U_k^n \in X$ for all $n \in \omega$ and $k \in n+1$.

Therefore, according to Proposition 3, Sec. 6, we deduce Axiom ICU, but this is impossible if LMTS is consistent. Consequently, our assumption is not true and $\mathbf{U}^* = \mathbf{U}$. □

We have proved that LMTS satisfies the following two properties:

- (1) it has the property of universal comprehension, which is expressed as the universality axiom **A6** $(\forall x \exists U(x \in U \wedge U \bowtie))$;
- (2) it has a countable metasequence (assembly) of all universes. Neither the $ZF + U$ theory nor the $ZF + U(\omega)$ theory (see Introduction) satisfy these two properties.

Moreover, Statement 1 implies that LMTS is strictly weaker than the $ZF + U$ and $ZF + U(\omega)$ theories. Therefore, LMTS satisfies Conditions (1)–(3) given in the Introduction. Thus, this theory is more natural for category theory than the $ZF + U$ and $ZF + U(\omega)$ theories.

REFERENCES

1. P. V. Andreev, E. I. Bunina, V. K. Zakharov, and A. V. Mikhalev, “The foundation of category theory within the framework of the local theory of sets,” In: *International Algebraic Conference. Abstracts of Reports* [in Russian], St. Petersburg (2002), pp. 10–11.
2. G. Blanc and A. Preller, “Lawvere’s basic theory of the category of categories,” *J. Symbol. Logic*, **40**, 14–18 (1975).
3. G. Blanc and M. R. Donnadiou, “Axiomatisation de la catégorie des catégories,” *Cahiers de topologie et géométrie différentielle*, **17**, 135–170 (1976).
4. N. Bourbaki, *Théorie des Ensembles*, Hermann, Paris (1960).
5. I. Bucur and A. Deleanu, *Introduction to the theory of categories and functors*, Mir, Moscow (1972).
6. N. da Costa, “On two systems of set theory,” *Proc. Koninkl. Nederl. Akad. Wet., Ser. A*, **68**, No. 1, 95–98 (1965).
7. N. da Costa, “Two formal systems of set theory,” *Proc. Koninkl. Nederl. Akad. Wet., Ser. A*, **70**, No. 1, 45–51 (1967).
8. N. da Costa and A. de Caroli, “Remarques sur les universes d’Ehresmann–Dedecker,” *C.R. Acad. Sci. Paris, Sér. A.*, **265**, 761–763 (1967).
9. P. Dedecker, “Introduction aux structures locales,” *Coll. Géom. Diff. Glob. Bruxelles*, 103–135 (1958).
10. C. Ehresmann, “Guttungen von lokalen strukturen,” *J.-Ber. Deutsch. Math. Verein*, **60**, No. 2, 49–77 (1957).
11. S. Eilenberg and S. MacLane, “General theory of natural equivalences,” *Trans. Amer. Math. Soc.*, **58**, 231–294 (1945).
12. S. Feferman, “Set-theoretical foundations of category theory,” *Lect. Notes Math.*, **106**, 201–247 (1969).
13. A. Fraenkel, “Zu den Grundlagen der Cantor–Zermeloschen Mengenlehre,” *Math. Ann.*, **86**, 230–237 (1922).
14. P. Gabriel, “Des catégories abéliennes,” *Bull. Soc. Math. France*, **90**, 323–448 (1962).
15. R. Topoi Goldblatt, *The Categorical Analysis of Logic*, North–Holland Publishing Company, Amsterdam–New York–Oxford (1979).
16. W. S. Hatcher, *The Logical Foundations of Mathematics*, Pergamon Press, Oxford (1982).
17. H. Herrlich and G. Strecker, *Category Theory. Introduction*, Heldermann Verlag, Berlin (1979).
18. J. Isbell, “Structure of categories,” *Bull. Amer. Math. Soc.*, **72**, 619–655 (1966).
19. T. J. Jech, *Lectures in Set Theory with Particular Emphasis on the Method of Forcing*, Springer–Verlag, Berlin–Heidelberg–New York (1971).
20. J. L. Kelley, *General Topology*, Van Nostrand Company, Toronto–New York–London (1957).
21. A. N. Kolmogorov and A. G. Dragalin, *Introduction in Mathematical Logic* [in Russian], MGU, Moscow (1982).
22. A. N. Kolmogorov and A. G. Dragalin, *Mathematical Logic. Additional Chapters*, [in Russian], MGU, Moscow (1984).
23. K. Kuratowski and A. Mostowski, *Set Theory*, Polish Scientific Publishers, Warszawa (1967).
24. F. W. Lawvere, “The category of categories as a foundation for mathematics,” In: *Proceedings of the Conference on Categorical Algebra*, Springer–Verlag, Berlin (1966), pp. 1–20.
25. S. MacLane, *Locally Small Categories and the Foundations of Set Theory*, Symposium, Warsaw (1959), pp. 25–43.
26. S. MacLane, “One universe as a foundation for category theory,” *Lect. Notes Math.*, **106**, 192–200 (1969).
27. S. MacLane, *Categories for Working Mathematician*, Springer–Verlag, Berlin (1971).
28. E. Mendelson *Introduction to Mathematical Logic*, Van Nostrand Company, Toronto–New York–London (1971).
29. J. Sonner, “The formal definition of categories,” *Math. Z.*, **80**, 163–176 (1962).

30. V. K. Zakharov and A. V. Mikhalev, "The local theory of classes and sets as a foundation for category theory," In: *Mathematical Methods and Applications. Proceedings of the 9th Mathematical Lectures MGSU* [in Russian], MGSU, Moscow (2002), pp. 91–94.
31. V. K. Zakharov and A. V. Mikhalev, "On the concept of a mathematical system," *Fund. Prikl. Mat.*, **4**, No. 3, 927–935 (1998).
32. V. K. Zakharov and A. V. Mikhalev, "Mathematical systems," In: *Proceedings of the International Algebraic Conference*, Walter De Gruyter Publishers, Berlin (2000), pp. 370–383.
33. E. Zermelo, "Untersuchungen über die Grundlagen der Mengenlehre, I," *Math. Ann.*, **65**, 261–281 (1908).