

Circular Kippenhahn curves beyond rotational invariance

This talk is devoted to finite-dimensional operators whose Kippenhahn curves consist of concentric circles centered at the origin. Equivalently, for such an operator A , the Hermitian pencil

$$\operatorname{Re}(e^{-i\theta}A)$$

has spectrum independent of θ . Such operators are said to have *Circularity property*. A familiar source of examples is given by *rotation invariant operators*:

Definition 1. An operator A is called rotation invariant if it is unitarily similar to $e^{-i\theta}A$ for every θ .

Clearly, rotation invariance is much stronger than the Circularity property. Another notion crucial for what follows is a *partial isometry*.

Definition 2. A linear operator $A \in M_n$ is called a partial isometry if it satisfies $\|Ax\| = \|x\|$ for all $x \perp \ker A$.

In 2015, Gau, Wang and Wu [1] posed the following conjecture.

Conjecture 1 (Gau–Wang–Wu). If A is a unitarily irreducible nilpotent partial isometry whose numerical range is a circular disc centered at the origin, then A is rotationally invariant.

They also verified this conjecture for partial isometries in dimensions up to 6.

In this talk, a new class of operators will be presented, which sits strictly between the class of operators having the Circularity property and the class of rotation invariant operators. We show that every operator of this class provides a counterexample to the above conjecture.

We show how Circularity property intertwines with the notion of partial isometry. Starting from a contraction C , one associates to it a partial isometry $\mathcal{A}(C)$, which is itself a contraction, and then iterates this procedure to obtain an infinite sequence of partial isometries. This sequence converges in a certain sense to the so-called Schaffer construction of a contraction. We give necessary and sufficient conditions on the initial contraction C for every element of this sequence to have the Circularity property, and we give a simple example of such an operator. In particular, the aforementioned property holds if and only if the defect pencil

$$\operatorname{Re}(e^{-i\theta}C) + \tau(I - C^*C)$$

has spectrum independent of θ .

There is another closely related conjecture by Gau, Wang and Wu:

Conjecture 2. If A is a partial isometry whose numerical range is a circular disc, then it's centered at the origin.

This conjecture is now settled. Namely, it holds for partial isometries of ranks up to 3 and does not hold for partial isometries starting with rank 4 - that was shown by Virtanen and O’Laughlin about two months ago. There are, however, certain refinements that still might be true. We recall existing approaches and settled cases, and demonstrate how techniques[3] lead to resolution of this Conjecture for rank three partial isometries.

References

- [1] H.-L. Gau, K.-Z. Wang and P. Y. Wu, *Circular numerical ranges of partial isometries*, Linear and Multilinear Algebra **64** (2016), no. 1, 14–35.

- [2] E. Shen, *On circular Kippenhahn curves and the Gau–Wang–Wu conjecture about nilpotent partial isometries*, Linear Algebra and its Applications 749, 2026, pp.208-241. <https://doi.org/10.1016/j.laa.2026.07.001>
- [3] N. Popov, E. Shen, I. Spitkovsky, *On Kippenhahn curves of low-rank partial isometries*, Integr. Equ. Oper. Theory 98, 32 (2026). <https://doi.org/10.1007/s00020-026-02846-w>