Spherical varieties: exam program

Exercises to solve at home for the final grade (enumerated according to the lecture notes):

Lecture 1: at least one of Exercises 4,5.
Lecture 3: at least one of Exercises 1,2.
Lecture 5: at least two of Exercises 2a,b,c; Exercise 3; at least two of Exercises 4a,b,c.
Lecture 6: at least one of Exercises 1,2.
Lecture 10: at least one of Exercises 1,2,3.
Lecture 11: Exercise 1.
Lecture 18: Exercise 2.
Lecture 19: Exercise 1; at least one of Exercises 2,3.
Lecture 20: at least one of Exercises 1a,b; at least one of Exercises 2a,b.
Lecture 21: at least two of Exercises 2a,b,c.
Lecture 23: Exercise 1.
Lecture 24: at least one of Exercises 1a,b.

Theoretical topics (for the oral exam, to be recounted with full proofs):

(1) Homogeneous fiber bundles and induced representations of algebraic groups. Frobenius reciprocity law.

(2) Homogeneous line bundles and their sections. Borel–Weil theorem. Multiplicities.

(3) Spherical homogeneous spaces: equivalent definitions of sphericality.

(4) Symmetric spaces are spherical.

(5) Normal varieties. The complement of an affine open subvariety is a union of prime divisors.

(6) Divisors and line bundles on normal varieties. G-linearization of line bundles.

(7) Sumihiro theorem. Linearization of a G-action on a normal quasiprojective variety.

(8) Spherical varieties: definition, finite number of G-orbits.

(9) A homogeneous space is spherical if any its equivariant open embedding contains finitely many orbits.

(10) Birational invariants of a spherical variety: weight lattice, rank, invariant valuations, colors.

(11) Two approximation lemmas on invariant valuations.

(12) Invariant valuations are uniquely determined by their restrictions to B-semi-invariant rational functions.

(13) Valuation cone of a spherical variety.

(14) Colored data of the symmetric space $G = (G \times G)/\operatorname{diag}(G)$.

(15) Simple spherical varieties: quasiprojectivity, B-stable prime divisors, open B-chart.

(16) The colored cone of a simple spherical variety: properties.

(17) A simple spherical variety is uniquely determined by its colored cone. Gordan lemma.

(18) Construction of a simple spherical variety from a colored cone: open B-chart, normality.

(19) Construction of a simple spherical variety from a colored cone: closed G-orbit, colored data.

(20) Elementary embeddings of spherical homogeneous spaces. Invariant valuations in the relative interior of a colored cone: geometric meaning.

(21) Colored faces of a colored cone and bigger G-orbits.

(22) The colored fan of a spherical variety: properties. Classification of spherical varieties by colored fans.

(23) Completeness criterion for a spherical variety.

(24) Toric varieties.

(25) Affinity criterion for a spherical variety.

(26) Algebraic monoids and equivariant group embeddings. Classification of reductive monoids.

(27) Local structure theorem.

(28) Toroidal varieties: local structure, criterion of smoothness.

(29) Every spherical variety is dominated by a toroidal one. The valuation cone is polyhedral.

(30) G-module structure of the space of sections of a G-line bundle over a spherical variety.

(31) Spherical double flag varieties and tensor product decompositions. Generalization of the Clebsch–Gordan formula.