CREMONA GROUP WORKSHOP

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Throughout these sessions we work over an algebraically closed field $\Bbbk$ of characteristic zero.

Definition. The Cremona group of rank n is

$$\mathrm{Cr}_n(\Bbbk) := \mathrm{Aut}_{\Bbbk}\, \Bbbk(x_1, \dots, x_n) = \mathrm{Bir}\, \mathbb{P}^n_{\Bbbk},$$

the group of birational self-maps of projective space.

Example. $\mathrm{Cr}_1(\Bbbk) = \mathrm{Aut}_{\Bbbk}(x) = \mathrm{Aut}\, \mathbb{P}^1_{\Bbbk} \simeq \mathrm{PGL}(2; \Bbbk)$, the group of Möbius transformations

$$x \longmapsto \frac{ax+b}{cx+d}.$$

All birational maps of $\mathbb{P}^1_{\Bbbk}$ are biregular.

For $n \geq 2$, we are interested in finite subgroups $G \subset \mathrm{Cr}_n(\Bbbk)$.

1. Examples of subgroups of the Cremona group

1.1. $\mathrm{GL}(n; \Bbbk) \subset \mathrm{Cr}_n(\Bbbk)$, $\mathrm{PGL}(n+1; \Bbbk) \subset \mathrm{Cr}_n(\Bbbk)$.

1.2. The *Cremona involution*

$$\tau_n : (x_0 : x_1 : \dots : x_n) \longmapsto (x_0^{-1} : \dots : x_n^{-1}).$$

For $n = 2$, this is the *standard quadratic Cremona involution*

$$\tau_2 : (x_0 : x_1 : x_2) \longmapsto (x_1 x_2 : x_0 x_2 : x_0 x_1),$$

and the map is undefined at $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$.

Theorem (Max Noether)**.** $\mathrm{Cr}_2(\Bbbk) = \langle \tau_2, \mathrm{PGL}(3; \Bbbk) \rangle$.

1.3. *Monomial Cremona transformations*: Consider a matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}(2; \mathbb{Z})$$

and the map

$$\chi : \Bbbk(x, y) \to \Bbbk(x, y), \qquad \chi(x, y) = (x^a y^c, x^b y^d).$$

This defines an embedding $\mathrm{GL}(2; \mathbb{Z}) \subset \mathrm{Cr}_2(\Bbbk)$.

1.4. *Affine transformations*: We have $\mathrm{Aut}\, \mathbb{A}^n \subset \mathrm{Cr}_n(\Bbbk)$.

Theorem (Jung)**.** *The group* $\mathrm{Aut}\, \mathbb{A}^2$ *is generated by affine and triangle automorphisms*

$$(x,\ y) \longmapsto (ax + f(y),\ by).$$

This is no longer true for $n \geq 3$. For example, the Nagata automorphism $\mathbb{A}^3 \to \mathbb{A}^3$ given by

$$\begin{array}{ccl} x_1 & \longmapsto & x_1 - 2x_2(x_1x_3 + x_2^2) - x_3(x_1x_3 + x_2^2)^2 \\ x_2 & \longmapsto & x_2 + x_3(x_1x_3 + x_2^2) \\ x_3 & \longmapsto & x_3 \end{array}$$

is not a triangle morphism. We denote by $T \subset \mathrm{Aut}\, \mathbb{A}^3$ the subgroup of tame automorphisms, which can be decomposed into affine and triangle morphisms. Shestakov and Umirbaev proved that the Nagata map is not tame.

1.5. *The Nagata group*: Consider a sufficiently general pencil $\mathcal{P} \subset |\mathscr{O}_{\mathbb{P}^2(3)}|$, $\mathcal{P} \simeq \mathbb{P}^1$ (a one-dimensional linear system) of plane cubic curves. The base locus of $\mathcal{P}$ is nine points $P_1, \ldots, P_9 \in \mathbb{P}^2$ in general position. Blowing up these points gives us $X \to \mathbb{P}^2$, and $\pi : X \to \mathbb{P}^1$ is an elliptic fibration.

Let $F_1, \ldots, F_9 \subset X$ be the exceptional divisors. A general fibre $E := X_\xi$, $\xi \in \mathbb{P}^1$ is an elliptic curve. Fixing F_1 as a base point, we have a group law on $(E, E \cap F_1) \simeq \mathrm{Pic}^0(E)$ and the divisors $E \cap (F_j - F_1)$ are independent in $\mathrm{Pic}^0(E)$. The translations $E \longrightarrow E$ given by

$$x \longmapsto x + (F_j - F_1)|_E$$

define birational maps $X \dashrightarrow X$. These extend to biregular maps $X \longrightarrow X$, and since X is rational, we have $\mathbb{Z}^8 \subset \mathrm{Aut}(X) \subset \mathrm{Cr}_2(\Bbbk)$.

1.6. *De Jonquière's transformations.* See below.

2. Outline of the proof of Noether's theorem.

Suppose $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational map. To distinguish the source and target, we write $\chi : X \dashrightarrow X'$, with $X \simeq X' \simeq \mathbb{P}^2$. Resolve indeterminacies of χ by

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ \mathbb{P}^2 \simeq X & & \overset{\chi}{\dashrightarrow} & & X' \simeq \mathbb{P}^2 \end{array}$$

so that $\chi \circ f = g$.

Let $\mathscr{H}' := |\mathscr{O}_{X'}(1)|$ be a base point free linear system on X', $\tilde{\mathscr{H}} := g^*\mathscr{H}'$ its pullback on $\tilde{X}$ and $\mathscr{H}$ its birational transform on X. If the map χ is not linear, then the base locus of $\mathscr{H}$ is non-empty and $\mathscr{H} \subset |\mathscr{O}_{\mathbb{P}^2}(d)|$, for some $d \geq 2$. Let

$$f : \tilde{X} = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = X$$

be a factorization into a sequence of blowups of points, let $E_i \subset X_i$ be the exceptional divisor of f_i, and let $E_i^* := (f_{i+1} \circ \cdots f_n)^*(E_i)$. If f is a blowup of distinct points $p_1, \ldots, p_n \in X$, these E_i^*'s are just components of the exceptional divisor of f. Let $\mathscr{H}_i$ be the birational (proper) transform of $\mathscr{H}$ on X_i. On each step we have

$$\mathscr{H}_i = f_i^* \mathscr{H}_{i-1} - m_i E_i,$$

where $m_i \geq 0$ is the multiplicity of $\mathscr{H}_{i-1}$ at the point $f_i(E_i)$. Then by induction we get

$$\tilde{\mathscr{H}} = f^*\mathscr{H} - \sum_i m_i E_i^*.$$

Comparing canonical divisors we also get

$$K_{\tilde{X}} = f^* K_X + \sum_i E_i^*.$$

It is easy to see

$$\tilde{\mathscr{H}}^2 = \mathscr{H}'^2 = 1, \quad \text{and} \quad (K_{\tilde{X}} + \tilde{\mathscr{H}}) \cdot \tilde{\mathscr{H}} = 2p_a(\tilde{\mathscr{H}}) - 2 = -2,$$

so $K_{\tilde{X}} \cdot \tilde{\mathscr{H}} = -3$. We now have two equalities

$$\sum_i m_i^2 = d^2 - 1 \quad \text{and} \quad \sum_i m_i = 3(d-1),$$

from which we obtain the *Noether-Fano inequality*

$$\exists i, j, k \quad \text{such that} \quad m_i + m_j + m_k > d.$$

We have the corresponding exceptional divisors E_i^*, E_j^*, E_k^*, contracting to points p_i, p_j, p_k. Denote by τ the standard Cremona involution with indeterminacy set (p_i, p_j, p_k), and let

$$\hat{\chi} := \chi \circ \tau : \hat{X} \longrightarrow X'$$

be the composite birational map, where $\tau : \hat{X} \to X$ and $\hat{H} \simeq \mathbb{P}^2$. This determines another linear system $\hat{\mathscr{H}}$ as the birational transform of $\mathscr{H}''$ under $\hat{\chi}$, and $\hat{\mathscr{H}} \subset |\mathscr{O}_{\hat{X}}(\hat{d})|$. For a general line $L \subset X'$,

$$\hat{\mathscr{H}} \cdot (\tau^{-1}\chi^{-1}(L)) = 2d - m_1 - m_2 - m_3 < d.$$

This process is called "untwisting of birational maps". Note that $\tau^{-1}\chi^{-1}(L)$ is a conic passing through p_i, p_j, p_k. By induction we keep lowering the degree until we get $d = 1$ and the composite is biregular. So we only need biregular maps and the Cremona involution.

Remark. The above arguments do not give a complete proof of the Noether theorem because we assumed that p_i, p_j, p_k are distinct "honest" points on $\mathbb{P}^2$ in general position. In general, we cannot assume this: for example, the set p_i, p_j, p_k can contain infinitely near points. So, our arguments work only for Cremona maps whose indeterminacy locus is in "general position".

3. De Jonquière's transformations.

As before, consider a birational map $\chi : X \dashrightarrow X'$, where $X \simeq \mathbb{P}^2 \simeq X'$, and let

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \overset{\chi}{\dashrightarrow} & X' \end{array}$$

be the resolution of its indeterminacies, i.e. $\chi \circ f = g$. We also let $\mathscr{H}' := |\mathscr{O}_X(1)|$, $\tilde{\mathscr{H}} := g^*\mathscr{H}'$ and $\mathscr{H}$ is the birational transform of $\mathscr{H}'$ to X. Then

$$\tilde{\mathscr{H}} = f^*\mathscr{H} - \sum_i m_i E_i^*$$

and

$$K_{\tilde{X}} = f^*K_X + \sum_i E_i^*.$$

Definition. We call the birational map χ de Jonquière if $m_0 = d - 1$.

Remark. We have the following equalities:

$$1 = \tilde{\mathscr{H}}^2 = d^2 - \sum_i m_i^2,$$

$$3 = -K_{\tilde{X}} \cdot \tilde{\mathscr{H}} = 3d - \sum_i m_i,$$

$$\sum_{i \neq 0} m_i = 2d - 2 = \sum_{i \neq 0} m_i^2.$$

Therefore, $m_1 = \cdots = m_{2d-2} = 1$.

Proposition. *The birational map χ is de Jonquière if and only if there exists a pencil of lines L'_t on X', $t \in \mathbb{P}^1$, such that $L_t := \chi^{-1}(L'_t)$ is also a pencil of lines on X.*

Proof. Let $\tilde{L}_t := g^* L'_t$. Then

$$\tilde{L}_t = f^* L_t - \sum_i k_i E_i^*$$

and

$$1 = \tilde{\mathscr{H}} \cdot \tilde{L}_t = dn - \sum m_i k_i,$$

where $n = \deg L_t$.

If χ is de Jonquière, then this equality becomes

$$1 = dn - (d-1)k_0 - \sum_{i \neq 0} k_i,$$

so $n = 1$ and L_t is indeed of degree one.

For the converse, let $n = 1$. Then we have

$$1 = d - \sum_i m_i k_i,$$

where k_i is either 0 or 1, since L_t is a pencil of lines. Thus $m_0 = d - 1$. □

3.1. Equations. Suppose χ is de Jonquière and $\mathscr{H}$ is as above. Let p_0 be the point for which $\mathrm{mult}_{p_0}(\mathscr{H}) = d - 1$. Let L be a line passing through p_0. There exists a divisor $C + L \in \mathscr{H}$, so that $\mathrm{mult}_{p_0}(C) = d - 2$. The curve C is given as

$$C = \{b(x_0, x_1, x_2) = 0\}, \quad \text{where} \quad p_0 = [0 : 0 : 1].$$

Let $S \in \mathscr{H}$ be given as

$$S = \{a(x_0, x_1, x_2) = 0\},$$

where

$$\begin{aligned} a &= a_d(x_1, x_2) + x_0 a_{d-1}(x_1, x_2) \quad \text{and} \\ b &= b_{d-1}(x_1, x_2) + x_0 b_{d-2}(x_1, x_2). \end{aligned}$$

This means that χ is given by

$$\chi : [x_0 : x_1 : x_2] \longmapsto [a(x_0, x_1, x_2) : b(x_0, x_1, x_2) x_1 b(x_0, x_1, x_2) x_2].$$

Going to affine coordinates by dividing out by x_2 and setting $x = x_0/x_2$, $y = x_1/x_2$, we can write

$$\chi : (x, y) \longmapsto \left(x, \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)}\right), \quad \text{where} \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0.$$

De Jonquière's involutions are de Jonquière maps which satisfy $x^2 = \mathrm{id}$. This implies $\alpha + \delta = 0$ and $\alpha^2 + \beta\gamma = 1$. We get

$$\chi(x, y) = \left(x, \frac{P(x)}{y}\right).$$

Moreover, we may assume that P is a polynomial of degree $2g + 1$, $g \geq 0$ without multiple roots. (We may have to change coordinates $y \mapsto \zeta y + \eta$, for regular functions ζ, η.) Then the fixed-point set of χ is

$$\mathrm{Fix}(\chi) = \{(x, y) \mid y^2 = P(x)\},$$

which is a hyperelliptic curve of genus g if $g \geq 2$.

Corollary. *For $g \geq 2$, we have a family of involutions parametrised by hyperelliptic curves, which are all non-conjugate in* $\mathrm{Cr}_2(\Bbbk)$.

Proof. Since the fixed-point set of χ is not rational, it cannot be contracted by rational maps, so all birational transformations must preserve the fixed-point locus. □

4. Finite subgroups of the Cremona group

Suppose that $G \subset \mathrm{Cr}_2(\Bbbk)$ is a finite subgroup and that G acts on X. We may assume that X is a projective surface, by virtue of the following reasoning. G acts regularly on a Zariski-open subset $U \subset \mathbb{P}^2$. Consider the quotient $\bar{U}/G$ of the closure of U by G. By taking the normalisation of $\bar{U}/G$ in $\Bbbk(U)$, we obtain a projective surface, so we may as well assume that G acts on a projective surface X.

Now we run the G-equivariant minimal model programme, removing G-orbits that are disjoint unions of (-1)-curves. In the output, which we now call X, only three different cases can occur:

(1) X is a minimal model if and only if K_X is nef if and only if there are no orbits of disjoint (-1)-curves. This is impossible, as X is rational.
(2) there is a G-equivariant fibration $f : X \to Z$ such that Z is a smooth curve, $|-K_X|$ is f-ample and $\rho(X/Z)^G = \mathrm{rk}\,\mathrm{Pic}(X/Z)^G = 1$.
(3) $|-K_X|$ is ample and $\rho(X)^G = 1$.

Proposition. *In the conic bundle case we have a G-minimal G-conic bundle. In the del Pezzo case X is a del Pezzo surface.*

We treat the two cases at length in the following two subsections.

4.1. The conic bundle case. If $-K_X$ is ample over Z, then there is an embedding $X \hookrightarrow \mathbb{P}(\mathscr{E})$, where $\mathscr{E} \to Z$ is a vector bundle of rank 3, such that $X_\eta \subset \mathbb{P}^2 = P(\mathscr{E}_\eta)$ is a reduced conic for all $\eta \in Z$. Note that the conic X_η must be reduced. Indeed, otherwise $X_\eta = 2C$, where $C \simeq \mathbb{P}^1$ and by the genus formula

$$2p_a(C)-2 = (K_X+C).C = K_X.C = \frac{1}{2}K_X.X_\eta = \frac{1}{2}(2p_a(X_\eta)-2) = -1,$$

a contradiction. Hence a general fibre of f is a smooth conic ($\simeq \mathbb{P}^1$) and special fibres are bouquets of two $\mathbb{P}^1$'s.

Remark. X is rational if and only if $Z \simeq \mathbb{P}^1$. Indeed, if X is rational, so Z is by Lüroth's theorem. Conversely, if $Z \simeq \mathbb{P}^1$, then the transcendence degree of $\mathbb{k}(Z) = \mathbb{k}(\mathbb{P}^1)$ equals to one, and since $\mathbb{k}$ is algebraically closed, $\mathbb{k}$ is a c_1-field (a c_1-field is a field such that any form $\phi(x_1, \ldots, x_n)$ with $\deg \phi < n$ represents 0.) Therefore, $\mathbb{k}(X) \simeq \mathbb{k}(Z)(t)$.

If the morphism f is smooth (i.e. f has no degenerate fibres), then we have the following.

Example (rational ruled (Hirzebruch) surfaces)**.** $\mathbb{F}_n$, $n \neq 0$. We can contract the $(-n)$-curve to get a birational map $\mathbb{F}_n \to \mathbb{P}(1,1,n)$. Then

$$\mathrm{Aut}(\mathbb{F}_n) \simeq \mathbb{k}^{n+1} \rtimes \mathrm{GL}(2;\mathbb{k})/\boldsymbol{\mu}_n$$

where $\mathbb{k}^{n+1}$ is regarded as the space $M_n \simeq \mathbb{k}^{n+1}$ of binary forms of degree n with natural action of $\mathrm{GL}(2;\mathbb{k})$.

For $n = 0$ we have $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and there is a split-exact short sequence

$$1 \longrightarrow \mathrm{PGL}(2;\mathbb{k}) \times \mathrm{PGL}(2;\mathbb{k}) \longrightarrow \mathrm{Aut}(\mathbb{F}_0) \longrightarrow \{1,\tau\} \longrightarrow 1.$$

In general f factors through a Hirzebruch surface:

$$f : X \xrightarrow{\sigma} \mathbb{F}_n \longrightarrow Z,$$

where σ is a birational (non-G-equivariant) morphism.

Example (Exceptional conic bundles)**.** Let $g \geq 1$. By definition an *exceptional conic bundle* is a conic bundle $f : X \to Z$ with $2g-2$ degenerate fibres and two disjointed sections F_i, $i = 1, 2$ such that $F_1^2 = F_2^2 = -(g+1)$.

Construction 1. Consider $\mathbb{P}^1 \times \mathbb{P}^1$. Fix a ruling $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and fix two different sections L_i, $i = 1, 2$. We have $L_1^2 = L_2^2 = L_1.L_2 = 0$. Take $g+1$ points $P_1, \ldots, P_{g+1}$ in L_1 and $g+1$ points $Q_1, \ldots, Q_{g+1}$ in L_2, and blow up all $2g+2$ points: $X \to \mathbb{P}^1 \times \mathbb{P}^1 \to Z$.

Construction 2. Let

$$Y \subset \mathbb{P}(1,1,g+1,g+1)$$

is given by

$$t_2 t_3 = F_{2g+2}(t_0,t_1),$$

and let $X \to Y$ be the minimal resolution. Then the projection

$$(t_0,t_1,t_2,t_3) \dashrightarrow (t_0,t_1)$$

induces a structure an exceptional conic bundle on X.

Now assume that X is G-minimal and let Σ be the set of singular fibres, whose size we denote by s. Then $\rho(X) = 2+s$, and by Noether's formula we have $K_X^2 = 8-s$. If $s = 0$, we are in the above case $X \simeq \mathbb{F}_n$. Note that X is not G-minimal if and only if X is a del Pezzo surface with $\rho(X)^G = 2$.

For $s = 1$, 2, 3, 5, X is a not G-minimal:

- The case $s = 1$ is trivial: $f : X \to \mathbb{P}^1$ has a unique section C with negative self-intersection number, so f cannot be G-minimal because C meets only one component of degenerate fibre.
- For $s = 2$ we have $K_X^2 = 6$ and the linear system $|-2K_X - F|$ (here F is the fibre) defines an equivariant contraction $X \to X' = \mathbb{P}^1 \times \mathbb{P}^1$, $K_{X'}^2 = 8$ of two (-1)-curves.
- For $s = 3$ we have $K_X^2 = 5$, and we use $|-K_X - F|$ to blow down $X \to \mathbb{P}^2$.
- For $s = 5$, $K_X^2 = 3$ and X is a cubic surface with a G-invariant line. This line can be contracted and we get a del Pezzo surface X' of degree 4.

Lemma. *Suppose $f : X \to Z$ has two sections $C_1, C_2 \subset X$ with $C_i^2 = -n$. Let s' be the number of components of Σ that meet both C_1 and C_2. Then*

$$2C_1.C_2 + 2n = s - s'.$$

In particular, $s \geq 2n + s' \geq 2n$.

We can use this lemma directly to show that the cases $s \leq 3$ cannot occur as G-minimal models. Our G-minimal surface X has Picard group $\mathrm{Pic}(X) \simeq \mathbb{Z}^{s+2}$. The group G acts on the Picard group with kernel

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \mathrm{Aut}(\mathrm{Pic}(X))$$

From now on we assume that $s \geq 4$. We distinguish two cases.

Case $G_0 \neq \{1\}$. Then G_0 fixes (-1)-curves and so G_0 fixes $s \geq 4$ singular fibres. So the image of $G_0 \to \mathrm{Aut}(Z) \simeq \mathrm{PGL}(2; \Bbbk)$ is trivial. Further, G_0 also fixes negative sections of f, and since it acts trivially on the base, it fixes these sections pointwise. On the other hand, G_0 acts faithfully on a general fibre $F \simeq \mathbb{P}^1$, so $G_0 \subset \mathrm{PGL}(2, \Bbbk)$. Since the intersection of F and a negative section is a fixed point, the group G_0 must be cyclic. Since general fibre is a $\mathbb{P}^1$ and G_0 acts cyclically, G_0 has exactly two fixed points in general fibre, and thus f has two G-invariant sections C_1, C_2. So $\mathrm{Fix}(G_0) \supset C_1 \cup C_2 =: C$. The curve C must be smooth, i.e. the disjoint union of two smooth, irreducible curves (namely the two sections C_1, C_2). This means that $f : X \to Z$ is an exceptional conic bundle.

Case $G_0 = \{1\}$. Then and $G \hookrightarrow \mathrm{Aut}(\mathrm{Pic}\, X)$. We have a short exact sequence

$$1 \longrightarrow G_F \longrightarrow G \longrightarrow G_B \longrightarrow 1,$$

where $G_B \subset \mathrm{Aut}(Z)$. We claim that the map $G_F \to (\boldsymbol{\mu}_2)^s$ into the group of permutations of the components of Σ is an injection. Indeed, otherwise some element $1 \neq \tau \in G_F$ acts trivially on the components of Σ. Since $\mathrm{Pic}(X)$ is generated by $-K_X$ and the classes of these components, τ trivially acts on $\mathrm{Pic}(X)$, a contradiction.

Further, the general fibre is $F \simeq \mathbb{P}^1$, so we also must have an embedding $G_F \hookrightarrow \mathrm{PGL}(2; \Bbbk)$. There are only two such possibilities: $G_F = \boldsymbol{\mu}_2$ and $G_F = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$.

Case $G_F = \boldsymbol{\mu}_2$. The fixed-point locus of G_F is a curve C and some points. Then $C \to Z$ is $2:1$, and C is smooth. In fact C is irreducible, since it cannot have two disjoint components: Doing so would force G_F to fix the components of the singular fibres, but that in turn would force G_F to act trivially on the Picard group (which is generated by the components of singular fibres and a section), which we assumed not to happen.

So C is a (generalized) hyperelliptic curve. Let P be a fixed point of G_F and $P \in F$, where F is a fiber. Consider three possibilities.

a) If $P \in F$ is a smooth point, then we have

$$0 \longrightarrow T_P F \longrightarrow T_P X \longrightarrow T_{f(p)} Z \longrightarrow 0.$$

Since $G_F = \boldsymbol{\mu}_2$ acts on $T_P X$ as $\mathrm{diag}(1, -1)$, $P \in C$ is not a ramification point of f.

b) If $P \in F$ is singular and $G_F = \boldsymbol{\mu}_2$ does not switch the components of F and thus acts as $\mathrm{diag}(-1, -1)$, then P is an isolated fixed point and $P \notin C$.

c) If G_F does switch the components of F, then $P = C \cap F$ is a ramification point.

In conclusion, we have a subset of fibres $\Sigma' \subset \Sigma$, and G_B fixes the sets Σ, Σ'.

Case $G_F = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. We have three non-trivial elements $\delta_1, \delta_2, \delta_3 \in G_F$. We argue as before to get three bisections $C_1 \neq C_2 \neq C_3 \neq C_1$ of δ_i-points. For each singular fibre F there are exactly two elements δ_i, $\delta_j \in G_F$ interchanging components of F. Indeed, let $F = F' \cup F''$ and let $\{P\} = F' \cap F''$. Then $G_F.P = P$. At least one of the δ_i must exchange the components (for otherwise G_F would be cyclic). We get a partition of Σ into three subsets $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ so that $C_i \to Z$ is ramified exactly over $\Sigma_j \cup \Sigma_k$, where $\{i, j, k\} = \{1, 2, 3\}$. Again, G_B fixes the partition Σ_i. One can show that in this case the quotient X/G_F is smooth and X/G_F

In both cases, we have a morphism $X/G_F \to Z$.

5. The del Pezzo case

Let X be a G-minimal G-del Pezzo surface. In this case, $-K_X$ is ample and $\rho(X)^G = 1$. We use the classification of del Pezzo surfaces. There are two well-known constructions.

I. Del Pezzo surfaces are rational. Hence either $X = \mathbb{P}^1 \times \mathbb{P}^2$ or X can be obtained as a blow-up $X \to \mathbb{P}^2$ is in $9-d$ points in general position, where $K_X^2 = d$. Here the morphism $X \to \mathbb{P}^2$ is not unique and is not G-equivariant.

Generalization. Embed $\mathbb{P}^3 \subset \mathbb{P}^9$, and blow up $0 \leq n \leq 7$ points in general position, $\tilde{\mathbb{P}}^3 \to \mathbb{P}^3$. Then $\tilde{\mathbb{P}}^3$ is a so-called del Pezzo threefold of degree $8-n$.

II. Let $d := K_X^2$. Then $d = \dim|-K_X|$.

- If $d = 1$, then $|-K_X|$ is an elliptic pencil with one base point P. Then we can realise X as a degree-6 hypersurface in $\mathbb{P}(1,1,2,3)$. The Galois involution of the projection $X \to \mathbb{P}(1,1,2)$ (which is is $2:1$), called the *Bertini involution.*
- If $d = 2$, then $|-K_X|$ also has one base point P, and we can realise X as a degree-4 hypersurface in $\mathbb{P}(1,1,1,2)$. There is a $2:1$-map $X \to \mathbb{P}^2$ whose Galois involution is called the *Geiser involution.*

If $d \geq 3$, $|-K_X|$ is very ample and X is a degree d subvariety of $\mathbb{P}^d$:

- For $d = 3$, X is a cubic hypersurface in $\mathbb{P}^3$.
- For $d = 4$, $X = X_{2\cdot 2} \subset \mathbb{P}^4$ (intersection of two quadrics).

- For $d=5$, $X=\mathrm{Gr}(2,5)\cap\mathbb{P}^5\subset\mathbb{P}^9$.
- For $d=6$, $X\subset\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ is a divisor of tridegree $(1,1,1)$.
- For $d=7$, $X=X_7\subset\mathbb{P}^7$.
- For $d=8$, $X=\mathbb{F}_1$ or $\mathbb{P}^1\times\mathbb{P}^1$.
- For $d=9$, $X=\mathbb{P}^2$.

Generally, we have $\mathrm{Pic}(X)=\mathbb{Z}^{10-d}$. The group G acts on $\mathrm{Pic}(X)$ so that $\mathrm{Pic}(X)^G=\mathbb{Z}$. The action preserves the intersection pairing and the class of $-K_X$. Let

$$N:=(K_X)^{\perp},\qquad \Delta:=\{\alpha\in N\mid \alpha^2=-2\}.$$

Then Δ is a root system in $N\otimes\mathbb{R}$ depending on d:

d	1	2	3	4	5	6
Δ	E_8	E_7	E_6	D_5	A_4	$A_1\times A_2$

6. Involutions of $\mathrm{Cr}_2(\Bbbk)$

Theorem. *Let $\tau\in\mathrm{Cr}_2(\Bbbk)$ be an involution. Then τ is conjugate to one of the following:*

(1) *A linear involution on $\mathbb{P}^2$.*
(2) *A de Jonquière's involution.*
(3) *A Geiser involution.*
(4) *A Bertini involution.*

The proof of this theorem is quite standard. We may assume that $G=\langle\tau\rangle$ acts on a G-minimal rational surface X. Then we consider two cases: where X has a conic bundle structure $f:X\to Z$ and where X is a del Pezzo surface with $\rho(X)^G=1$.

The conic bundle case. Assume that X has a structure of a minimal (G-equivariant) conic bundle $f:X\to Z$. If f is a $\mathbb{P}^1$-fibration, then $X\simeq\mathbb{F}_n$ for some n. By applying elementary transformations with centers at fixed points we get $n=1$, i.e. $X\simeq\mathbb{F}_1$. Then contracting the negative section we get a linear involution on $\mathbb{P}^2$. If f has degenerate fibers and G trivially acts on $\mathrm{Pic}(X)$ (i.e. $G=G_0$), then f is an exceptional conic bundle. In this case f is not G-minimal, a contradiction. Finally, we assume that f has degenerate fibers, $G\neq G_0$, and $G_F=G$ (i.e. G trivially acts on the base). Then τ switches components of all degenerate fibers. Hence the set of τ-fixed points is a smooth curve C. The induced map $C\to Z=\mathbb{P}^1$. Clearly, there is a birational map

$X \dashrightarrow \mathbb{F}_1$ preserving a general fibre. This induces a fiberwise birational action of τ on $\mathbb{F}_1$. Contracting the negative section we get a de Jonquière's involution on $\mathbb{P}^2$. Thus τ can be written as

$$\tau : (x, y) \longmapsto \left(x, \frac{P(x)}{y} \right),$$

where P is a polynomial of degree $2g+1$, $g \geq 0$ without multiple roots. If $g = 0$, then τ is conjugate to a linear involution.

The del Pezzo case Thus we assume that X is a del Pezzo surface with $\rho(X)^G = 1$. We have

$$-\tau^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X), \qquad K_X \longrightarrow -K_X.$$

Then $-\tau^*(x) = x + \lambda K_X$ for some $\lambda \in \mathbb{Q}$. We compute:

$$-x.K_X = x.K_X + \lambda K_X^2 \lambda, \quad \text{so} \quad \lambda = -\frac{2x.K_X}{K_X^2}$$

Taking x to be a (-1)-curve, we have

$$-\tau^*(x) = x - \frac{2x.K_X}{K_X^2} K_X,$$

and so $K_X^2 = 1$ or 2. If τ' is a Bertini or Geiser involution, then $\tau \circ \tau'$ acts trivially on $\mathrm{Pic}(X)$. But then $\tau \circ \tau'$ preserves 7 or 8 points, and we have $\tau \circ \tau' = \mathrm{id}$, so $\tau = \tau'$.

Here is a geometric explanation of the involution. For $d = 2$, $X \to \mathbb{P}^2$ is the blow-up in 7 points. Fix one more point P. We have a pencil of elliptic curves through those eight points, and this pencil has one base point, P'. The involution exchanges P and P'. For $d = 1$, $X \to \mathbb{P}^2$ is the blow-up in 8 points. Fixing one more point P, there is a unique elliptic curve through those nine points, and letting P be the base point for the group law on that curve, the involution is the group inverse map.

Thus we may assume that X contains no (-1)-curves. Then there are two possibilities.

- For $d = 9$, $X = \mathbb{P}^2$, and τ is a linear involution.
- For $d = 8$, $X = \mathbb{P}^1 \times \mathbb{P}^1$, and τ exchanges the two factors. In suitable non-homogeneous coordinates (x, y) on $\mathbb{P}^1 \times \mathbb{P}^1$ the involution has the form $\tau(x, y) \mapsto (y, x)$. Thus it is conjugate to linear one.

7. Finite subgroups, continued

Suppose $G \subset \mathrm{Cr}_2(\Bbbk)$ is a finite subgroup.

7.1. Simple groups. We begin by considering the case where G is simple. If $G = \mathfrak{A}_5$, then there are a lot of embeddings $G \hookrightarrow \mathrm{Cr}_2(\Bbbk)$ induced by $G \hookrightarrow \mathrm{PGL}(2;\Bbbk) \simeq \mathrm{Cr}_1(\Bbbk)$. Furthermore, $G \hookrightarrow \mathrm{PGL}(3;\Bbbk)$, which already acts biregularly. So assume $G \not\simeq \mathfrak{A}_5$.

If G acts on a conic bundle $f : X \to Z$ then G fits to an exact sequence

$$1 \longrightarrow G_F \longrightarrow G \xrightarrow{f_*} G_Z \longrightarrow 1.$$

Since G is simple, there is an embedding of G into $\mathrm{Aut}(Z)$ or $\mathrm{Aut}(F)$, where F is a general fibre. On the other hand, G is not embeddable to $\mathrm{PGL}(2,\Bbbk)$, a contradiction.

Assume thus that X is a del Pezzo surface. We consider the various cases according to $d = K_X^2$.

- The case $d = 1$ cannot occur, as $|-K_X|$ has one base point P, and G has to act on $T_{P,X}$ effectively. Hence $G \subset \mathrm{GL}(T_{P,X}) \simeq \mathrm{GL}(2;\Bbbk)$. This contradicts the classification of finite subgroups in $\mathrm{GL}(2;\Bbbk)$.
- For $d = 2$, the anti-canonical map $X \to \mathbb{P}^2$ is a double cover whose branch divisor $B \subset \mathbb{P}^2$ is a smooth quartic. The action of G in X descends to $\mathbb{P}^2$ so that B is G-stable. Therefore, $G \subset \mathrm{Aut}(B)$. According to the Hurwitz bound $|G| \leq 168$. Then we have $G \simeq \mathrm{PSL}(2;\mathbb{F}_7)$, with $|G| = 168$.
- For $d = 3$, X is a cubic in $\mathbb{P}^3$. We have $\mathrm{Pic}(X) = \mathbb{Z}^7$ and $G \subset W(E_6) \cap \mathrm{SL}(6,\mathbb{R})$. Hence the order of G divides $25920 = 2^6 \cdot 3 \cdot 5$. On the other hand, G faithfully acts on $H^0(X;-K_X) \simeq \Bbbk^4$. Combining these we get a contradiction.
- For $d = 4$, $X = X_{2\cdot 2} = Q_1 \cap Q_2 \subset \mathbb{P}^4$. Then G acts on the pencil of quadrics $\langle Q_1, Q_2\rangle \simeq \mathbb{P}^1$. Since $G \not\simeq \mathfrak{A}_5$, this action is trivial. Hence, there is a G-stable degenerate quadric $Q' \in \langle Q_1, Q_2\rangle$. This Q' must be a cone over $\mathbb{P}^1 \times \mathbb{P}^1$. Thus G acts effectively on $\mathbb{P}^1 \times \mathbb{P}^1$. Since G is simple, $G \subset \mathrm{Aut}(\mathbb{P}^1)$, a contradiction.
- For $d = 5$, consider the (faithful) action of G on $\mathrm{Pic}(X)$. $\mathrm{Pic}(X)$ contains a root system of type A_4, so $G \hookrightarrow W(A_4) \simeq \mathfrak{S}_5$ and $G \simeq \mathfrak{A}_5$, a contradiction.
- For $6 \leq d \leq 8$, we have $2 \leq \rho(X) \leq 4$. Since the action of $\mathrm{Pic}(X) \simeq \mathbb{Z}^{\rho(X)}$ is non-trivial, we have a contradiction.
- For $d = 9$, $X = \mathbb{P}^2$. So $G \subset \mathrm{PGL}(3;\Bbbk)$, and by the classification of finite subgroups in $\mathrm{PGL}(2;\Bbbk)$ the group G is either $\mathfrak{A}_6$ or $\mathrm{PSL}(2;\mathbb{F}_7)$.

Thus we have proved the following.

Theorem. *Let $G \subset \mathrm{Cr}_2(\Bbbk)$ be a finite simple group. Then either $G \simeq \mathfrak{A}_5$, or G is conjugate to one of the following actions:*

(1) *$G \simeq \mathrm{PSL}(2;\mathbb{F}_7)$ is the Klein group acting on $\mathbb{P}^2$,*
(2) *$G \simeq \mathrm{PSL}(2;\mathbb{F}_7)$ is the Klein group acting on some special del Pezzo surface of degree 2,*
(3) *$G \simeq \mathfrak{A}_6$ is the Valentiner group acting on $\mathbb{P}^2$,*

7.2. p-elementary abelian groups. We say that G is *p-elementary abelian* group if $G \simeq (\boldsymbol{\mu}_p)^r$ for some r and in this case r is called the *rank* of G.

Theorem. *Let $G \subset \mathrm{Cr}_2(\Bbbk)$ be a p-elementary abelian subgroup and let $r = \mathrm{rk}(G)$ be its rank.*

(1) *If $p \geq 5$, then $r \leq 2$, and if $r = 2$ then G is conjugate to a subgroup of $\mathrm{PGL}(3;\Bbbk)$.*
(2) *If $p = 3$, then $r \leq 3$, and if $r = 3$ then G is conjugate to a group acting on the Fermat cubic*
$$\left\{\sum_i x_i^3 = 0\right\} \subset \mathbb{P}^3.$$
(3) *If $p = 2$, then $r \leq 4$, and if $r = 4$, then either G acts on*
$$\left\{\sum_i x_i^2 = \sum_i \lambda x_i^2 = 0\right\} \subset \mathbb{P}^4,$$
or X is some special conic bundle.

If G acts on a conic bundle $f : X \to Z \simeq \mathbb{P}^1$ then, as above, G fits to an exact sequence
$$1 \longrightarrow G_F \longrightarrow G \xrightarrow{f_*} G_Z \longrightarrow 1.$$
where G_F, $G_Z \subset \mathbb{P}^1$. We have $\mathrm{rk}(G_F)$, $\mathrm{rk}(G_Z) \leq 1 + \delta_{2,p}$. Hence $\mathrm{rk}(G) \leq 2 + 2\delta_{2,p}$ in this case.

Assume that G acts on a del Pezzo surface X with $\rho(X)^G = 1$.

As above, we consider the various cases according to $d = K_X^2$.

- If $d = 1$, then G faithfully acts on $T_P X \simeq \Bbbk^2$ and so $\mathrm{rk}(G) \leq 2$.
- If $d = 2$ and $p \neq 2$, then G acts on $H^0(X; -K_X) \simeq \Bbbk^3$.
- If $d = 3$, then G acts on $H^0(X; -K_X) \simeq \Bbbk^4$, and $r \leq 3$.
- For $d = 4$, $X = X_{2\cdot 2} = Q_1 \cap Q_2 \subset \mathbb{P}^4$. Then G acts on the pencil of quadrics $\langle Q_1, Q_2\rangle \simeq \mathbb{P}^1$. Since $G \not\simeq \mathfrak{A}_5$, this action is trivial. Hence, there is a G-stable degenerate quadric $Q' \in \langle Q_1, Q_2\rangle$. This Q' must be a cone over $\mathbb{P}^1 \times \mathbb{P}^1$. Thus G acts effectively on $\mathbb{P}^1 \times \mathbb{P}^1$. Since G is simple, $G \subset \mathrm{Aut}(\mathbb{P}^1)$, a contradiction.

- For $d = 5$, consider the (faithful) action of G on $\mathrm{Pic}(X)$. Then $G \hookrightarrow W(A_4) \simeq \mathfrak{S}_5$ and $\mathrm{rk}(G) \leq 2$.
- For $6 \leq d \leq 8$, we have $2 \leq \rho(X) \leq 4$. Since the action of $\mathrm{Pic}(X) \simeq \mathbb{Z}^{\rho(X)}$ is non-trivial, we have a contradiction.
- For $d = 9$, $X = \mathbb{P}^2$. So $G \subset \mathrm{PGL}(3; \Bbbk)$, and by the classification of finite subgroups in $\mathrm{PGL}(2; \Bbbk)$ $\mathrm{rk}(G) \leq$.